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Pure and Applied: Christopher Clavius’s Unifying Approach to Jesuit Mathematics Pedagogy

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Pure and Applied: Christopher Clavius's Unifying Approach to Jesuit Mathematics Pedagogy

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

History (Science Studies)

by

Audrey Marie Price

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Professor Robert S. Westman, Chair
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Professor Cathy Gere
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2017

The Dissertation of Audrey Marie Price is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2017

DEDICATION

To Peggy, Lydia, and Dan

EPIGRAPH

Just as it is necessary that he who wants to read first learns the letters of the alphabet, and having continually repeated them, makes use of them in pronouncing everything out loud, so it is necessary that he who wants to become familiar with all of the mathematical disciplines must first understand these elements of geometry perfectly and fully.

Christopher Clavius on Euclid's *Elements*

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ACKNOWLEDGEMENTS

When I began my graduate career I only had the vaguest idea of what writing a dissertation would entail. This volume could never have come into existence without the constant support of numerous friends and colleagues.

I am grateful to have had a wonderful committee to guide me through the research and writing process. Without them, this project would not have been possible. Early on, Robert Westman and Tal Golan introduced me to a vast literature on early modern science that helped shape the way I read my sources. With each draft he read, Bob pushed me to make my ideas more precise and to more clearly position my work within the literature. I am especially indebted to John Marino who suggested Jesuit mathematics as a research topic and guided me to my first sources. When I first began my research, John worked through challenging Latin passages with me until I reached the point at which I could sit and read Clavius's textbooks. I only wish that he had lived to help me through the writing, too. Nancy Cartwright and Cathy Gere broadened my horizons within science studies and history of science and have helped me understand how my project fits into those fields. Their support in the finishing stages of writing has been incomparable. Ulrike Strasser was one of the first professors to read a very early draft and gave me the confidence that I could develop it into a strong dissertation. As my work became more technical, the presence of a mathematician seemed essential. I cannot thank Daniel Wulbert enough for providing insights into the nuances of the mathematics.

I will never be able to express my thanks adequately to Stanley Chodorow who read the first draft of every chapter of this dissertation and helped me come to understand what I wanted to say and how best to say it. From Stan, I learned the essential skill of revision. My undergraduate adviser, Karen Parshall, has also given me invaluable guidance on my writing and support as I have worked to express my ideas clearly. Without her, I never would have begun this project, let alone finished it.

I am tremendously grateful for the financial and intellectual support provided by the faculty, staff, and teaching assistants at the Humanities Program in Revelle College. Steve Cox and Antony Lyon always had their doors open to me when I needed someone to bounce ideas off of. Many of my colleagues there, especially Alessandra Brivio, became good friends. My classes in Humanities were often a source of joy, and teaching was one of my favorite parts of graduate school.

Research for this dissertation began with a summer at the Huntington Library, to whose staff I am grateful for making it easy to work with sixteenth-century texts and for making it possible to acquire several images to support my text. After that first summer, I did my research online. And while the men and women behind the Google Books database remain anonymous, I would be remiss if I did not express my tremendous gratitude for their efforts in digitizing so many texts. Through Google Books, I was able to conduct my research from the comfort of my own home.

Finally, writing is often an all-consuming task. I am so thankful to my family for their constant patience and support. My husband, Daniel Price, has put up with my long hours spent researching and writing. Without his reminders to take the occasional break, I would have lost my mind. My father, Klaus Lackner, has patiently read numerous drafts of every chapter, worked through many of the mathematical proofs with me, and provided hours of insightful conversations about the ways in which Clavius's mathematics relates to mathematics as it is taught and used today. My sisters, Claire and Maureen, have both edited and proofread parts of the dissertation. I owe my mother, Peggy Lackner, more than I can ever repay. When I was in high school, she taught me how to write. Now she has been a constant source of every kind of support imaginable. I spent several months at my parents' home so that I could focus on writing while my mother did everything else, including taking care of my one-year-old daughter, who I hope will someday forgive me for sneaking away to write while she played with her Oma.

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ABSTRACT OF THE DISSERTATION

Pure and Applied:
Christopher Clavius's Unifying Approach to Jesuit Mathematics Pedagogy

by

Audrey Marie Price

Doctor of Philosophy in History (Science Studies)

University of California, San Diego, 2017

Professor Robert Westman, Chair

This dissertation examines the pedagogical project of Christopher Clavius (1538-1612) as a key step in the development of modern mathematics. In it, I show that Clavius united two contemporary approaches to mathematics: one that saw the field as an abstract way of discovering universal truths, and one that saw the field as an art, that is a tool for practical purposes. To do so, he combined pure and applied

mathematics throughout his textbooks. The union of mathematics as a science and mathematics as an art was motivated by the needs of the nascent Jesuit school system in which Clavius was the professor of mathematics at the flagship school, the Collegio Romano. This unification permeated Clavius's work, leading him to write textbooks on practical mathematics in addition to his commentaries on pure mathematics and theoretical astronomy. Moreover, Clavius combined the different aspects of mathematics within his individual textbooks. This is apparent in his 1574 commentary on Euclid's *Elements,* a text that formed the foundation for Jesuit mathematics education. In this textbook, Clavius's changes to and commentary on the Euclidean text along with his diagrams show the pure abstract forms of mathematics to have potential applications in both sciences, like theoretical astronomy, and arts, like cartography. Through a comparison of Clavius's commentary on Euclid to two other closely contemporary commentaries on the same text, one by Federico Commandino (1509-1575) and the other by Sir Henry Billingsley (d. 1606), I show that Clavius's combination of the abstract and physical facets of mathematics created an image of mathematics on par with philosophy as well as a versatile tool for philosophers and artisans alike. This vision of mathematics combines those found in Commandino's and Billingsley's commentaries, which respectively emphasize mathematics as a science and mathematics as an art. In so doing, Clavius provided his readers with a realist approach to mathematics, paving the way for increasingly more mathematical descriptions of the world that emerged during the Scientific Revolution and that relied on progressive advances in abstract mathematics.

Introduction

Known by his contemporaries as the Euclid of their times, Christopher Clavius, the mathematics professor at the Jesuits' Collegio Romano in the latter half of the sixteenth century, was a prolific author and pedagogue. Despite his contemporary fame, Clavius is largely absent from grand narratives of the Scientific Revolution, since, as one of the last Ptolemaic astronomers, he landed on the wrong side of history. Furthermore, even when Clavius is included in narratives of the Scientific Revolution, he is often portrayed as backwards-thinking, or, at best, one among many promoters of mathematics, ignoring his own contemporary reputation. This is especially true in early surveys of the Scientific Revolution, including Herbert Butterfield's *Origins of Modern Science* and Alexandre Koyré's *From the Closed World to the Infinite Universe.* And, while he often appears in studies of men like Copernicus, Tycho Brahe, Galileo, and Kepler, he has rarely been the principal subject of historical study.[1] Even in his excellent, recent study of Clavius's astronomy, James Lattis still

[1] Clavius is notably absent from Thomas Kuhn's *The Copernican Revolution* (Thomas Kuhn, *The Copernican Revolution: Planetary Astronomy in the Development of Western Thought.* (Cambridge, MA: Harvard University Press, 1957) and Herbert Butterfield's *The Origins of Modern Science* (Herbert Butterfield. *The Origins of Modern Science, Revised Edition.* New York: The Free Press, 1957.) In *The Scientific Renaissance* Marie Boas Hall mentions Clavius and his students only as arbiters of Galileo's discoveries; they accepted his telescopic discoveries as real, but rejected his interpretations of them. Marie Boas Hall, *The Scientific Renaissance: 1450-1630,* (New York: Dover Publications, Inc, 1994), 323-326. More recently, Clavius appears in such narratives, but usually only as an example of one of many scholars promoting mathematics. See Peter Dear, *Revolutionizing the Sciences: European Knowledge and Its Ambitions, 1500-1700*, *Second Edition.* (Princeton: Princeton University Press, 2009), 65-66. Floris Cohen also includes Clavius, but he does make explicit in his narrative that Clavius's promotion of mixed mathematics was not a fundamental break with Aristotelian natural philosophy and did not necessarily lead to the Scientific Revolution. H. Floris Cohen, *How Modern Science Came Into the World: Four Civilizations, One 17th-Century Breakthrough,* (Amsterdam: Amsterdam University Press, 2010), 143-151.

gives his subject significance as one of a dying breed of astronomer-theologians determined to defend Ptolemy, explicitly stating that “Clavius was the last important Ptolemaic astronomer,” one who “helped set the standards by which innovators, such as Copernicus and Galileo, would be judged.”[2] Historians often extend similar treatment to Clavius’s students, several of whom were involved in disputes with Galileo. In fact, in discussions of Galileo, Clavius’s students, especially Oratio Grassi and Christoph Grienberger, are often used as foils to show off Galileo’s rare talent for uncovering the truth.[3]

And yet, as Lattis also expresses, during his lifetime, Clavius was a well-respected mathematician in his own right, and, as the professor of mathematics at the Jesuits’ Roman college, he was an influential figure. Detailed studies of sixteenth-century mathematics do often include Clavius as a prominent figure in the mathematical culture of his day. For example, Antonella Romano presents Clavius as

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[2] James Lattis, *Between Copernicus and Galileo: Christoph Clavius and the Collapse of Ptolemaic Cosmology* (Chicago: University of Chicago Press, 1994), xiv.

[3] For example, see Dijksterhuis’s discussion of Grassi and Galileo’s dispute over what he terms Galileo’s “avowal of atomistic ideas.” In that example Galileo expressed an idea that “describe[s] very accurately what was henceforth to be a fundamental principle in the mechanistic conception of the world” while Grassi denied such “perilous” ideas because they conflicted with “the dogma of the Eucharist.” E.J Dijksterhuis, *The Mechanization of the World Picture*. trans. C. Dikshoorn (Oxford: The Clarendon Press, 1964), 423-424. More recently, in *Galileo’s Telescope,* Massimo Bucciantini and his co-authors, while acknowledging that Clavius and his students were well-respected mathematicians, point out the Jesuits’ conservatism regarding the composition of the moon. They described Clavius’s position that the moon did not have a uniform density as a thesis that “offered easy refuge to those who wanted to the back the notion of the basic difference between celestial and sublunar bodies.” (p. 205). In other words, the Jesuits were firmly rooted in defending ancient Aristotelian notions, even when confronted with evidence that others believed readily contradicted those notions. See Massimo Bucciantini, Michele Camerota, and Franco Giudice, *Galileo’s Telescope: A European Story*, trans. Catherine Bolton. (Cambridge, MA: Harvard University Press, 2015). Of Course, as William Shea’s description of Galileo’s *Assayer* shows, categorizing the Jesuits as “ancients” and Galileo as a modern goes back to Galileo himself. See William R. Shea, *Galileo’s Intellectual Revolution: Middle Period, 1610-1632*, (New York: Science History Publications, 1977), 75.

the primary voice for the promotion of mathematics within the Society of Jesus, and William Wallace identifies Clavius and his students as a significant source for Galileo at the start of the latter's career.[4] These readings agree with Lattis's claim that it was his role as a pedagogue that made Clavius a significant contributor to the astronomical discourse of his day, especially as a founder of school of astronomy with international reach within the Jesuit system.[5] Such a characterization of Clavius begins to illuminate his importance. However, sixteenth-century mathematics included a great deal more than astronomy, and Clavius's significance as a pedagogue needs to be studied in the broader discipline, including the quadrivial sciences of geometry, arithmetic, music, and astronomy and a variety of additional mixed sciences, such as geography, perspective, and mechanics and their corresponding mathematical arts. In this dissertation, I begin that project by examining the first edition of Clavius's commentary on Euclid's *Elements* as the earliest text in which he outlined the vision of mathematics that defined his entire pedagogical project.

As is common with many sixteenth-century figures of scientific interest, little is known of Clavius's early life. Besides his birth in Bamberg in 1538, we can only speculate about him until he was received into the Society of Jesus in Rome in 1555. Ordained in 1564, he completed his vows specific to the Jesuit Order in 1575. In his brief biography of Clavius, Lattis suggests that the young German was likely attracted

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[4] Antonella Romano, *La Contre-Réforme Mathématique: Constitution et Diffusion d'une Culture Mathématique Jésuite à la Renaissance.* (Rome: École Française de Rome, 1999); William Wallace. *Galileo and His Sources: The Heritage of the Collegio Romano in Galielo's Science.* (Princeton: Princeton University Press, 1984).
[5] Lattis, *Between Copernicus and Galileo,* 219.

to the Society by enthusiastic visiting preachers, including the Jesuit Peter Canisius (1521-1597), who were attempting to secure the Catholic Church's hold over German principalities before the Peace of Augsburg (1555) formally recognized the Lutheran presence in the Holy Roman Empire.[6]

Whatever his motivation for joining, Clavius found an opportunity to pursue an education in the Society of Jesus. Nearly immediately upon entering the Society, Clavius was sent to study in Coimbra, where he learned grammar and rhetoric and may have begun his training in philosophy. By 1561 he was back in Rome where he completed the philosophy and theology course required of the Jesuit novitiate. It is not clear where or when Clavius began his study of mathematics, although it is possible that he met the mathematician and astronomer Pedro Nuñez (1502-1578) while at Coimbra. It is also possible that he studied with mathematically-inclined Jesuits during his time in Rome. According to Lattis, Clavius himself claimed that he was self-taught.

Shortly after his return to Rome, Clavius's studies in mathematics paid off as he was asked first to teach the subject at the Collegio Romano (starting in 1563) and later to serve on the pope's commission on calendar reform (starting in the early 1570s). While the development of the Gregorian calendar, which is still in use today, has proven to be Clavius's most long-lasting work, most of his career was devoted to his work in pedagogy. Clavius became the professor of mathematics at Rome in 1563.

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[6] Ibid., 13-14. I have drawn the following biographical sketch from Lattis's work. An annotated chronology of Clavius's life with a more complete record of his travels can be found in the first volume of Clavius's correspondence edited by Ugo Baldini and P.D. Napolitani. *Christoph Clavius: Corrispondenza* ed. U. Baldini and P.D. Napolitani (Pisa: Universtià di Pisa, 1992), Vol. 1.

Although he was not always the instructor for the general mathematics course, he was active in the teaching an advanced mathematics to select students in his academy until his death in 1612.[7] At various times, he held the status of *scriptor*, which granted him a reprieve from teaching in order to write for the benefit of the Society. While his first textbook, his commentary on Sacrobosco's *Sphere* was published in 1570, he began writing more prolifically after 1574 when he spent a few months in Messina working with Francesco Maurolico (1494-1575), a humanist and geometer known for his work translating and editing Greek mathematical texts.[8] There he helped his older colleague with the printing of textbooks that the Jesuits teaching in Messina had requested. Clavius left Messina with a number of manuscripts on a variety of mathematical topics, including Euclid, gnomonics, the 1572 nova, and optics. He later incorporated many of these manuscripts into his own textbooks.[9] The heart of his pedagogical project, Clavius's textbooks are the best source for understanding his goals for mathematics within the emergent Jesuit school system.

While Lattis's examination of Clavius's introductory astronomy textbook, the commentary on Sacrobosco's *Sphere*, begins a much-needed examination of both Clavius as a teacher and of the role his pedagogy played in sixteenth- and seventeenth-century mathematics, Lattis's focus on astronomy means that he makes only brief

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[7] When Clavius was not the instructor, that task was assigned to one of his collaborators in his academy. Christopher Grienberger and Otto van Malecote were among those who taught the general course. See Ugo Baldini, "The Academy of Mathematics of the Collegio Romano from 1553 to 1612" in *Jesuit Science and the Republic of Letters* ed. Mordechai Feingold, (Cambridge MA: The MIT Press, 2003), 48; 72-74.

[8] Paul Lawrence Rose, *The Italian Renaissance of Mathematics: Studies on Humanists and Mathematicians from Petrarch to Galileo*, (Geneva: Librairie Droz, 1975), 159-179.

[9] This entire biographical sketch has been drawn from Lattis, *Between Copernicus and Galileo,* 1-29.

mention of Clavius's other textbooks and their role in the Jesuit curriculum. Moreover, although Clavius, from his position as the mathematics professor at the Collegio Romano during the half-century in which the Jesuits created their curriculum, had tremendous influence on the mathematics portion of the *Ratio Studiorum,* i.e. the Jesuits' official curriculum, his pedagogical project went beyond what that course of study accommodated and is best exemplified in the collection of textbooks he wrote to cover the wide range of mathematical topics he suggested for Jesuit schools.

From 1570, when he published his first edition of his commentary on Sacrobosco's *Sphere*, until 1608, when he published his *Algebra*, Clavius wrote over a dozen textbooks on topics from introductory geometry and astronomy to practical geometry and arithmetic, timekeeping, and the construction and use of various astronomical instruments. He continued to revise his work, publishing several editions of many of his books before they appeared in their final forms in the first four (of five) volumes of the *Opera Mathematica* published over the years 1611 and 1612, the year of his death. These 1611 and 1612 versions represent his final word on what constituted a nearly complete mathematics curriculum. (The fifth, non-pedagogical volume holds his work on calendar reform.)

In the *Opera,* he organized his texts by category rather than order of study. Volume one contains all of his texts on theoretical geometry, the branch of mathematics Clavius believed to be foundational to all other mathematical studies. It contains his commentaries on Euclid's *Elements* and on Theodosius's *Sphere*, as well as treatises on sines, rectilinear triangles, and spherical triangles. Volume two encompasses the texts relevant to concerns of a mundane nature: two explicitly practical texts – the *Geometria practica* and the *Epitome arithmeticae practicae* – and the *Algebra,* which Clavius viewed as a means to extend mathematical study, especially arithmetic.[10] Volumes three and four consist of his various works on astronomy, which Clavius believed to be the pinnacle of the mathematical sciences. As he put it in his commentary to the *Sphere* of Sacrobosco, "Moreover, of these four mathematical sciences [the quadrivium] (from which, indeed, all others dealing in any way whatever with quantities flow and are propagated), astronomy is obviously the broadest on account of the multitude of things which it considers; and on account of that it is the most worthy."[11] The commentary on Sacrobosco and the *Astrolabium* in the third volume serve as the introduction to astronomy, and *Gnomonics,* and two horology books in the fourth offer a study of the applications of astronomy to timekeeping.[12]

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[10] In his preface to the *Algebra* Clavius said that this study treated all of mathematics, but he spent most of his preface explaining how algebra went beyond arithmetic because, unlike the quadrivial study of number, algebra possessed the ability to explain things. See Christopher Clavius, *Algebra* (Rome: Bartholomaeum Zannetum, 1608), 1-3.; N.B. The list of topics in each volume found at the beginning of Volume 1 claims that Volume 2 would also contain Clavius's response to Joseph Scaliger on the topic of cyclometry (the study of the measurement and use of circles), but the treatise is not found in second volume. Christopher Clavius, *Opera Mathematica V tomis distributa ab auctore denuo correcta, et plurimis locis aucta* (Mainz: Antonius Hierat and Reinhardus Eltz, 1612).)(4*v*.

[11] Evidence for the high esteem in which Clavius held astronomy can be found throughout his work. It is perhaps most pronounced in his preface to his commentary on the *Sphere* of Sacrobosco, in which he outlines his thoughts on the history and nature of astronomy. Christopher Clavius, *In Sphaeram Ioannis de Sacro Bosco Commentarius* (Rome: Victorium Helianum, 1570), 2. "Harum autem quatuor scientiarum Mathematicarum (ex quibus quidem omnes aliae quoque modo de quantitate agentes manant, ac quas considerat: & ob id dignissima."

[12] Clavius had also written a third book on horology, the *Compendium brevissimum* of 1603, that did not make it into the *Opera Mathematica*. Christopher Clavius, *Compendium brevissimum describendorum horologiorum horizontalium ac declinantium* (Rome: Aloysium Zannetum, 1603).

While it is clear from the composition of Clavius's *Opera Mathematica* that he saw astronomy as the fulfillment of the promise of mathematics, it also is apparent that Clavius saw mathematics as both an abstract study and a practical discipline in which pure geometry served as the basis for developing mathematics as a versatile means to approach the study and manipulation of the universe. The combination of theoretical and practical components of mathematics is already present in astronomy.[13] The commentary on Sacrobosco provided instruction on how to see the order in the celestial sphere and on how to understand man's place in it, while the various books on timekeeping showed one way in which astronomy could be applied to daily life. But, astronomy, as a mixed science, was itself an application of pure geometry. The commentary on Euclid provided the tools to understand abstract mathematical ideas, including equants and epicycles in astronomy, and illustrated a standard of demonstration that yielded certain knowledge. And, pure mathematics could also be turned to studies and manipulations of the physical world, as was done practical geometry and arithmetic. By writing on each of these topics, Clavius used his pedagogical texts to create a complete picture of the mathematical sciences and arts of his day, a project that went far beyond providing the means to continue the study of astronomy or a defense of a Thomistic Aristotelian cosmology. His work therefore needs to be contextualized within the sixteenth-century development of the mathematical disciplines. Such a contextualization places Clavius at the center of the

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[13] For a discussion of the two sides to astronomy, see Robert Westman, *The Copernican Question: Prognostication, Skepticism, and the Celestial Order* (Berkeley: University of California Press, 2011), 34-43.

Scientific Revolution historiography. In fact, Ugo Baldini has claimed that Clavius's textbooks, which were developed to present a complete summation of "all the important contributions" to a topic, represent "an essential contribution" to the development of "the modern scientific textbook" and the shift from the use of classics to the use of textbooks.[14] As this dissertation will show, Clavius's commentary on *The Elements* is very similar to a modern textbook in that it defines its field through its presentation and commentary on significant contributions to the study of Euclidean geometry. And, the definition that Clavius provided for his discipline united pure and mixed studies, thereby making possible the development of a realist-mathematical science based on the abstract concepts of pure mathematics.

Any discussion of the Scientific Revolution must acknowledge the challenge of encapsulating a two-hundred-year period of intellectual development in a single term, let alone a term that is as loaded as "Scientific Revolution." Indeed, to convey such challenges, Steven Shapin famously opened his book *The Scientific Revolution* with the provocative claim, "There was no such thing as the Scientific Revolution, and this is a book about it."[15] The use of the word "scientific" suggests that the sixteenth and seventeenth centuries are properly seen as the time in which modern science was born, but in those centuries, the word "scientia" meant all systematic knowledge. And natural philosophy, which is often taken as the early modern ancestor of modern science, included elements of theological study and practices like alchemy that would

[14] Ugo Baldini, "The Academy of Mathematics," 67. Of course, Clavius's commentary on Euclid's *Elements*, was a commentary on a classic text. However, Clavius's additions are so extensive that his students could hardly claim to be studying from a classical version of *The Elements.*
[15] Steven Shapin, *The Scientific Revolution* (Chicago: The University of Chicago Press, 1996), 1.

not be considered part of science today.[16] Floris Cohen has recently defined natural philosophy as any knowledge structure "of comprehensiveness, of an aimed-for totality, of prestructured patterns emerging from very general, basic principles, with an anchoring in the phenomenal world being sought in assorted bits and pieces of apparently well fitting empirical evidence." [17] To see sixteenth- and seventeenth-century natural philosophy as a direct analog to modern science is to anachronistically emphasize those elements of that knowledge structure that reflect features of modern science. To avoid such confusion, Cohen uses the term "nature-knowledge" to cover the variety of studies he discusses in his description of the Scientific Revolution.[18] The use of the word "revolution" suggests dramatic and sudden change, but two centuries is a long time and continuity can be identified between medieval and early modern knowledge production.[19] Furthermore, narratives of the Scientific Revolution once focused on a history of ideas removed from their social context.[20] But as many

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[16] Rupert Hall saw the Scientific Revolution as the disassociation of a rational science from "magic and esoteric mystery," which suggests that during the sixteenth and seventeenth centuries magic was still an essential part of knowledge. Rupert Hall, *The Scientific Revolution 1500-1800: The Formation of the Modern Scientific Attitude, Second Edition.* (Boston: Beacon Press, 1962), xii. Lynn Thorndike devoted eight volumes to an in-depth examination of the relationship between magic and science. Two of these are on the sixteenth century in which he discusses astrology, alchemy, cabala, and divination among other occult interests of men of the period, including such luminaries as Pico della Mirandola, Francesco Barozzi, and Francis Bacon. In this work he comes to the conclusion that natural knowledge was still very much informed by belief in "occult virtues." Lynn Thorndike, *A History of Magic and Experimental Science*, Vols. V and VI. (New York: Columbia University Press, 1941), 591.
[17] Cohen, *How Modern Science Came Into the World:* 9.
[18] Ibid., xviii-xix.
[19]See Pierre Duhem, "From *To Save the Phenomena:* Essay on the Concept of Physical Theory from Plato to Galileo." In *Essays in the History and Philosophy of Science*," Translated by Roger Ariew and Peter Barker, 131-156 (Indianapolis: Hackett Publishing Company, 1996).
[20] See William Whewell, *History of the Inductive Sciences,* (London: John W. Parker, West Strand, 1837); Alexandre Koyre, *From the Closed World to the Infinite Universe* (Baltimore: The Johns Hopkins University Press, 1957); Herbert Butterfield, *The Origins of Modern Science, Revised Edition* (New York: The Free Press, 1957).

more recent historians, including Shapin, have argued a revolution of ideas is meaningless without the context of its society.[21] Thus, the history of the Scientific Revolution must also account for the social and cultural contexts in which ideas were explored, even when the details of those contexts may not seem remotely modern, let alone relevant to modern science.

But despite the difficulties associated with using the phrase "the Scientific Revolution" to describe changes in the pursuit of knowledge during the sixteenth and seventeenth centuries, to deny its existence is to throw the baby out with the bath water. While no single story could ever hope to cover all of the nuances of the development of modern science, the concept of the Scientific Revolution provides a framework in which historians can explore the labyrinthine paths pursued by mathematicians, natural philosophers, magicians, metaphysicians, and artisans as they sought to make sense of their universe. As is always the case, some avenues of study proved more fruitful than others and attracted a great deal of attention and plenty of students over long periods of time. Among these paths are those that form the basis for the traditional narratives of the Scientific Revolution, such as the rise of the inductive method and the mathematization and mechanization of knowledge and knowledge production.[22] Other paths, including those that seem to bear little relation to modern science, failed over time to attract adherents because they seemed to lead nowhere or because cultural interests shifted the focus of philosophers to other studies. Among those paths historians can find the pursuit of alchemy and the more mundane

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[21] Shapin, *Scientific Revolution,* 4.
[22] Whewell, *History;* Dijksterhuis, *Mechanization.*

study of natural history.[23] Still other paths, including Clavius's defense of Ptolemaic astronomy, appear from a modern perspective to be dead ends. Through the beginnings, ends, and interconnections of these paths, a picture emerges of a period of intellectual unrest during which definitions of every facet of knowledge – what counted as knowledge, how and where knowledge was produced, who could create knowledge, who could possess knowledge, and why knowledge was worth pursuing - changed.[24] The phrase "the Scientific Revolution" thus takes on meaning not as a clear narrative of inevitable, linear progress from medieval to modern modes of thought but as a periodization for a time, from approximately 1500 to 1700, of significant cultural and intellectual shifts in the understanding of knowledge from which modern science emerged.[25]

This is not to say that the changes that took place in the sixteenth and seventeenth centuries leading to modern science were a series of coincidental lucky breaks. Cohen has convincingly argued for coherence in both his *The Scientific Revolution: A Historiographical Inquiry* and his *How Modern Science Came into the World.* In his efforts to explain how and why modern science emerged in Europe and why the seventeenth-century changes to the pursuit of knowledge gave way to an "as-

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[23] Brian P. Copenhaver, "Natural Magic, Hermetism, and Occultism in Early Modern Science," in *Reappraisals of the Scientific Revolution,* ed. David C. Lindberg and Robert S. Westman*,* (Cambridge: Cambridge University Press, 1990), 261-301.
[24] See Peter Dear, *Revolutionizing the Sciences*. Dear bookended his text with chapters titled "What was worth knowing" at the start of the sixteenth century (the first chapter) and at the start of the eighteenth century (the last chapter), allowing him to effectively illustrate changes in the state of knowledge that occurred in the two centuries he studied.
[25] See H. Floris Cohen, *The Scientific Revolution: A Historiographical Inquiry,* (Chicago: The University of Chicago Press, 1994), 2. As Cohen explains, the concept of the Scientific Revolution emerged in the first half of the twentieth century as "an analytical tool expressly forged for grasping the essence of the emergence of modern science."

yet-unbroken chain of scientific growth," Cohen points to various features of European culture, including the value Europeans placed on manual labor and the value Renaissance artists gave to mathematics, which he called the "European Coloring."[26] Those features of European culture serve as the background to what he identifies as the first three revolutionary transformations: the mathematization of nature, the rise of a natural philosophy based on corpuscularian thought, and the rise of a Baconian "fact-finding, practice-oriented mode of experimental science."[27] These three transformations led to three more revolutionary changes, the last of which, called "The Newtonian Synthesis," allowed natural philosophy to give way to modern science, a precise mathematical study of nature.[28] In this narrative of multiple revolutions, coherence emerges in large part from the changing role of mathematics within natural philosophy. Indeed, the development of a realist-mathematical approach to natural knowledge is the first of the revolutionary transformations Cohen identifies. It lays the groundwork for the "universe of precision," which Cohen calls "one constitutive hallmark of our modern world."[29]

Clavius emerges in this transformation as an early contributor to the Scientific Revolution for his efforts "to enrich Aristotle's doctrine with a dash of mathematical science." He did so by expanding the role of "mixed mathematics," those branches of study with both quantitative and physical components. Mixed mathematics later proved fruitful for those who sought to use mathematics to further nature-knowledge,

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[26] Cohen, *How Modern Science,* xv; Cohen, *Scientific Revolution,* 509.
[27] Cohen, *How Modern Science*., xvi.
[28] Ibid., 637-716.
[29] Ibid., 160.

including such luminaries as Galileo, Kepler, and Descartes.[30] In my dissertation, I examine Clavius's approach to mathematics pedagogy in order to clarify how he used pure mathematics to promote "mixed mathematics" as he sought to establish a place for his discipline within the Jesuits' Aristotelian curriculum, the means through which he became an influential figure in the Scientific Revolution.

The importance of pedagogy in the development of new theories has long been acknowledged within the history of science. When Thomas Kuhn outlined the production of scientific knowledge, he gave pedagogy a central role in the establishment of "normal science" as puzzle-solving based on a particular paradigm. He asserted that the content of the paradigm was conveyed to new scientists through textbooks and practice problem-solving.[31] Furthermore, in his narrative, a new paradigm can only be established when it is accepted into the schools in new textbooks, and when new generations of scientists, trained in the new paradigm, replace the aging subscribers to the old paradigm.[32] Even if one rejects the notion that Clavius's work was in any way "revolutionary," his textbooks remain a valuable source for understanding the ways in which mathematics was taught to rising generations of scholars at the start of the Scientific Revolution.

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[30] Ibid., 143-147. Cohen is rightfully cautious about attributing too much to Clavius's work. In and of itself, Clavius's work could well have led to nothing more than small expansions of Greek mathematics. (See pp. 148-151). But he does acknowledge that "Clavius's example may well have served to some limited extent as one stimulus among others for both Kepler and Galileo when embarking on the adventure of making mathematical science realist" (212).
[31] Thomas Kuhn, *The Structure of Scientific Revolutions, Second Edition* (Chicago: University of Chicago Press, 1970), 46.
[32] Kuhn, *Structure of Scientific Revolutions.* See pages 136-140 for the role of textbooks in making revolutions invisible and pages 151-152 for the replacement of older generations.

Such a reading could lead to the claim that Clavius's work an example of Kuhnian "normal science." However, a close examination of Clavius's textbooks reveals that it would be a mistake to present his work as a simple representation of a pre-Scientific Revolution mode of thought. As Lattis has shown, Clavius's commentary on Sacrobosco's *Sphere* engages deeply with the contemporary discourse on astronomy, including both the Copernican and the Tychonic hypotheses. As I will show, Clavius' commentary on Euclid's *Elements* was a product of his engagement with a variety of contemporary approaches to mathematics. Therefore, his work is better seen as a contribution to the rise of mathematics that marked the beginning stages of the Scientific Revolution than as an example of a pre-Revolution approach to mathematics. Moreover its value as pedagogy lies in its potential for providing new scholars with a vision of mathematics as a model of knowledge that allowed them to contribute to the changes to knowledge production that were taking place during the Scientific Revolution. Accepting Cohen's view that the first revolutionary transformation was the rise of a realist-mathematical science, that is the development of a mathematical approach to describing the physical structure of the universe, Clavius's textbooks, especially his commentary on Euclid, can be seen as part of the efforts to rewrite textbooks that necessarily accompanied that revolution.[33]

Clavius's curriculum and his textbooks can only be understood against the backdrop of the well-documented developments that were taking place in mathematics during the sixteenth century, especially the acceptance of mathematics' ability to make

[33] See Kuhn, *Structure of Scientific Revolutions,* 137 for the argument that all revolutions are accompanied by the rewriting of textbooks.

true claims about the world. In the sixteenth century, mathematics was defined to include both sciences and arts. Within the quadrivium, which comprised the quantitative half of the liberal arts, geometry and arithmetic represented purely mathematical knowledge while astronomy and music belonged to the "mixed" branches of mathematics which combined mathematical and physical subjects. Other such mixed mathematical studies included geography, perspective, and mechanics. In the sixteenth century, building on medieval arguments by scholars like Robert Grosseteste (1175-1253) and Roger Bacon (1214-1292), those engaged in the study of mathematics argued that their field was a discipline in its own right with the ability to make true claims about the world.[34] Thus, the mathematical *arts* of the quadrivium could be, and were, called the mathematical sciences. Yet, the practical value of mathematics to the physical world implied by the term "arts" was also widely recognized by sixteenth-century scholars. Indeed, the mixed branches of mathematics, which often had immediate practical applications, were called "mathematical arts." And so, in that period, scholars writing about mathematics treated the field as both a science, that is a source of universal truths, and an art, that is a source of practical value. And while most scholars recognized both facets of their discipline, they often emphasized one or the other in any given work, keeping a distinction between mathematics as a science and mathematics as an art and illustrating which part of the field they each felt was more valuable.

[34] See Roger Ariew, "Christopher Clavius and the Classification of Sciences" *Synthese,* 83 (1990): 293-300.

The distinction between mathematics as a science and mathematics as an art has been replicated in Scientific Revolution historiography. One narrative of the Scientific Revolution tells of mathematization of knowledge from Copernicus to Newton. Copernicus's *De Revolutionibus* is often taken to mark the beginning of the Scientific Revolution, yet in its dedicatory letter written by Andreas Osiander, the reader was reminded that hypotheses need only provide a means to calculate positions of the planets that accorded with observation, and did not have to be true representations of the structure of the universe.[35] In contrast, Newton's *Principia*, often seen as the culmination of the Scientific Revolution, is a mathematical description of the universe. Thus, the Scientific Revolution can be read as the period in which mathematics gained status as a discipline that could reveal philosophical truths about the universe. That is precisely the argument Dijksterhuis outlined in his *Mechanization of the World Picture.* He concluded, "The mechanization of the world-picture during the transition from ancient to classical science meant the introduction of a description of nature with the aid of the mathematical concepts of classical mechanics; it marks the beginning of the mathematization of science, which continues at an ever-increasing pace in the twentieth century."[36]

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[35] Osiander's comment was a standard disputation device, presenting Copernicus's work as one theory among a variety of theories of the universe. However, it made it possible for readers of Copernicus's text to accept the utility of his mathematical model while ignoring or denying the philosophical claims of heliocentrism, thereby separating mathematics from philosophy. (For a discussion of one such interpretation of Copernicus's work, Philip Melanchthon's, see Robert S. Westman, "The Melanchthon Circle, Rheticus, and the Wittenberg Interpretation of the Copernican Theory," *Isis* 66 (June, 1975), 172-174.) In another article, Westman argued that Osiander's letter denied astronomers – i.e. mathematicians – the right to draw conclusions about natural philosophy. See Robert Westman, "The Astronomer's Role in the Sixteenth Century: A Preliminary Study," *History of Science* 18 (1980): 108-109.

[36] Dijksterhuis, 501.

Studies of mathematics itself also tend to focus on the rise of mathematics within the hierarchy of disciplines, an opportunity the discipline was afforded because of its dual classification as the quadrivial half of the liberal arts and, in Aristotle's works, as one of three branches of philosophy (the others being divine and natural).[37] As Robert Westman discusses in *The Copernican Question,* disciplines in the sixteenth century were ranked hierarchically based on a variety of factors, including their certainty, antiquity, and the measure of abstraction of their subject matter. Together these features contributed to what sixteenth-century scholars called the nobility of their disciplines. As the word "nobility" suggests, the hierarchy of disciplines was ordered based on the potential each discipline had to lead its students to virtue and appreciation for the divine. Certainty, abstraction, and antiquity could all add to a discipline's nobility, as could other factors, such as their practical value and their moral dignity. Moreover, as Westman points out, "praising or satirizing the professions depended on which of these criteria were favored and in which combination."[38]

As I will discuss in Chapter 1, during the sixteenth century, mathematicians used the certainty, abstraction, and antiquity of their discipline to argue for its nobility, and, thus, its status alongside or above natural philosophy within the hierarchy of disciplines. Such arguments contradicted the claim that mathematics was subordinate

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[37] For a discussion of medieval classification of disciplines, and specifically Thomas Aquinas's efforts to address the difficulty of mathematics appearing as both a liberal art and a philosophical science, see Ralph McInery, "Beyond the Liberal Arts" in *The Seven Liberal Arts in the Middle Ages* ed. David L. Wagner (Bloomington: Indiana University Press, 1983), 248-272. For Thomas Aquinas' resolution of the dual nature of mathematics, see especially page 252.
[38] Robert Westman, *The Copernican Question,* 30.

to natural philosophy because the former could only describe phenomena while the latter could explain them.[39] Paul Rose's *The Italian Renaissance of Mathematics* traces mathematicians' claims for the nobility of their discipline through an examination of the restoration of ancient mathematics in sixteenth-century Italy as he makes the case for the existence of mathematical humanism. His discussion begins with Regiomontanus, who carried out an extensive project of translation of Greek mathematics using texts from Cardinal Bessarion's library in the fifteenth century, and ends with a discussion of Galileo, the relationship of mathematics to physics, and the question of the certitude of mathematics that encapsulates the sixteenth-century elevation of the discipline.[40] In this narrative, mathematics rises from the status of a lower discipline to an academic study of the physical world. However, this version of the role of mathematics in the Scientific Revolution only shows the place of mathematics within the philosophy curriculum of universities. It risks removing mathematics from its social and cultural context and ignoring the practical value of mathematics, which was often little esteemed by university-based philosophers and humanists.

While the debate between Aristotelians and mathematicians over the status of mathematics was restricted to the educated Latinate classes, non-Latinate craftsmen also took an interest in the discipline during the sixteenth century. Indeed, another narrative of the Scientific Revolution offers the mathematization of various activities

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[39]See Marcus Hellyer, *Catholic Physics: Jesuit Natural Philosophy in Early Modern Germany* (Notre Dame: University of Notre Dame Press, 2005), 116-117.
[40] Rose, *The Italian Renaissance of Mathematics*.

of craftsmen and engineers as a social project that shifted knowledge production from the contemplative philosophizing of medieval universities to the active practices of marketplaces, workshops, mines, and newly discovered territories, thereby uniting mathematics and physics within the arts. This narrative often emerges in studies of the social context in which the "great" ideas of the traditional progress narrative emerged. For example, E. G. R. Taylor described her *The Mathematical Practitioners of Tudor and Stuart England* as a "chronicle of lesser men…but for whom great scientists would always remain sterile in their generation." In it she observed, "the history of any aspect of applied science and technics…must also be a history of attitudes, of the contemporary climate of opinion."[41] More recently, Deborah Harkness offered a study of knowledge production in London. She claims, "The foundations of the Scientific Revolution in Elizabethan London depended on three interrelated social endeavors: forging communities, establishing literacies, and engaging in hands-on practices."[42] In her work, mathematics is featured as the key to mechanical ingenuity and new inventions for the English artisan class. On a broader scale, Marie Boas Hall, for whom the most significant change of the Scientific Revolution was the change in the social identity of a "scientist," devoted an entire chapter to the uses of mathematics, including astronomy, navigation, cartography, and various aspects of mechanics and studies of motion, in her book, *The Scientific*

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[41] E.G.R Taylor, *The Mathematical Practitioners of Tudor and Stuart England* (Cambridge: The University Press, 1968). (Reprint of the original 1954 edition.) For the first quotation see page xi. For the second, page 4.

[42] Deborah Harkness, *The Jewel House: Elizabethan London and the Scientific Revolution* (New Haven: Yale University Press, 2007), 6.

Renaissance. According to Hall, the widespread study of these fields was the result of "the popularization of science and the new awareness of the needs of the technical man," allowing mathematics to clearly illustrate what she believed to be the crucial shift of the Scientific Revolution, namely, the shift in identity of a "scientist" from a "classical scholar" in 1450, to a "new kind of learned man or a technical craftsman" by 1630.[43] Each of these more applications-oriented readings, however, risk downplaying the role mathematics had in the academies and its connections to natural philosophy.

Clavius's pedagogical project united these two major strands of the development of mathematics, namely the elevation of mathematics within the hierarchy of disciplines and the application of mathematics to worldly affairs. His project was motivated by a desire to give a complete mathematics education to all Jesuit students, some of whom would go on to become scholars and represent the Society of Jesus in philosophical discourse in Europe, and others of whom would become international missionaries and might need knowledge of practical mathematics. Thus, the multifaceted mathematics curriculum he wrote for the Jesuit schools was a reflection of the schools' own varied purposes as the Jesuits' primary missionary activity, and can only be understood in the context of Jesuit schools. While the most obvious purpose for Jesuit schools was to provide training in the catechism to all Catholic boys, it did not take long for them to shift from catechetical training grounds to intellectual academies. The shift can be in large part attributed to the

[43] Hall, *Scientific Renaissance,* 345.

Jesuits preference for teaching the sons of nobles, whom they believed could then lead others to follow the catechism through their example. Thus, even though the lack of fees meant that Jesuit schools attracted sons of artisans and merchants for whom a Latin education could open future opportunities, the Jesuit curriculum became focused on the needs of the uppermost echelons of society, through whose education they hoped to stem the tide of Protestantism. In this light, the development of the mathematics portion of the Jesuit curriculum is an example of Shapin and Schaffer's argument in *Leviathan and the Air Pump* that "Solutions to the problem of knowledge are solutions to the problem of social order."[44] In order to achieve their social goal of strengthening the Catholic Church, the Jesuits had to devise a curriculum that could meet the various needs of their patrons while inspiring faith in the Catholic Church. Because the place of mathematics was contested, that portion of the curriculum offers valuable insight into how the Jesuits sought to use knowledge to promote their faith and a Catholic social order that could withstand the spread of Protestantism.

The Jesuits took a two-pronged approach to knowledge in their curriculum. First, they sought to provide an education suitable to sixteenth-century standards for learned gentlemen. This curriculum was humanistic and began with grammar before teaching humanities and rhetoric. At the Jesuit universities, like the Collegio Romano, students could go on to study logic, philosophy (including mathematics), theology, and Hebrew.[45] The rigor of the advanced portion of the curriculum was determined by

[44] Steven Shapin and Simon Schaffer, *Leviathan and the Air Pump: Hobbes, Boyle and the Experimental Life*, (Princeton: Princeton University Press, 1985), 332.
[45] Paul Grendler, *Schooling in Renaissance Italy: Literacy and Learning, 1300-1600*. (Baltimore: The Johns Hopkins University Press, 1989), 377-381. For a discussion of how Jesuit universities differed

the Jesuits' mission to become active and respected contributors to the intellectual discourse of their times. In this way, they hoped to appeal to the leaders whose sons they sought to teach. Public academic exercises, including orations and disputations, served as venues for Jesuits to display their knowledge and skill and so to convince local leaders that their schools were superior to other options, including Protestant schools.[46] Furthermore, as Martha Baldwin has shown, the Order also depended on the scholarly publications of their members to secure its reputation and patronage for its schools.[47] The Jesuits were so actively engaged in the pursuit of natural knowledge that Mordechai Feingold has recently argued that "by and large, the scholarly activities and aspirations of Jesuits were indistinguishable from those of other contemporary savants, secular or ordained, irrespective of denomination."[48] In order to facilitate such work, at least some Jesuit schools needed to be equipped to train future scholars. In mathematics, Clavius headed a special academy for mathematics students at the Collegio Romano, which, as Ugo Baldini has shown, was as much a research group as it was a classroom. Baldini argues that students in the academy not only followed Clavius's rigorous mathematics curriculum, they also produced original mathematics research, on a variety of topics including geometrical topics, most notably statics,

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from other Italian universities, including the elevation of Thomas Aquinas's work above that of Peter Lombard's in the theology course and the extension of the philosophy curriculum to include metaphysics and logic, see George Gannss, *Saint Ignatius's Idea of a Jesuit University,* (Milwaukee: The Marquette University Press, 1956), 153-187.

[46] Ibid., 368.

[47] Martha Baldwin, "Pious Ambition: Natural Philosophy and the Jesuit Quest for the Patronage of Printed Books in the Seventeenth Century" in *Jesuit Science and the Republic of Letters* ed. Mordechai Feingold, (Cambridge MA: The MIT Press, 2003), 285-321.

[48] Mordechai Feingold, "Jesuits: Savants" in *Jesuit Science and the Republic of Letters* ed. Mordechai Feingold, (Cambridge MA: The MIT Press, 2003), 2.

algebra, and topics like gnomonics that related to calendrical mathematics. The academy also actively worked with numerous scholars interested in mathematics, including Galileo.[49]

Second, since the Jesuits used their schools "not as one ministry among many, but as a super-category equivalent to that into which all the other *consueta ministeria* [customary ministries] fell," they found it expedient to include some practical training to fulfill the requirements of their ministries.[50] Those ministries included preaching, the various corporal works of mercy, administering the sacraments, and "any other works of charity, according to what will seem expedient of the glory of God and the common good."[51] The latter goal - contributing to the common good - led their schools to become integral parts of the cities in which they were established. As John O'Malley notes, the schools should be understood as "civic institutions – usually requested by the city, in some form paid for by the city, established to serve the families of the city."[52] From the point of view of the mathematics curriculum, that civic role hinged on the practical branches of mixed mathematics, including geography, hydrography, and various components of military engineering. Antonella Romano showed in her study of the center-periphery relationships between the norms prescribed by Roman Jesuits and French educational practice that French Jesuits

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[49] Ugo Baldini, "The Academy of Mathematics," 47-98. Baldini's description presents an image of a research group not unlike a modern laboratory, in which the professor outlined topics of interest and worked with the students to develop original contributions to those topics.
[50] John O'Malley, *The First Jesuits,* (Cambridge, MA: Harvard University Press, 1993), 200.
[51]George Ganss, trans. *The Constitutions of the Society of Jesus,* (Saint Louis: The Institute of Jesuit Sources, 1970), 66-67.
[52] John O'Malley, "Introduction" in *The Jesuits II: Cultures Sciences, and the Arts 1540-1773*, eds. John W. O'Malley, S.J., Gauvin Alexander Bailey, Steven J. Harris, and T. Frank Kennedy, S.J. (Toronto: University of Toronto Press, 2006), xxxi.

studied practical mathematics to meet the needs of their royal patrons.[53] In turn, the patrons maintained their local financial support of the schools. The Jesuit schools also needed a curriculum that included practical mathematics in order to train their own missionaries to Asia and the Americas. Before a mission was even set up, practical mathematics – the demonstration of astrolabes and clocks, for example - was used to impress local leaders, as studies of Matteo Ricci's career in China illustrate.[54] Once missions were established (a task which benefitted from mathematical knowledge for the sake of timekeeping as well as building and establishing agriculture), the discipline was clearly necessary in some of the variety of intellectual pursuits in which Jesuit missionaries engaged. For example, nearly 800 works of geography and natural history, many of which were written by missionaries or at least informed by data that they sent back to Europe, were published before the Society's suppression in 1773.[55] Natural history did not necessarily require mathematics, but geography and any accompanying cartography did. Astronomy was also used by missionaries, who could – in addition to enhancing European studies by providing observations from geographically distant points – use their own observations to establish timekeeping devices, develop calendars and almanacs. In order to facilitate such activities, Jesuit colleges needed to provide rigorous training in mathematics.

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53 Romano, *La Contre-Réforme Mathématique,* 3.

54 See Peter Engelfriet, *Euclid in China: The Genesis of the First Chinese Translation of Euclid's Elements, Books I-VI (Jihe Yuanben, Beijing, 1607) and Its Reception Up to 1723* (Leiden: Brill Academic Publisher, 1998), 56-98.

55 Steven J. Harris, "Mapping Jesuit Science: The Role of Travel in the Geography of Knowledge: in in *The Jesuits: Cultures Sciences, and the Arts 1540-1773*, eds. John W. O'Malley, S.J., Gauvin Alexander Bailey, Steven J. Harris, and T. Frank Kennedy, S.J. (Toronto: University of Toronto Press, 2006), 212-240.

When Clavius proposed his curriculum and his textbooks as the mathematics program for Jesuit colleges, he kept the various needs of his school system in mind. In this dissertation I show Clavius's combination of pure and mixed mathematics to be the result of a pedagogical decision that positioned Jesuit mathematical training at the intersection of two sixteenth-century approaches to mathematics: one treated mathematics as a science, that is a contemplative source of truth, placing it alongside or even above natural philosophy in the hierarchy of disciplines, while the other conceived of it as an art, that is a tool with which to actively manipulate the physical world for personal and social benefit. Peter Dear has described these two approaches of mathematics as two discourses. In his view, "One of them was 'natural philosophical,' in the sense of its being contemplative and aimed at understanding the natural world; the other was instrumental and was geared toward the production of practical effects, whether to do with moving weights or improving agriculture."[56] Thus, through pure mathematics, Clavius created a union between these "two mutually supportive, but analytically distinct, enterprises or discourses," and his pedagogical work provides a window into the complexities of early modern mathematics and offers insights into how mathematics eventually gained a central position in the describing and manipulating the world. And while Clavius's own interest in the status of his discipline within the philosophy curriculum remains clear throughout his work, his inclusion of practical study shows how mathematics was necessary to both contemplative philosophy and active practices of artisans, thereby, allowing his work

[56] Peter Dear, "What is History of Science the History of?" in *Isis.* Vol. 96, no. 3 (September 2005), 397.

to connect the two narratives that the historiography of the Scientific Revolution has provided.

The influence of Clavius's project is evident in the work of later Jesuit scholars. For example, Dear has shown that Giuseppe Biancani (1566-1624), a member of Clavius's academy in the late 1590s, expanded on Clavius's arguments for the status of mathematics as a science (as opposed to an art). Dear also explores the same themes in Christopher Scheiner's work on astronomy and optics. Scheiner (1573-1650), who did not arrive in Rome until after Clavius had died, had studied mathematics at Ingolstadt, illustrating that Clavius's ideas were carried throughout the Jesuit school system. In fact, Albert Krayer's reconstruction of the astronomical library at the Jesuit university in Mainz shows that that university held nine of Clavius's texts on that topic in 1630.[57] Krayer also published Otto Cattenius's lecture notes from the school year of 1610/1611. Those notes indicate a close reliance on Clavius's texts in the classroom.[58] With those texts came Clavius's vision of mathematics as both a source of true knowledge about the universe and as a utilitarian pursuit. As Marcus Hellyer has shown, although Jesuit mathematicians clearly discussed the nature of the universe, they also maintained a practical emphasis in their teaching.[59]

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[57] Albert Krayer, *Mathematik im Studienplan der Jesuiten: Die Vorlesung von Otto Cattenius an Der Universität Mainz (1610/1611),* (Stuttgart: Franz Steiner Verlag, 1991), 373-374. Krayer's reconstruction of the astronomical texts held by the library at the University of Mainz in 1630 shows that they held nine astronomy titles by Clavius, with multiple copies bringing the total to eighteen books. Presumably at least some of Clavius's non-astronomical works were also held in the library.
[58] Ibid., 181 – 360.
[59] Hellyer, *Catholic Physics*, 119-137.

Nor was Clavius's influence limited to other Jesuit priests. Roger Ariew has argued that Clavius's combination of pure and mixed mathematics and his elevation of the discipline establishes the context in which René Descartes, who studied at the Jesuit college of La Flèche in the second decade of the seventeenth century, developed his understanding of mathematics. Ariew noted that Descartes himself is supposed to have claimed that Clavius's *Algebra* was his only education in the subject.[60] Furthermore, Clavius's combination of mathematical sciences and arts was key to the development of a realist-mathematical science, a revolutionary transformation in which Galileo was a central figure. And, like Descartes, Galileo can be seen to have been influenced by the Jesuits, especially Clavius, who, on account of the high esteem in which his contemporaries held him, was among the scholars to whom Galileo reached out when he was establishing his career.[61] Wallace has argued that Clavius provided his younger contemporary with at least one set of lecture notes on logic from the Collegio Romano. He has also suggested that Galileo requested and received other notes on mathematical topics from Clavius and his students.[62] While we may never know how much material Clavius provided to Galileo, the former's defense of the status of mathematics against natural philosophers almost certainly appealed to

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[60] Roger Ariew, "“Christopher Clavius, *The Promotion of Mathematics.”* In *Descartes' Meditations: Background Source Materials,* ed. Roger Ariew, John Cottingham, and Tom Sorell, (Cambridge: Cambridge University Press, 1998), 24. The source Ariew included was not the *Algebra*, but rather a brief document Clavius wrote outlining the ways in which the Society could support mathematics within the schools and the importance of doing so.

[61] William Wallace, "Galileo's Jesuit Connections and Their Influence on His Science," in *Jesuit Science and the Republic of Letters*, ed. by Mordechai Feingold, (Cambridge, MA: The MIT Press, 2003), 103-104. Wallace notes that Clavius even helped Galileo secure a teaching position in the late 1580s.

[62] Ibid., 104.

Galileo and helped him to cement his own "general understanding of what a science [including mathematics] was," and may well have contributed to his realist approach to mathematical descriptions of the universe.[63]

Although Descartes' exposure to Clavius's work occurred after the latter's death and Galileo's contact with Clavius began only in the late 1580s, it is possible to see the combination of contemplative and practical components of mathematics that defined Clavius's pedagogical project in his early textbooks. In order to sharpen our understanding of this relationship, I have chosen to compare his 1574 commentary on Euclid's *Elements* with two closely contemporary commentaries on the same ancient text: the 1572 Latin commentary by the Italian humanist Federico Commandino (1509-1575) and the 1570 English commentary by the English haberdasher Henry Billingsley (d. 1606). Each of those commentaries represents an approach to mathematics aligned with one of the two strands of the discipline presented side-by-side in Clavius's work. Commandino's Latin translation of Euclid was part of his efforts as a mathematician and a humanist to restore his chosen discipline to its ancient dignity through the translation of Greek texts.[64] It showed mathematics to be a contemplative study and the branch of philosophy intermediate between divine and natural philosophies.[65] Billingsley's Euclid, as the first edition of *The Elements*

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[63] Wallace, *Galileo and His Sources,* 148; See 136-148 for Clavius's position on mathematics and how he and Biancani argued against Jesuit philosophers for the status of mathematics.
[64] Rose, *Italian Renaissance,* 185-221.
[65] Commandino, *3*v*. "Hinc triplex illud philosophiae genus, Divinum, quod quidem ut nomine, ita & re duo reliqua supra qua dici potest, antecellit; Naturale, quod tertium est, ac postremas ordine, ac dignitate habet partes, & medium, quod mathematicum appellatur: quoniam solum vere disci, ac sciri potest, ob summam rei subiecta constantiam, & certam demonstrandi rationem."

intended for the non-Latinate merchant class of London, showed mathematics to be a practical discipline with potential concrete uses for artisans. Many of those uses are enumerated in John Dee's "Mathematicall Preface." Dee (1527-1608) was well-known to his contemporaries as a philosopher with expertise in mathematics. His studies ranged from geometry to astrology and alchemy to cabala and hermetic philosophy. The preface he wrote to accompany Billingsley's commentary offers an analysis of the discipline of mathematics, including Dee's version of its many branches, both theoretical and practical. While the preface does combine contemplative and instrumental approaches to mathematics, as a survey of various branches of mathematics Dee identifies, it spends a great deal more space on the practical branches of mixed mathematics than on pure mathematics. As such, it is a fitting preface to Billingsley's practically-oriented text.

While a study of a single edition of just one of Clavius's texts cannot hope to provide a complete picture of his pedagogical project, an in-depth discussion of *The Elements* can throw Clavius's goals for mathematics pedagogy into relief because Euclid's role as a foundational text for all mathematical knowledge made it the subject of numerous sixteenth-century commentaries, including Clavius's. That role can be attributed to the structure and content of the text, which dates to the end of the fourth century BCE when Euclid of Alexandria, about whom little is known, is believed to have compiled the mathematical theorems of other more ancient scholars and added some of his own to create a comprehensive text on the foundations of geometry.[66]

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[66] Shuntaro Ito, *The Medieval Translation of* The Data *of Euclid,* (Tokyo: University of Tokyo Press, 1980), 7. *The Elements* was not Euclid's only work. Books titled *Optics,* and *Catoptrics*, and *The Data*

Divided into thirteen books, Euclid's work covers geometry from the definition of a point to the construction of solid figures. The first six books study plane geometry. Three books (seven through nine) develop number theory and one book (ten) is devoted to the study of commensurability of magnitudes. Books eleven through thirteen begin the study of solid geometry, ending with the construction of the five Platonic solids.[67] Thus, the original text introduced a variety of mathematical topics which could be used as the foundation to studies in plane geometry (and its applications, such as optics), number theory, and solid geometry. Over the years some of those further studies were appended to the text. The two books added by Hypsicles of Alexandria in the second century BCE, exploring the relationships of the Platonic solids to each other, were usually included with the Euclidean text in the sixteenth century. In 1566 the French mathematician Franciscus Flussas Candalla (1512-1594 aka Comte François Foix de Candale), added a sixteenth book further comparing the Platonic solids to one another. Some later commentators, including Clavius, appended that book to their own editions of Euclid to provide a more complete study of solid bodies. The utility of that study is attested to by Kepler's use of it in his *Mysterium Cosmographicum* in which he considered the structure of the cosmos in terms of the five regular solids inscribed in one another.[68]

(a study of givens in geometrical problems) are also attributed to him based on the claims of other ancient authors and similarities between styles of the texts. However, all that is known about the man himself is that he lived in Alexandria sometime between the time of Plato and the time of Archimedes.

[67] The Platonic solids are a tetrahedron (four faces), a cube (six faces), an octahedron (eight faces), a dodecahedron (twelve faces), and an icosahedron (twenty faces). These are the only five solids that meet the criteria of being regular solids, meaning that all edges and angles are equal, and having the same number of faces of the polyhedron meet at each vertex.

[68] Johannes Kepler, *Prodromus dissertationum cosmographicarum contiens mysterium cosomographicum de admirabili proportione orbium coelestium.* (Frankfurt: Erasmus Kempferus,

In the mid-sixteenth century, there was something of a revival of interest in *The Elements.* Although the text has a long history in the university mathematics education, it was not until after 1453 when the fall of Constantinople brought Greek scholars, most notably Cardinal Bessarion who was the patron of Regiomontanus, to Western Europe, that European scholars had the opportunity to translate Euclid from Greek versions.[69] The first such translation by Bartolomeo Zamberti (1473-1539) appeared in 1505 in Venice. It was based on a fourth-century CE commentary by Theon of Alexandria found among the texts that Bessarion and other scholars had brought to the West.[70] However, sixteenth-century scholars also had medieval versions of the text available to them. Indeed, many of the commentators from that time likely first learned *The Elements* from a version of Campanus of Novara's thirteenth-century translation. Campanus's version, which was based on Arabic texts, was the most common version of *The Elements,* and the source for most Latin

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1621), p. 52. He mentions Candalla's addition in his thirteenth chapter where he describes the inscription of the Platonic solids into one another. He begins the section with a discussion of propositions in the fifteenth book for which he cites Campanus, the most used medieval translator of Euclid. It is not clear from his text which version or versions of Euclid he used. The reference to Campanus is as follows, "Iam recta connectens centra figurae & basis est radius, sive semidiameter inscripti per ultimam lib. 15. Campani in Euclidem." (p. 50). The reference to Candalla reads, "His adde quae Candalla, & quae alii de corporibus iam demonstrarunt, ut quod potentia NM diametiehtis in sphaera…." (p. 52).

[69] Lon R. Shelby, "Geometry" in *The Seven Liberal Arts in the Middle Ages*, ed. David L. Wagner (Bloomington: Indiana University Press, 1983), 205. In the Middle Ages mathematics education was limited in universities. As Shelby explains, Aristotelian studies were the primary focus, so mathematics took a backseat. Still, for those scholars interested in theoretical geometry, Euclid was the main source. Prior to the twelfth-century, when Adelard of Bath, Hermann of Carinthia and Gerard of Cremona translated the text from Arabic into Latin, medieval universities often taught from remaining fragments of Boethius's fifth-century translation. These were superseded in the thirteenth century by Campanus of Novara's translation.

[70] Cardinal Bessarion was a patron of Peurbach, through whom he became patron to Regiomontanus. The latter's extensive translation project was based on the cardinal's library. See Ernst Zinner, *Regiomontanus: His Life and Work* trans. Ezra Brown (Amsterdam: Elsevier Science Publishers, 1990),13, 29, 51-52.

editions, including the first printed edition of 1482.[71] Sixteenth-century commentators, therefore, grappled with choosing between or combining medieval and ancient sources, making a commentary on *The Elements* a means through which mathematical scholars could demonstrate their own erudition. Furthermore by the mid-sixteenth-century, the elevation of the status mathematics within universities and the rise of a desire for mathematical literacy among artisans had created such a demand for these texts that new editions poured out of print shops. In the introduction to his own 1908 version of Euclid, Thomas Heath lists twenty-two editions, in Greek, Latin, and vernacular languages, published across Europe in the sixteenth century. Twenty of those were published between 1533, when the first printed Greek edition appeared, and 1575, when Commandino's Italian translation of his earlier commentary appeared.[72]

With so many versions of Euclid's text available, it is not immediately clear why Clavius felt he needed to write his own commentary on *The Elements* for Jesuit students. He actually gave two justifications for his commentary in his letter to the reader; one reflected his interest in mathematics as a contemplative study, and the other pointed to the utility of mathematics. First, he claimed (without any examples)

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[71] H.L.L. Busard, *Campanus of Novara and Euclid's* Elements. (Germany: Franz Steiner Verlag, 2005), 32.

[72] Thomas Heath, *The Thirteen Books of Euclid's Elements, Introduction and Books 1 and 2*. (London: Cambridge University Press, 1908), 97- 113; Heath's list is not complete. He selected those which he felt were important editions for the transmission of the Greek text in one way or another. Thus, only Clavius's 1574 Euclid appears in Heath's Introduction, even though the 1589 version was substantially changed. 1575 certainly was not the last year of the sixteenth century in which versions of *The Elements* were printed, but because Heath observed that Commandino's commentary became the source for many future publications, it gave him a convenient point at which to move his list forward to establish his work's place in the translations of Euclid.

that it was unfortunate that most editions of *The Elements* were deeply flawed and difficult to read and understand. Since he argued that all of mathematics was built on the foundation of *The Elements* and that those who had not mastered Euclid would never be able to understand other mathematical authors, such as Archimedes and Ptolemy, Clavius took it upon himself to create an accurate and clear edition.[73] Regarding editions based on Campanus's texts, he expressed concern that in places the authors listed the propositions according the order found in Arabic, rather than Greek, texts, thereby corrupting Euclid's method, which he later claimed gave mathematics the certainty necessary to allow it to uncover universal truths. Furthermore, in typical humanist fashion, he also said that even versions based on the Greek text of Theon of Alexandria, were incomplete and corrupted through errors made by copyists.[74] His edition sought to restore the original Greek order and complete the text. Second, he

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[73] Christopher Clavius, *Euclidis Elementorum* (Rome: Vincentium Accoltum, 1574), a4*v*. "Cum enim longa, diuturnoque experientia nobis esset perspectum, atque exploratum, eam esse utilitatem, atque adeo necessitatem horum elementorum, ut frustra quisquam se speret, ipsorum praesidio, acutissimas, subtilissmasque Archimedis, Apollonii, Theodosii, Menelai, Ptolomaei, caeterorumque illustrium Mathematicorum demonstrationes posse percipere."

[74] Ibid.,a5*r*. "Sed alter secutus in omnibus est traditionem Arabum, qui magna ex parte Euclidis ordinem, ac methodum perverterunt, verbaque propositionum eiusdem locis non paucis immutarunt, ut verus, germanusque auctoris sensus perdifficile possit intelligi; id quod maxime in decimo libro perspicitur: Alter (Theonem intelligo) pene innumeris mendis, uitii que incuria librariorum ita est depravatus, & propter nota graecas, quae in eius demonstrationibus adhibentur, obscuras illas, ac male expressas adeo impeditus, ut magnam difficultatem inexercitatis ingeniis, perplexitatemque gignat. Quo fit, ut Euclidem sine maximo labore, ac studio nemo percipiat." The first Latin translation based on Greek texts, Bartolomeo Zamberti's 1505 version, may have been based on the work of Theon and is likely the version that Clavius had in mind for this critique, as it was at the center of a debate over the merit of translations based on Greek texts versus the merits of those based on the Arabic texts in the early years of the seventeenth century. (Heath, *Thirteen Books,* 98-100). The concern over errors that had been introduced into ancient texts through mistakes made in copying, translating, and interpreting texts was common to all kinds of humanism. Biblical humanists, notably Robert Estienne, were devoted to the pursuit of more accurate scriptures based on Hebrew and Greek manuscripts through which they sought to correct the Vulgate. See Basil Hall, "Bibilical Scholarship: Editions and Commentaries" in *The Cambridge History of The Bible: The West from the Reformation to the Present Day* ed. S.L. Greenslade (Cambridge: The University Press, 1963), 39-93. For Estienne see especially pp. 63-67.

claimed that "on account of the singular utility" of Euclid, the text should be used as a handbook. Therefore, he made the unique choice to have his commentary printed in two small volumes so that it could be easily carried by its owner even on long journeys to far-flung Jesuit missions.[75]

However, neither of the justifications Clavius gave can explain the final version of his commentary. Despite his promise to create an accurate version of Euclid's text, it is misleading to describe Clavius's commentary as a translation of *The Elements.* Rather, it is a new text composed of the propositions and proofs found in ancient and modern versions of *The Elements* with the addition of several of Clavius's own proofs. His text contains references to numerous other versions of Euclid, ancient, medieval, and contemporary. His most frequent citations are to Theon of Alexandria, Campanus of Novara, and Federico Commandino, but he also referenced numerous other mathematicians of his own era, including Francesco Barozzi (1537-1604), John Dee (1527-1608), Peter Ramus (1515-1572), and Jacques Peletier (1517-1582). Such extensive citation of his own contemporaries, far beyond what is found in other commentaries of the day, fit his goals as a teacher by providing his students with the means to create a picture of the current state of discipline.[76] Furthermore, Clavius

[75] Clavius, *Euclidis Elementorum,* a5*v*. "Nam cum Euclides, propter singularem utilitatem, instar enchiridii, manibus semper debeat circumgestari, neque unquam deponi ab his, qui fructum aliquem serium ex hoc suavi Matheseos studio capere volunt, in eoque progredi; id vero in hunc diem exemplaribus omnibus maiore forma impressis, necdum factum videamus; hoc nostra editio certe, si nhil aliud, attulerit commodi, atque emolumenti. Sunt enim hi nostri commentarii in universum Euclidem conscripti commodiore nunc forma, quam vulgo caeteri, (id quod magnopere a nobis, qui nos audierunt, efflagitabant,) volumineque editi, ut facile iam queant, nulloque negotio, e loco in locum, cum res tulerit, ferri atque portari."

[76] Clavius's citations of Theon and Campanus are hardly unusual. Most commentators relied on some combination of versions of Theon's and Campanus's texts and pointed out the older scholars' changes to what was believed to be Euclid's original text. This is very much in-keeping with the practices of

did not always leave the proofs or comments exactly as he found them in any of his sources. Besides occasionally pointing out and correcting mistakes in others' works, he often compressed or even entirely rewrote proofs in an effort to improve their clarity and make them more useful to his readers.[77] In chapter three, I will discuss how Clavius changed the demonstration to the Pythagorean Theorem to make it more accessible. And while Clavius's encyclopedic treatment of Euclidean proofs fits with his desire to create a useful handbook, his justification neither explained what he meant by the utility of geometry nor when he expected his students to need a handbook on geometry. In this study, I will bring Clavius's goals into focus by comparing his commentary to those of Commandino and Billingsley.

Of the three texts considered here, Commandino's was the closest to a simple translation of the Greek text. His translation was made from Simon Gyrnaeus's 1533 Greek edition, which was based upon Theon's text.[78] Commandino's goal fit the humanist project of restoring ancient knowledge, and, consequently, he made far fewer additions than Clavius, and those he did make were drawn exclusively from ancient sources. Anthony Grafton has shown that such practices were common to

authors at the time, most of whom claimed to engage in extensive study of ancient texts to develop their own intellectual habits. (While Campanus was a medieval scholar, versions of his commentary on the ancient text of *The Elements* were relatively easily accessible in the sixteenth century. The first translation based on Theon's Greek text (by Bartolomeo Zamberti) had only become available in 1505. In some cases references to Campanus were purely critical of divergences between his text and Theon's.) See Anthony Grafton, *Commerce with the Classics: Ancient Books and Renaissance Readers* (Ann Arbor: The University of Michigan Press, 1997) for a discussion of various approaches, including adding notes drawn from other ancient texts and seeking to correct errors in the Greek, humanist scholars took to reading ancient texts.

[77] Thomas Heath praises Clavius's efforts to improve the clarity of proofs with his concluding evaluation of the text: "Altogether his [Clavius's] book is a most useful work." Heath, *Thirteen Books*, 105.

[78] Ibid., 105.

humanist scholars.[79] And Paul Rose has shown that mathematical humanists, like their counterparts in biblical humanism, believed a return to the Greek texts was a means to rediscover "the eternal truths of God, man and the universe" that had been lost through neglect and misinterpretations and miscopies of texts in the Middle Ages.[80] Such a project was inherently suspicious of changes to ancient texts. Indeed, in his dedication letter for *The Elements,* Commandino assured his patron that remaining faithful to the ancient proofs made his edition superior to those of Candalla and other recent commentators who had eschewed the Greek demonstrations in favor of their own.[81]

Commandino's desire to restore mathematics as a source of truths about the world was not uncommon among mathematically inclined humanists of the sixteenth century. Similar sentiments had been expressed over a century earlier in Regiomontanus's famed 1464 Padua *Oration*, in which the German humanist described the discipline of mathematics from its ancient origins to the activities of its modern practitioners, praising its value for the erudition of any learned man.[82] Since

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[79] Citing only ancient sources fits with the humanist interest in using ancient knowledge as their own intellectual foundations, and he seems to have been in good company in using ancient additions to the original text to create his own commentary. Anthony Grafton notes in *Commerce with the Classics* that Lorenzo Valla's marginal glosses on his translation of Thucydides were copied from the scholia of Greek commentaries on the text. (Grafton, *Commerce*, 17).

[80] Rose, *Italian Renaissance,* 6. Rose provides a helpful discussion of the relationships between humanists and mathematicians in his first chapter in which he describes Renaissance attitudes towards mathematics.

[81] Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), *2*v* - *3*r*. "At Candalla vir & generis nobilitate, & rerum cognitione insignis, licet omnes Elementorum libros, qui postulari a latinis videbantur, latinos fecerit, locupletaueritque, parum tamen (ut audio) eo nomine commendatur, quod longius iter ab Eulide averterit; & demonstrations quae in graecis codicibus habentur, velut inelegantes, & mancas suis appositis reicerit."

[82] For a brief discussion of the Oration, including a summary, see Zinner, *Regiomontanus*, pp. 69-74. Another excellent discussion is found in J. Byrne, "A Humanist History of Mathematics?

that lecture was first published in 1537, it is possible that Commandino had come across it while he studied for his medical degree, a course that would have required some study of mathematics, at the University of Ferrara.[83] However, while Commandino received a doctorate in medicine, he did not cultivate a practice as a physician, choosing instead to pursue a career in various courts as a mathematician (though his skills as a physician were sometimes required), and, ultimately settling in the Urbino court where he was employed by the duke as a tutor.[84] He may even have received a pension from his patron. That position gave him access to the Duke's substantial library, the time to translate numerous texts, and a number of noble students who could assist with his restoration project.[85] When his commentary on *The Elements* was published in 1572, he had already published works of Archimedes, Ptolemy, and Apollonius and had commentaries on works of Aristarchus, Hero of

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Regiomontanus's Padua Oration in Context" in *The Journal of the History of Ideas,* Vol. 67, No. 1, January 2006, 57-60.

[83] Paul Rose identified a miscellany published in Nuremberg in 1537 as the source that preserved the Oration through today. See Rose, 95 and n.55 on p. 113. The source is *Rudimenta Astronomica Alfragrani. Item Albategnius astronomus peritissimus De Motu Stellarum* (trs. Plato of Tivloi)… *cum… additionibus Joannis de Regiomonte. Item Oratio Introductoria in Omnes Scientia Mathematicas Joannis de Regiomonte Patavii habita, cum Alfraganum publice praelegeret. Eiusdem Utilissima Introductio in Elementa Euclidis,* (Nuremberg, Johannes Petreius, 1537).

[84] Such a career path was not uncommon for someone interested in mathematics. For a discussion of the connection between mathematics and medicine see Robert Westman, "The Astronomer's Role in the 16th Century: A Preliminary Survey," *History of Science* (1980), 119-120. Alex Marr locates sixteenth- and seventeenth-century mathematical practitioners in the "court, the battlefield, the marketplace, the workshop, and the studio." According to Marr, this diversity of places of practice is part of the reason that the early modern era lacked a clear definition of "mathematician." See Alexander Marr, *Between Raphael and Galileo: Mutio Oddi and the Mathematical Culture of Late Renaissance Italy*, (Chicago: University of Chicago Press, 2011), 16.

[85] For a brief discussion of the duke's library, including Cardinal Bessarion's contributions to it, see Rose, *Italian Renaissance*, 54-55. For the duke's patronage of Commandino, see pages 202-206. In his dedication to his 1566 translation of Apollonius, Commandino explicitly praised the dukes of Urbino for their library (Rose 203). Before Commandino returned to his native Urbino, he had worked in Verona, Rome, and Parma. His primary patron during those years was the cardinal Ranuccio Farnese (1530-1565).

Alexandria, Theodosius, and Pappus underway.[86] After Commandino's death in 1575, his students, especially Guidobaldo del Monte, continued Commandino's mathematical work and began to build on the Greek texts that their master had restored. Rose notes that they focused on mechanics and the works of Archimedes, but their interest seems to have remained in abstract mathematical truths instead of the practical of mathematics, and Djiksterhuis observed that the mechanical work done by Commandino benefitted "mathematics rather than mechanics."[87] Del Monte himself was unconcerned by the precise physical details that could allow the construction and use of the machines he described, preferring to keep "mechanical sciences" found in his theoretical writings and "mechanical arts" required to actually construct machines separate from one another.[88]

Even though Billingsley and Commandino relied on the same 1533 version of *The Elements* for their translations, the Englishman's text presented a vision of mathematics at odds with that found in Commandino's. Where Commandino and his students were interested in mathematics as a branch of philosophy intermediate between divine and natural philosophy and, thus, as a source of universal truths, Billingsley intended his readers to use his translation of Euclid's *Elements* as a foundation to improve mathematical arts. In sixteenth-century England those arts

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[86] Ibid., 185-221. In his first chapter on the Urbino school, Rose discusses all of Commandino's numerous translations and restorations of Greek mathematics texts.
[87] Ibid., 222-242. The chapter is titled "The Urbino School II: Guidobaldo dal Monte and the Archimedean Renaissance." Rose claims that "Guidobaldo's attitude to mathematical instruments paralleled his attitude towards machines (as will be seen). Through these material devices, he felt, abstract mathematical truth could be made completely visible" (p. 224). Djiksterhuis, *Mechanization,* 324.
[88] M. Henninger-Voss, "Working Machines and Noble Mechanics: Guidobaldo del Monte and the Translation of Knowledge." *Isis,* Vol. 91, No. 2 (June, 2000), pp. 233-259.

included navigation, astrology, horology, gunnery and other military applications, cartography (and geography), and practical mensuration.[89] Thus, where Commandino added little to the text, Billingsley, seeking to provide his reader with as much potentially useful material as he could, added commentary of his own and included the comments of numerous other authors, including Campanus and Candalla, on most of the propositions in Euclid's text. Billingsley's vision of mathematics as the foundation of various arts fit into a broader pattern of increased interest of merchants and artisans in the discipline as a means to profit.[90] He himself was a merchant active in the civic life of London, who, on his title page, styled himself as "H. Billingsley, Citizen of London."[91] He served as Sheriff and Lord Mayor of London in the 1580s and 1590s, and starting in 1589, he was one of the Queen's four customs farmers for the port of London.[92] He also established scholarships for poor students to attend St. John's College, Cambridge, where he had studied in the early 1550s, and gave property to the same college.[93] His edition of Euclid can be seen as an early attempt to provide useful services to his fellow Londoners, the "good wittes" among his colleagues who could use the mathematical arts to improve their various crafts.[94]

Unlike his contemporary English mathematical writers Thomas and Leonard Digges and Robert Recorde, Billingsley did not have an established reputation for

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[89] For a discussion of the various mathematical arts practiced in England in the latter half of the sixteenth-century see Taylor, *Mathematical Practitioners*, 7-48.
[90] Harkness, *Jewel House,* 98.
[91] Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), frontispiece.
[92] Custom farmers collected the taxes due on imported goods and paid a rent to the Crown for the privilege of receiving the customs money.
[93] Heath, *Thirteen Books,* 110.
[94] Billingsley, "To the Reader," ii*v.*

knowledge of mathematics when he published his commentary.[95] In order to give the book legitimacy in the eyes of potential readers, John Dee, who was well-known for his expertise in mathematics, was asked to write the preface.[96] According to Nicholas Clulee, it is not clear whether Billingsley or the printer, John Daye, approached Dee, but either way, the request led to some collaboration between Billingsley and Dee in the production of the first English edition.[97] Dee not only wrote the preface, he also provided several additions to the text, most of which appear in the later books on solid geometry. While Billingsley explicitly had a non-university based audience in mind, Dee's "Mathematicall Preface" was targeted to both Billingsley's intended readers and university philosophers. In it he set out to give a complete description of mathematics, both its science and its arts. He claimed to be the first to give a description that included all mathematical arts.[98]

Today, historians can read the "Mathematicall Preface" as a treatise on early modern philosophy of mathematics, and in that capacity, it has been the subject of a great deal of study as part of efforts to understand John Dee's complicated position within the Scientific Revolution. Clulee has elegantly analyzed the preface to show

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[95] The limited role Billingsley played in mathematical practice of is day is apparent in the narrative portion of Taylor's *Mathematical Practitioners* in which he is only mentioned in passing (p. 34). While he is included in her list of biographies, his commentary on *The Elements* is not included in her list of works (p. 171). It is only mentioned in her entry for John Dee's "Mathematicall Preface" (p. 320).

[96] Although both Billingsley and Dee had studied at St. John's College, they seem to have been in Cambridge at different times. John Dee was there in the 1540s, and Billingsley didn't start his studies until 1551. Thus it is likely that they did not meet until they were both working in London.

[97] Nicholas Clulee, *John Dee's Natural Philosophy: Between Science and Religion*. (London: Routledge, 1988), 146.

[98] John Dee, "Mathematicall Preface" in Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), second page (no number or indicator).

that Dee's intent in studying mathematics was to understand nature and the divine through "the hidden springs and ultimate reasons behind the processes and very existence of the cosmos."[99] That is, Dee himself understood mathematics as an abstract source for the discovery of truths about the universe. And, therefore, despite the fact that he acknowledged that Billingsley's intended audience would be more interested in the practical uses of mathematics than in its philosophical import as a source of universal truths, he frequently digressed from his stated purpose of "recit[ing] describ[ing] and declar[ing] a great Number of Artes, from our two Mathematicall fountains [arithmetic and geometry], derived into the fields of *Nature*" to offer his observations on the links between mathematics and natural philosophy.[100] The preface's philosophical location of mathematics as the intermediate branch of a tripartite division of knowledge into the supernatural, the mathematical, and the natural exactly mimics that found in the work of the Neoplatonist Proclus (412-485). The resemblance has been used in arguments that Dee was a philosopher whose Neoplatonic commitments combined mathematics and experience allowing him to create a philosophy that was a precursor to modern science.[101] The preface also contains traces of Dee's interest in hermeticism and alchemy, to which some

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[99] Clulee., 231. Lynn Thorndike's description of John Dee's interest in the soul and the "imaginative spirit" offers a similar vision of Dee's interest uncovering the hidden secrets of the universe through any means available to him, including mathematics and natural magic. See Lynn Thorndike *History of Magic and Experimental Science,* vol. 6, p. 391.

[100] Dee, "Mathematicall Preface," in Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), a.iii. For a brief discussion of Dee's digressions see Clulee, *John Dee's Natural Philosophy,* 146-148.

[101] See I.R.F Calder, "John Dee Studied as an English Neoplatonist" (PhD diss., London University 1952) for an in-depth study of the role Dee's interests as unified by his central Neoplatonist mathematical idealism.

historians, notably Frances Yates, have credited his significance.[102] Nevertheless, the majority of the "Mathematical Preface" is devoted to Dee's description of the uses he identified for mathematics as a promise of future profit for the intended non-Latinate merchant readers. Thus, because it was written as a preface for Billingsley's English Euclid, the "Mathematicall Preface," emphasizes the mathematical arts, including astrology, navigation, geography, navigation, and architecture, as "arts of social utility" for which geometry is the foundation.[103] In this dissertation, I will study the preface in its intended role as a preface to an introductory mathematics text for a non-Latinate audience, which, following Dee, will lead me to emphasize the practical applications of mathematics as opposed to its status within the hierarchy of disciplines.

As this dissertation will show, Clavius positioned his approach to *The Elements* between those of his two contemporaries, using arguments for both the nobility and utility of mathematics as a means to justify its study. In order to more completely understand the vision of mathematics Clavius presented in his commentary on Euclid, I have divided my dissertation into two parts. In the first part, I study Clavius's pedagogical project as an attempt to create a complete mathematics curriculum for the Jesuit schools. In the first chapter, I focus on arguments for the nobility of mathematics. I establish his ideal curriculum as a part of the sixteenth-century debate over the status of mathematics within the hierarchy of disciplines. This chapter explores the promotion, by Clavius and others, of the study of mathematics as

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[102] Clulee., *John Dee's Natural Philosophy,* 8, 166-167. For a helpful discussion of the Yates thesis see Cohen, *Scientific Revolution*, 169-176.
[103] Clulee, *John Dee's Natural Philosophy*, 148.

a source of certain knowledge that could bridge the concrete world of physics and the abstract ideas of metaphysics. Such arguments were commonly made by mathematical humanists, including Commandino, and they appear in Clavius's preface to *The Elements.* As a result of his conviction that mathematics was a source of truths about the physical and the divine, he created an ambitious ideal for Jesuit schools: a three-year mathematics curriculum with an emphasis on astronomy, the branch of mathematics that he believed was the most noble because of the proximity of the heavenly bodies to the divine.

In the second chapter I turn to the role of practical mathematics in Clavius's curriculum. Here I explore how Clavius understood practical aspects of his discipline as part of his vision of mathematics as a noble intellectual pursuit. Thus, his arguments for practical mathematics were designed to make his discipline appealing to the princes whom the Jesuits hoped to teach, not the craftsmen to whom Dee and Billingsley sought to appeal. Nevertheless, like the English authors, Clavius presented practical mathematics as a social good. That argument and the Jesuits' own needs for the applications of mathematics led the authors of the *Ratio Studiorum* to maintain a place for a branch of practical mathematics in their curriculum, even as they diminished the place of mathematics in each successive draft.

Chapter three begins the second part of my dissertation, in which I offer a close comparison of the three versions of *The Elements*. In this chapter, I examine the structures of the three commentaries, the aids which the authors provided the readers, and the mathematical content of the first book. Since all three authors assert that

Euclid's text had not been meaningfully changed since it was written in the fourth century BCE, the differences between the mathematical content of each text, in everything from the definitions to the proofs, are indicative of the differences in their approaches to mathematics and help to illustrate what they each hoped the reader would gain from the text. In this chapter I argue that Clavius's approach to mathematics emerges as a combination of the approaches of his contemporaries, making his text a work that could be used to teach mathematics to all Jesuit students, whatever their backgrounds or projected career paths.

In the fourth chapter I study the relationships that each author developed between geometry and arithmetic. As the two branches of the quadrivium classified as pure mathematics, each of these fields of mathematics could be argued to have a foundational role in the study of mathematics. However, humanists seemed to favor geometry in their arguments for the status of mathematics as a branch of philosophy, and arithmetic frequently appeared in cases for the utility of mathematics. Thus, the relationship described between geometry and arithmetic indicates the value each author ascribed to mathematics. Indeed, Commandino favored geometry, reducing arithmetic to a tool for its study. Billingsley argued the opposite, saying that the necessity of number to the study of magnitude made arithmetic the single foundational branch for all of mathematics. In this chapter I argue that Clavius again takes a middle road between his two contemporaries. For him, arithmetic and geometry are analogous fields of study, and each can serve as an aid to the other. In the context of *The Elements*, arithmetic aids geometry.

Finally, in the fifth chapter I look at the use of visualizations in *The Elements* to understand how each author saw the place of mathematics within the hierarchy of disciplines. This chapter relies on the images each author presents for the definitions and two propositions in the first book as well as the definitions and one proposition in the solid geometry books. In this chapter I argue that where Billingsley's use of visualizations grounds mathematics in the concrete physical world, and Commandino's images are mere aids to the development of abstract, universal ideas, Clavius's images unite universal ideas and physical objects in mathematics, allowing him to place the discipline at the intersection between metaphysics and physics.

Throughout my study I have attempted to take a "moderate historicist" approach by examining how Clavius, a well-respected mathematician of his day, understood and presented his own discipline.[104] This dissertation is thus a case study of Clavius as one mathematician who sought to combine mathematical theory and practice for a specific institutional reason: the needs of the Jesuit schools. It illustrates how a mathematical scholar with no explicitly "revolutionary" goals united two contemporary trends in his discipline that historians have since recognized as essential to the Scientific Revolution.[105] Driven by the Jesuits' needs, Clavius used his

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[104] John H. Zammito, *A Nice Derangement of Epistemes: Post-Positivism in the Study of Science from Quine to Latour.* (Chicago: The University of Chicago Press, 2004), 5. Zammito critiques postmodern theory, both in the philosophy of language and in social constructivism, as hyperbolic and advocates a "more moderate historicism" to counteract the cycles of extravagant theorizing and scaling back he identifies in the history of philosophy of science and science studies. I interpret this to mean attempting to describe our actors and time periods in terms of what they believed they were doing rather than fitting them into a larger theory of the development of science.

[105] Peter Dear, "What is History of Science the History of?" 390-406; Peter Dear, *Discipline and Experience: The Mathematical Way in the Scientific Revolution*. (Chicago: The University of Chicago Press, 1995). In both of these sources Dear discusses the combination of theory and practice. In the article, it is the early modern separation of *theorica* and *practica* that Dear sees as a major stumbling

pedagogical program and textbooks to use pure mathematics as a unifying foundation for the mathematical sciences, which he identified as noble, and the mathematical arts, whose utility was increasingly recognized in the sixteenth century.

Because *The Elements* was the first text Clavius required in his curricula and the text most likely to be taught in Jesuit schools it became the focus of my study as the text that introduced Clavius's vision of his discipline to students. While much remains to be done to understand how Clavius's readers worked through and understood his texts, comparing his commentary on Euclid's *Elements* to two other important contemporary works allowed me to uncover differences in the presentation of the foundations of mathematics. These differences illustrate that, in his efforts to make Euclid accessible to a variety of students, Clavius practiced what he preached and combined the theoretical and the practical components of his discipline throughout his commentary.

Clavius's sustained combination of these two facets of mathematics makes his work a case study in the intersection of the intellectual and social components of the Scientific Revolution. Intellectual histories of the Scientific Revolution focus on the nobility of mathematics and its rise within the hierarchy of disciplines. Social histories are more likely to focus on the increased interest in practical mathematics.

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block to identifying "science" before the nineteenth century, and thus early modern efforts to combine the two elements of knowledge appear to be the reason that modern historians identified the seventeenth century as the Scientific Revolution. In the book, Dear traces the origins of experimental science to an early modern notion of experience as a general understanding of "*how things happen* in nature, rather than a statement of *how something had happened* on a particular occasion." In chapter 2 he studies Jesuits' use of experience in an effort to make the case that astronomy and optics were sciences by an Aristotelian standard. In the process Dear describes, the Jesuit mathematicians relied on Clavius's combination of mathematical arts and mathematical sciences.

Because his work unites the two approaches to mathematics, it illustrates the limits of relying on any one historical narrative. This dissertation, therefore, offers Clavius's pedagogical work as one bridge between the philosophical and social narratives of the role of mathematics in the Scientific Revolution and seeks to present a more complete picture of the place of mathematics in early modern thought.

Chapter One

Christopher Clavius and the Jesuit Mathematics Curriculum: A Mathematician's Ideal

"Indeed, if the nobility and excellence of a science is to be judged by the certitude of the demonstrations of which it makes use, the mathematical disciplines would without doubt have the foremost place among all others."[1]

Christopher Clavius, *Euclidis Elementorum,* 1574

In 1599, the Society of Jesus published the definitive draft of the *Ratio Studiorum*. Intended as the curriculum for all Jesuit schools around the world, it was produced over a period of nearly twenty years with many Jesuit priests contributing expertise, opinions, and advice. During the drafting process, Christopher Clavius, the most eminent sixteenth-century Jesuit mathematician and the professor of mathematics at the Jesuits' flagship Collegio Romano, exerted the greatest influence on the position of mathematics in the curriculum. Indeed, while today he is primarily remembered for his work on calendar reform in the early 1580s, the vast majority of his work was dedicated to developing a complete curriculum for Jesuit schools. By the time he began the calendar reform, he had been teaching mathematics almost continuously for

[1] Christopher Clavius, *Euclidis Elementorum Libri XV Accessit XVI de solidorum Regularium comparatione* (Rome: Vincentium Accoltum, 1574), b*v*. "Si vero nobilitas, atque praestantia scientiae ex certitudine demonstrationum, quibus utitur, sit iudicanda, haud dubie Mathematicae disciplinae inter caeteras omnes principem habebunt locum."

eighteen years, and had published his two most famous commentaries, the *Commentary on the "Sphere" of Sacroboso* (first edition 1570) and the *Commentary on Euclid's Elements* (first edition 1574), both of which were intended as textbooks for all Jesuit schools.[2] In these and other textbooks written to accompany his mathematics curriculum, Clavius embraced contemporary arguments by humanist mathematicians outside the Order about the status of mathematics as a discipline comparable to natural philosophy. Based on these arguments, he established his ideal course of study, in which mathematics from geometry to astronomy was presented as a means to prepare students for the study of theology and understanding the divine.

Clavius's curriculum was built on the work of his Jesuit predecessors, Jerome Nadal (1507-1580) and Balthazar Torres (1518-1561), who had created numerous opportunities for Jesuit students to pursue mathematics as part of the Society's ambition to save souls through education. Their suggested programs of mathematics lasted over two years and covered a wide range of topics, giving Clavius a precedent to suggest an even broader three-year course of study. Clavius's curriculum, for which he wrote several of the accompanying textbooks, covered topics from geometry and practical arithmetic to perspective and astronomy. In his estimation, mathematics, especially in the sub-branch of astronomy, could aid all Jesuit students in their pursuit of knowledge of the divine. After all, astronomy studied the heavens, the part of

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[2]James Lattis, *Between Copernicus and Galileo: Christoph Clavius and the Collapse of Ptolemaic Cosmology* (Chicago: The University of Chicago Press, 1994), 3. While Lattis's interest is in Clavius's work on astronomy, he does note that the mathematician was known for his work as a pedagogue. In fact, Lattis points out that Clavius's contemporaries knew him as "the Euclid of his times" because his corpus of textbooks covered nearly all branches of early modern mathematics.

Creation that best illustrated the nobility of the divine.[3] This argument contributed to the ongoing debate over the status of his discipline, the question of the certitude of mathematics. That debate arose from the clash between the traditional view of mathematics - propounded by the Aristotelian Alessandro Piccolomini (1508-1579) - as a lower discipline that could describe but not explain the natural world and a renaissance during the fifteenth and sixteenth centuries of the view of ancient mathematics – expressed by the Platonist Francesco Barozzi (1537-1604) – as a source of sure truths. Treatises, written by these two authors in 1547 and 1560 respectively, spurred Jesuit authors to consider the issues at length in order to determine the place of mathematics in their curriculum.[4] Indeed, Clavius's arguments for the status of mathematics as intermediate between the physical and divine were clearly based on those found in Barozzi's text as well as in the works of other mathematically inclined humanists.

This chapter examines Clavius's vision of mathematics and its ideal role in Jesuit schools as a product of both earlier Jesuit mathematics curricula and the debates over the status of the discipline. I will trace the development of the mathematics curriculum in Jesuit colleges before Clavius assumed his professorship in Rome in

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[3] Christopher Clavius, *In Sphaeram Ioannis de Sacro Bosco Commentarius* (Rome: Victorium Helianum, 1570), 8-9. "Tertio, quem corpora coelestia sunt propinquiora nobilissimo ac primo enti, puta, Deo glorioso; Immo secundum Averroem corpus coleste est mediator, ac ligamentum superiorum cum inferioribus, & locus aeternorum, ac divinorum; omnes etenim philosophi, ac nationes etiam quantumuis barbarae, in coelo Deum tanquam in sede collocant propria. Quamvis enim Deus non huic vel illi loco sit alligatus, sed ubivis locorum (quod nullis aliis convenit rebus) existat; ponitur tamen in coelo, tanquam in nobiliori mundi parte, ubi maxime suam omnipotentiam, & bonitatem manifestat, ut Theologi asserunt."

[4] Piccolomini's 1547 treatise is the *Commentarium de certitudine mathematicarum disciplinarum,* and Francesco Barozzi's 1560 response is entitled *Opusculum in quo uno Oratio et duae Quaestiones altera de certitudine, et altera de meditate mathematicarum continentur.*

order to understand how he built on the work of previous mathematics teachers. Then I will turn to the renaissance of mathematics and consequent debate over the certitude and status of mathematics as presented in fifteenth- and sixteenth-century mathematical texts, including arguments philosophers made against mathematics. Using the prefaces to his commentaries on Euclid's *Elements* and Sacrobosco's *Sphere,* I will examine how Clavius placed his work within those contemporary debates. Finally, using suggestions Clavius made for how mathematics should be taught, I will discuss his vision for how the Society could make his ideal curriculum possible and how it could take full advantage of the benefits he believed the study of mathematics offered. Before considering the place of mathematics in the Jesuit schools, however, it is necessary briefly to discuss the goals and organization of the school system.

Jesuit Schools

Although the Order's founder, Ignatius Loyola, had hoped to create a missionary order, the Society of Jesus saw teaching as part of its apostolic mission from its formal inception. The papal bull creating the Society made provisions for colleges and universities to train its future members.[5] It also drew the Jesuits towards the education of laymen. Indeed, Pope Paul III mandated that one who wished to join the Society of Jesus must "purpose to become a member of a society principally instituted to work for the advancement of souls in Christian life and doctrine, and for

[5] "The Bull of Institution, 1540" in *The Catholic Reformation,* ed. John Olin, (New York: Fordham University Press, 1992), 207.

the propagation of the faith by public preaching and the ministry of God's Word, by spiritual exercise and works of charity, *more particularly by grounding in Christianity boys and unlettered persons,* and by hearing the confessions of the faithful, aiming in all things their spiritual consolation."[6] Moreover, members were to strive "above all things" for "the instruction of boys and ignorant persons in the knowledge of Christian doctrine, of the Ten Commandments, *and other such rudiments as shall be suitable,* having regard to the circumstances of persons, places, and times." The Jesuits interpreted the phrase on "grounding boys and unlettered persons" in Christianity as an exhortation to teach, and believed that the other rudiments were grammar and rhetoric. When they deemed it appropriate to offer more advanced education, they added philosophy and more advanced lessons in theology.

Despite the papal exhortation to teach, the Jesuits took several years to realize their school system, but, once begun, it came quickly to define their primary activity. Initially, the Society of Jesus only sent members to teach theology at a variety of existing universities. In some cases, the Jesuits established residential colleges loosely connected to universities to provide for future members of the Order who were studying there, but no classes were offered at these colleges.[7] In 1546, the Jesuit teaching mission took on a new form when the Duke of Gandia received approval for the use of ecclesiastical funds to found a Jesuit college in his Spanish realm, where there was no existing university.[8] For the first time, Jesuit priests at this new school

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[6] For this and the quotation that follows, see ibid., 204-205 (with my emphasis).
[7] John W. O'Malley, *The First Jesuits* (Cambridge, MA: Harvard University Press, 1993), 202.
[8] This Duke, Francisco de Borja, eventually joined the Society of Jesus and became the General of the order in 1565.

were teaching more than theology, and they were working with students who would not necessarily join the Order.[9] From this point on, teaching became the overarching mission of the Jesuits, and every Jesuit had to assist.[10] A short two years later, the Society of Jesus was asked to open a school in Messina, Sicily, and, just a few months after that, another in Palermo. Both schools were approved, and more followed.[11] By 1556, the Jesuits had opened thirty-three schools; by 1581 the number had increased to 150, and by 1599, when the definitive *Ratio Studiorum* was published, there were 245 schools, including a few in America and Asia.[12]

By the turn of the seventeenth century, even if the Jesuits had intended to pursue other activities, they could not spare any resources, especially human resources, from the tasks required to run the vast number of schools they had created. In fact, the Jesuits consistently struggled with teacher shortages as both the number and size of their schools rapidly increased. By 1600 there were 8,500 Jesuits, most of whom staffed the various schools. A Jesuit school offering the full curriculum from grammar to theology could not be run by fewer than fourteen Jesuits, unless some held two positions, which was usually not possible. The 1599 *Ratio Studiorum* called for a rector, one or two prefects of study, seven professors in the higher faculties (sacred scripture, Hebrew, scholastic theology, cases of conscience, philosophy, moral philosophy, and mathematics), and five teachers in the lower studies of grammar and

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9 O'Malley, *First Jesuits,* 203.
10 Ibid., 200.
11 Ibid., 205-206.
12 Allan P. Farrell, Introduction, *The Jesuit* Ratio Studiorum *of 1599,* trans. Allan P. Farrell (Washington DC: Conference of Major Superiors of Jesuits, 1970), iii.

rhetoric. These lower studies could be, and often were, divided into multiple classes in large schools.[13] Even so, because schools usually had well over 1,000 students, classes were large, around 200 students, so it was impossible for a single teacher to monitor the work of each student.[14] To help mitigate the chronic teacher shortage, instead of increasing the number of classes taught, large classes were divided into groups of ten students, in which one member acted as captain and monitored the work of his group-mates.[15]

Giving students leadership roles may also have been seen as a way to further the Jesuits' goal of training students to become leaders in their local societies.[16] Their hope was that graduates of their colleges would know how to live "satisfying lives as Christian gentlemen," that is that they would seek to perfect their souls through virtuous living and would spread that knowledge to others through the examples they set.[17] It was with these goals in mind, that the early Jesuits, especially Ignatius Loyola, considered what the best curriculum would be. Ignatius himself had been educated in a medley of Spanish universities, where he was exposed to an older medieval curriculum, and at the University of Paris where he studied a humanist curriculum. In his guidelines for the Order's schools, Ignatius attempted to take the best from both traditions.[18] Ultimately, this resulted in a curriculum that was

[13] Ibid., 8-9, 25-46.
[14] John Donohue, *Jesuit Education: An Essay on the Foundations of its Idea* (New York: Fordham University Press, 1963), 63-66.
[15] Ibid., 66.
[16] O'Malley, *First Jesuits* 211-213.
[17] George Ganss, *St. Ignatius's Idea of a Jesuit University: A Study in the History of Catholic Education* (Milwaukee: The Marquette University Press, 1956), 166.
[18]Ibid., 11-17.

substantially similar to that taught at secular humanist schools, focusing on Latin and Greek and teaching from classical sources. However, unlike those schools, the Jesuits colleges emphasized the importance of theology, and they were careful to expurgate morally questionable sources.[19]

In Part Four of the *Constitutions,* Ignatius outlined a curriculum that described what should be taught (both to students intending to join the Society, known as scholastics, and to those who were not called to the Order, called externs) as well as the method for running the schools to best ensure that Jesuit teaching could lead students to salvation.[20] In order to make the schools attractive to externs, teachers were to take an active interest in their students and their local setting, so that the curriculum could be adapted to local customs.[21] And, in keeping with traditions from medieval universities, students at all but the lowest levels were required to participate in public disputations, and students who intended to receive a degree had to take public examinations.[22] While Ignatius left the curriculum of externs open to the demands of local needs, his work was fairly detailed with respect to the program of study for the scholastics. He insisted on the grammar and rhetoric of various languages – including, but not limited to, Latin and Greek, without which no other study was

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[19] Francesco Ceasaro, "Quest for Identity: The Ideals of Jesuit Education in the Sixteenth Century," in *The Jesuit Tradition in Education and Missions: A 450-Year Perspective*, ed. Christopher Chapple (Scranton: University of Scranton Press, 1993), 18.
[20] According to Ignatius, the method of teaching was just as essential as the content of the curriculum to enabling students to reach the ultimate goal of salvation and to ensuring the success of Jesuit schools. See Ignatius Loyola, "Part Four of the *Constitutions* of the Society of Jesus," in *St. Ignatius's Idea of a Jesuit University: A Study in the History of Catholic Education* (Milwaukee: The Marquette University Press, 1956), 291.
[21] Ibid., 333.
[22] Ibid., 313-317.

possible.[23] Usually a total of four years was spent in the humanities curriculum: two on grammar and two on rhetoric. Once a student advanced to the higher faculties of the arts, he would spend three years on natural sciences that could be learned from reason (as opposed to revelation): logic, natural philosophy, and metaphysics. Finally students would proceed to a four year course in scholastic and positive theology and sacred scripture.[24] Some students could study some topics in greater detail as ordained by their superiors.[25] This course of study became the foundation for the *Ratio Studiorum*.

Mathematics in the Jesuit Schools before the Ratio Studiorum

In the sixteenth century, mathematics was composed of many branches. The quadrivium (geometry, arithmetic, music, and astronomy) was the Latin standard of mathematics education in universities where, because mathematics received less scrutiny from humanists than the trivium, teachers were still using medieval texts, most notably those written by Boethius.[26] Geometry and arithmetic were considered pure mathematics because of their lack of concrete objects of study. Astronomy and music, "sensible" subjects, which were accessible through the senses, were considered mixed mathematics. However, even the astronomy and music curricula treated their

[23] Sometimes Hebrew was added to Latin and Greek.
[24] Loyola, Part Four, 310. For more information on the time spent on each portion of the curriculum see Ganss, 47-50.
[25] Loyola, Part Four, 307-308.
[26] Paul Grendler, *Schooling in Renaissance Italy: Literacy and Learning 1300-1600* (Baltimore: Johns Hopkins University Press, 1989), 309. The texts used likely included Boethius's *De arithmetica* and his fragments of Euclid's *Elements.*

sensible subjects as abstract entities.[27] Understanding the motion of celestial bodies required the use of unobservable geometrical concepts such as equants and epicycles. The study of music dealt with abstract ratios and the harmony of the soul and the celestial spheres more than with practical instrument-making or the harmony of notes. In vernacular schools, one could find more practically oriented mathematics, especially "abbaco," the practical arithmetic that enabled the advanced bookkeeping of Renaissance merchants.[28] Other branches of practical, or mixed, mathematics, such as perspective, geodesy, and mechanics, could have been learned while studying philosophy when studying for a trade, but were not likely to have been available in any formal mathematical schooling.[29]

Despite the variety of topics encompassed under the heading of mathematics, the discipline was mentioned only once in the *Constitutions,* and there was no indication as to which branches of it should be taught. Ignatius included mathematics

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[27] For a discussion of the role of theoretical astronomy within the quadrivium see Claudia Kren, "Astronomy: in *The Seven Liberal Arts in the Middle Ages* ed. David L. Wagner, (Bloomington: Indiana University Press, 1983), 218-247. See also David C. Lindberg, *The Beginnings of Western Science,* Second Edition, (Chicago: The University of Chicago Press, 1992), 261-270 for comments on the role of astronomical theory in medieval education. For a discussion of music as an abstract discipline see Theodore Karp, "Music" in *The Seven Liberal Arts in the Middle Ages* ed. David L. Wagner, (Bloomington: Indiana University Press, 1983), pp. 174-178. Karp describes music as an abstract discipline and points out that according to medieval philosophers including Boethius and Cassiodorus, "it was not essential to be able either to play, sing, or compose in order to qualify as a musician!" (p. 177).

[28] Ibid., 306-309. Some vernacular schools also taught surveyor's geometry. See J.V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance* (Oxford: Oxford University Press, 1997), 14-16.

[29] For a discussion of perspective as part of both mathematics and physics see David C. Lindberg, *Roger Bacon and the Origins of Perspectiva in the Middle Ages: A Critical Edition and English Translation of Bacon's Perspectiva with Introduction and Notes* (Oxford: Clarendon Press, 1996), xxxiv-xli. Lindberg points out that Aristotle "placed optics on the boundary between physics and mathematics." Robert Grosseteste (c. 1168-1253) was one of the first Western scholars to call for the mathematization of optics, and Roger Bacon (c. 1214 – 1292) carried it out, though he combined it with a physical analysis (see page lii), keeping optics within the realm of philosophy.

as part of the arts and natural sciences curriculum, but explicitly limited it to the "moderation appropriate to secure the end we are seeking."[30] The nature of this "appropriate moderation" for the education of youths as well as for the betterment of their souls and society remained unclear. This gave the mathematics teachers in Rome the opportunity to create extensive programs of study with the justification that each branch of mathematics could somehow be of use to the study of the divine or in preparation for the potential roles of Jesuit missionaries. Indeed, in the 1550s when Jerome Nadal, who had taught mathematics while he was studying both in Alcalá and in Paris before the founding of the Order, was tasked with promulgating Ignatius's *Constitutions* to the Jesuit schools, he gave "appropriate moderation" a very liberal interpretation and created rigorous mathematics courses.[31]

It is not clear what sparked Nadal's interest in mathematics education. Jesús Luis Paradinas Fuentes believes that Nadal was of Jewish descent and that the importance of astronomy to Hebrew culture gave him early exposure to and interest in mathematics.[32] Antonella Romano suggests that, because of its practical implications, mathematics had provided a competitive edge for Jesuit colleges in the early 1540s in Padua and Venice, something Nadal would have known.[33] Both explanations can be

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[30] Loyola, Part Four, 332.

[31] Ganss, *St. Ignatius's Idea,* p. 285; Antonella Romano, *La Contre-Réforme Mathématique: Constitution et Diffusion d'une Culture Mathématique Jésuite à la Renaissance.* (Rome: École Française de Rome, 1999), pp. 50-51.

[32] Jesús Luis Paradinas Fuentes, "Las Matematicas en La Ratio Studiorum de los Jesuitas," Fundación Orotava de Historia de la Ciencia," *LLULL: Revista de la Sociedad Española de Historia de las Ciencias y de las Técnicas* 35, no. 75, (2012): 136, https://dialnet.unirioja.es/servlet/articulo?codigo=3943923.

[33] Romano, *La Contre-Réforme Mathématique,* 47-51. Those colleges were exclusively for Jesuit novices and were associated with existing universities. The competitive edge that came from the

supported by Nadal's rigorous program of study for mathematics, which covered pure mathematics (geometry and arithmetic) and a variety of branches of mixed mathematics, with an emphasis on astronomy. Nadal's plan, a scaled back version of which was implemented in the college he helped found at Messina, called for three mathematics lessons a day and suggested textbooks from recent as well as more traditional authors, including Orontius Finé, Peurbach, and Euclid.[34] Mathematics complemented other philosophy courses in all but the first of the four years of philosophy studies. Second-year philosophy students began their mathematics training with Euclid, practical arithmetic, and the principles of astronomy. In their third year they studied music and perspective, with the theory of the planets and the astrolabe following in the last. Nadal ended his program with the observation that upon completion of this course of study, philosophy students would "know at least the principles of all of mathematics," but the inclusion of practical arithmetic and perspective betrays a utilitarian interest beyond that desire.[35]

Nadal's position as an administrator (and as one of the founding Jesuits) gave him the means to promote mathematics throughout the Jesuit school system, but because he was no longer in the trenches as an instructor when he wrote his curriculum, it was only an ideal. In practice, Nadal's colleague Balthazar Torres, who held the mathematics chair at the Collegio Romano from 1553 to 1561, offered a

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inclusion of mathematics in the Parisian model of a humanist liberal arts education espoused by the Jesuits could thus gain the Society of Jesus an increase in numbers.

[34] Ibid., 55.

[35] Fuentes, "Las Matematicas," 137. The quotation from Nadal is in Fuentes' text in Spanish. The English translation is mine based on Fuentes' Spanish.

slightly modified program of study. Torres's two-and-a-half-year formal course included a logic lesson every morning and every afternoon some topic from pure or mixed mathematics. Much of what he suggested followed Nadal's plan closely, although it was somewhat shortened, at the expense of astronomy, to fit into a half year less.[36] He particularly emphasized the practical branches of his discipline by teaching all mathematics topics in one class.[37] Thus, even as students first learned foundational pure mathematics, they could see potential applications being developed by the more advanced students.[38] However, perhaps fearing that such a class structure would not be conducive to advanced study, Torres suggested that talented students continue to study mathematics for a third year working closely with the professor in private study or small groups instead of sitting in a formal class. Such study groups were possible because in the *Constitutions*, Ignatius had provided for dispensations from certain duties (usually teaching the lower disciplines) for students who showed talent in a particular area of study to focus on that field.[39] These dispensations only applied to those studying the higher disciplines. While Torres's small groups were likely intended to help alleviate the difficulties colleges faced in finding capable mathematics professors, by establishing the means for students to devote their efforts

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[36] Romano, *La Contre-Réforme Mathématique*, 77.
[37] Ibid., 74. While Ignatius intended the Collegio Romano to be a model for all other Jesuit schools (Ganss, *St. Ignatius's Idea,* 34), it is not clear that mathematics instructors elsewhere followed Torres' curriculum. Indeed, the *Constitutions* left the curriculum open for adaptation to local needs. Romano has shown that, at least in France, the mathematics curriculum followed the desires of local leaders. For my purposes, however, it is sufficient to study the Collegio Romano's mathematics curriculum because that is the curriculum Clavius inherited when he took over the professorship in Rome.
[38] Romano, *La Contre-Réforme Mathématique*, 80; Fuentes, "Las Matematicas," 139. Of course, the flip side is that the more advanced students would always have easy access to the foundational material in pure mathematics that made the applications possible.
[39] Loyola, Part Four, 335.

to mathematics at the expense of other areas of study, they also claimed an elevated position for mathematics.[40]

The chair of mathematics at the Collegio Romano passed to Christopher Clavius in 1563. He took up the torch of Nadal and Torres and continued to promote the study of mathematics, creating an even more extensive curriculum. In the early 1580s, he wrote three possible mathematics curricula, each of which was designed for students of different levels and covered successively smaller amounts of mathematics.[41] The most ambitious program was designed for those who were interested in pursuing the most perfect knowledge of mathematics; the second was for students who were not going to use mathematical knowledge extensively, but who still needed a firm foundation; the third was the shortest, requiring only two years of study, and provided what Clavius deemed to be only a basic knowledge of mathematics.[42] Clavius clearly did not believe that this last curriculum was truly sufficient. He titled it “Third, briefest order adapted to fit even a course of mathematics that should be completed in two years,” a time-span Clavius ultimately felt was too short. At the end of that curriculum he suggested that “if the students are capable and desire to learn,”

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[40] Romano, *“La Contre-Reforme Mathematique,”* 68-70. Even with a mathematics academy in Rome, there remained a shortage of qualified teachers for the discipline, which meant that mathematics remained a marginal subject in most Jesuit colleges.

[41] Ibid., 103. As will be discussed in the next chapter, even Clavius’s least rigorous suggested curriculum was not adopted by the Jesuit schools. Thus, the only place in which his curriculum was taught was in his continuation of Torres’ private academy in Rome. The development of that academy will be briefly discussed at the end of this chapter.

[42] Christopher Clavius, “Ordo servandus in addiscendis disciplinis mathematicis (1581),” in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 110-115.

then the more rigorous second curriculum could be taught in two years, and, if they so desired, students could continue their study independently.[43]

In the first curriculum, the extent of Clavius's ambition for mathematics students is clear. They should be exposed to much of then-known mathematics: all fifteen books of Euclid and the sixteenth book from the 1566 commentary of Francois Flussas Candalla, practical arithmetic, Sacrobosco's *Sphere* (or another introduction to astronomy), use of various astronomical instruments including quadrants, speculative arithmetic, speculative music, algebra, trigonometry, spherical geometry (found in Clavius's commentary on *The Sphere of Theodosius)*, the structure of the astrolabe, horology, geography, measurement of plane and solid figures, perspective, various phenomena and problems of astronomy, the motions of the planets (including the Alphonsine Tables), the works of Archimedes on circles as well as mean proportions, doubling the cube, and squaring the circle, mechanics, and a study of cylindrical sections and their relationship to ellipses.[44] These topics were broken down into

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[43] Ibid., 114-115; "Ordo tertius brevissimus et ad cursum mathematices, qui duobus annis absolve debet, accommodatus." "Probo autem magis secundum ordinem quam hunc, si duobus annis absolvi posset. Posset autem, si discipuli sint capaces et cupidi discendi."

[44] Ibid., 110-113. Speculative arithmetic stands in contrast to practical arithmetic. It was the study of number theory, as opposed to the operations on those numbers. This included numerology, as that field pertained to the study of scripture. For speculative arithmetic Clavius suggested the work of the thirteenth century mathematician Jordanus de Nemore. Similarly, speculative music means music theory instead of musical practice. For that topic, he suggested the work of Jacques Lefèvre d'Étaples, who wrote a treatise on music theory in four books, *Musica libris quatuor demonstrata*. Since these authors were connected in his curriculum it is likely that Clavius had read the 1496 or 1514 printing of d'Étaples work on music with Jordanus's work on arithmetic and a treatise on arithmetic by Boethius. The 1514 printing also included a second brief treatise by d'Étaples on arithmetic. (*In hoc opere contenta. Arithmetica decem libris demonstrata. Musica libris demonstrata quattuor. Epitome in libros arithmeticos divi Severini Boetii. Rithmimachie ludus qui et pugna numerorum appellat.* (Paris: Henrici Stephani: 1514). For what I have called trigonometry, Clavius wrote "Tractatus sinuum, una cum eorum usu circa varia phaenomena et problemata ad primum mobile spectantia, absque demonstrationiubus." I am interpreting this item to cover most of what we would recognize today as basic trigonometry which is all based on the sine function.

twenty-two pedagogical units, for each of which Clavius indicated possible textbooks. He had clearly read extensively in his own mathematics education citing a wide range of authors, including Francisco Maurolico (whom he had visited in 1574), Michael Stifelius, Jacques Peletier, Regiomontanus, and Ptolemy.[45]

The second curriculum was a little shorter with only nineteen pedagogical units. He left out the works of Archimedes, mechanics, and cylindrical sections. He also shortened one of his units on Euclid's *Elements* to eliminate books seven through ten, which deal with number theory. He was likely trying to avoid repeating material learned in the speculative arithmetic unit. While Clavius clearly preferred the first curriculum to the second, he did not seem as dissatisfied with it as he did with the third option, which included only fifteen pedagogical units. (In this curriculum he eliminated *The Sphere* of Theodosius, phenomena and problems of astronomy, the motions of the planets, and the unit on speculative arithmetic and music.) However, Clavius's dissatisfaction with the third program should not disguise how much material it covers. In fact, even he seems to have feared that this third program would be rejected as too much material to squeeze into two years. He therefore outlined explicitly how much time should be spent on each section of the curriculum. In the first few months of the school year, from the beginning of term until the end of January, students were to learn the first four books of Euclid. Then, until Easter they were to study practical arithmetic, the sphere and its applications to ecclesiastical

[45] Ibid., 110-113. For Clavius's visit to Maurolico see Lattis, *Between Copernicus and Galileo,* 19. Ugo Baldini dates Clavius's trip to Messina, where Maurolico lived and worked, to the months between April and September of 1574. See Ugo Baldini, *Christoph Clavius Corrispondenza* Vol. 1, Parte 1*,* ed. U. Baldini and P. D. Napolitani, (Pisa: Universtia di Pisa, 1992), 46.

computation, and the fifth and sixth books of Euclid. From Easter to Pentecost students were taught geometric and astronomical forms, and finally, until the end of the year they studied perspective and horology.[46] In the first year of study alone, they would have covered almost as wide a variety of mathematics as Nadal had laid out in his entire three-year course. The second year was equally rigorous, continuing with two more books of Euclid, and adding trigonometry, geography, the use of astrolabes, conic sections, the theory of planets (including the use of the Alphonsine Tables), the classical Greek problems of squaring the circle and doubling the cube, practical algebra, and finding the area of planar and solid figures. It is clear from this outline, that the second year was designed to emphasize mixed mathematics by focusing on the practical applications – especially the practical uses of astronomy - of the pure mathematics that had been learned in the first year.

While Clavius did offer a variety of textbooks for each topic in his curricula, one author recurs more than any other - Clavius himself. Although at the time the curricula were written he had only published his commentaries on Euclid's *Elements* and Sacrobosco's *Sphere,* he mentions sixteen texts that he was in the process of writing or that he planned to write. These cover almost every branch of his course of study, leaving only speculative arithmetic and speculative music exclusively to other authors (namely, Jordanus of Nemore and Jacques Lefèvre d'Étaples).[47] While Clavius did not write all of the texts he planned, he did manage to cover most of the material listed in his curriculum. Notably missing are the promised texts on

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[46] Clavius, "Ordo servandus,"114.
[47] Ibid., 110-113. See note 44 for a brief explanation of speculative arithmetic and speculative music.

geography, perspective, and mechanics. He did, however, write the *Geometria practica,* which could be seen as a foundation for all three of those topics.[48] Clavius's textbooks provided a legacy that would continue to promote the study of mathematics even after his death. Jesuits around the world used them both in their schools and in their missionary work. Non-Jesuits throughout Europe, even in Protestant states, read and taught from them.[49] While the textbooks effectively covered the content of Clavius's curriculum, they did not specify how that curriculum could best be taught or what the status of mathematics should be. Before turning to how Clavius envisioned the implementation of his ideal curriculum, it will be helpful to study how he positioned his work in the contemporary conversation surrounding the status of mathematics beyond the Jesuit Order.

The Renaissance of Mathematics and the Question of Its Certitude

In the fifteenth and sixteenth centuries mathematics was the subject of an ongoing renaissance and restoration. For many mathematicians, finding, interpreting, and using recently rediscovered texts of Aristotle, Archimedes, Pappus, Euclid, Apollonius, and Proclus, among others was an obsession.[50] According to those

[48] The *Geometria practica* published in 1604, was one of the last books Clavius wrote. He suffered an illness around that time. It is possible that he wrote it to cover all of the topics for which he had not yet written textbooks because he feared he would not live long enough to complete individual texts for each topic. Indeed, geography is represented in this book through the numerous problems on measuring altitudes of mountains and various distances. Since these measurements rely on how the human eye sees distances, the reader must also consider problems of perspective. Mechanics is begun through discussions of how to relate shapes to each other, necessary considerations in building any machine.
[49] Lattis, *Between Copernicus and Galileo,* 5.
[50] Paul Rose, *The Italian Renaissance of Mathematics: Studies on Humanists and Mathematicians from Petrarch to Galileo* (Geneva: Librairie Droz, 1975), 1-2.

involved in this endeavor, greater comprehension of mathematics brought with it a greater appreciation for its knowledge and the desire to endow that knowledge with a status comparable to its perceived worth. As more and more texts were translated and discussed, the mathematicians and humanists of the Renaissance were faced with a changing understanding of the relationships between mathematics and other forms of knowledge.

In medieval European universities, mathematics had been regarded as a lower discipline. Its primary function was to prepare scholars for the study of the higher forms of knowledge - theology, law and medicine. Indeed, mathematics professors were paid less than other professors, and they usually treated the position as a stepping stone to the more prestigious and lucrative chair in medicine.[51] In the fifteenth and sixteenth centuries, mathematicians challenged their subordinate status. They argued that their discipline was able to make sure claims about the world, and that this ability granted it epistemological status comparable to that of natural philosophy. One of the earliest mathematicians to make such arguments was Regiomontanus, a German mathematician who had studied astronomy and logic at the University of Leipzig in the late 1440s and had then studied at the University of Vienna in the 1450s.[52] There he worked with Georg Peurbach, whose knowledge of Greek and relationships with humanists enabled him to provide Regiomontanus with the skills and connections to

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[51] Robert Westman, "The Astronomer's Role in the 16th Century: A Preliminary Survey," *History of Science* (1980), 119-120.
[52] Rose, *Italian Renaissance,* 90-93.

become a significant mathematical humanist.[53] Combining these two courses of study, Regiomontanus undertook a restoration of mathematics through the translation of Greek mathematical texts. He also established arguments in defense of mathematics in a 1464 oration at the University of Padua with which he began series of lectures he gave on astrology.[54] He first argued that the study of mathematics was necessary to understand Aristotelian philosophy, but his lecture went far beyond a promise of utility to future study.[55] It actually established a place for mathematics above natural philosophy in the hierarchy of disciplines. According to Regiomontanus, of the disciplines accessible to human reason, only mathematics had the certainty necessary to grant insight into the divine.[56] Of course, of the branches of mathematics, astronomy was naturally the best suited to the study of the divine because its subject was the heavens, and, in the form of astrology, it could provide direct insights into God's plan.

To illustrate the discipline's certainty, Regiomontanus traced the history of the four branches of mathematics from the ancient to the early modern world with both mythological histories and references to the works of various mathematicians. Many

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[53] For Peurbach's humanist connections see Franz Wawrik "Österreichishe Kartographische Leistungen im 15. Un 16. Jahrhundert," pp. 105-106 and Dieter Wuttke, "Verhältnis Humanismus un Naturwissenschaften," pp. 134-135, in *Der Weg der Naturwissenschaft von Johannes von Gmunden zu Johannes Kepler*, ed. Günther Hamann and Helmuth Grössing, *Verlag der Österreichisch Akademie der Wissenschaften*, vol. 497, 1988. For Regiomontanus's significance as a mathematical humanist see Rose, *Italian Renaissance,* 90-117; Noel M. Swerdlow, "The Recovery of the Exact Sciences of Antiquity" in *Rome Reborn: The Vatican Library and Renaissance Culture*, ed. Anthony Grafton, (Washington, D.C: Library of Congress, 1993), 125-167.

[54] It also happens to be the only one of the series that has survived.

[55] For Regiomontanus's arguments about the utility of mathematics in his "Oration," see J. Byrne, "A Humanist History of Mathematics? Regiomontanus's Padua Oration in Context" in *The Journal of the History of Ideas,* Vol. 67, No. 1, January 2006, 57-60.

[56] Ibid., 60-61

of the works he cited were Greek treatises which he had either translated or hoped to translate, but he also named some from the medieval and contemporary eras. In his view, there was a continuous evolution of mathematics through all of the texts he named, which was made possible because mathematicians readily acknowledged the work of their predecessors as accurate. Contrasting the wide acceptance of the truth of ancient mathematics with the constant disagreement among supporters of divergent versions of Aristotelianism, he argued that mathematics had a certainty natural philosophy could never possess.[57]

By the mid-sixteenth century, Regiomontanus's arguments had become part of the mathematical discourse of the day through a miscellany printed in Nuremberg by Johannes Petreius in 1537 as well as in Erasmus Reinhold's edition of the *Oration* printed in Wittenberg in 1549.[58] In fact, the influence of his *Oration* can be found in the works of other humanist mathematicians who sought to ennoble their discipline throughout the sixteenth century, including those of Clavius. Mathematicians working

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[57] For more on Regiomontanus's arguments see Ibid., 41-61 and Rose, *Italian Renaissance,* 90-117. Byrne argues that Regiomontanus's humanism was very much combined with the university mathematics of the mid-fifteenth century. Notably, the histories Regiomontanus provided were not filled with references to ancient sources, and may well have been based on the medieval work of Isidore of Seville. In Regiomontanus's accounts, geometry can be traced to ancient Egyptians need to measure land after the flooding of the Nile. Arithmetic traces its origins to Pythagoras. Astronomy is given two possible origins, one, Hebraic, with God giving it to Abraham. The other is Greek and based on the myth of Prometheus. He traced the scientific foundations of mathematics to the Greeks, namely Hipparchus and Ptolemy.

[58] *Rudimenta Astronomica Algranagi. Item Albategnius astronomus peritissimus De Motu Stellarum* (trs. Plato of Tivloi)… *cum… additionibus Joannis de Regiomonte. Item Oratio Introductoria in Omnes Scientia Mathematicas Joannis de Regiomonte Patavii habita, cum Alfraganum publice praelegeret. Eiusdem Utilissima Introductio in Elementa Euclidis,* (Nuremberg: Johannes Petreius, 1537). Johannes Petreius is the same printer who published the first edition of Copernicus's *De Revolutionibus* in 1543. See also Ed. Erasmus Rheinhold *Oratio de Johannes Regiomontano Mathematico, in Renunciatione gradus Magisterii philosophicis, recitata ab Erasmo Rheinhold Salueldensi Mathematum professore,* (Wittenberg: Vitum Creutzer, 1549).

outside of schools, notably, Federico Commandino and his students, also embraced Regiomontanus's position. They dedicated themselves to the revival of ancient Greek mathematics through the translation of Greek texts, a method they deemed viable precisely because mathematics contained certain, universal truths.[59] Indeed, Commandino seems to have been drawn to mathematics, despite his training in medicine, precisely because mathematics could provide the certainty that the early deaths of several family members had convinced him medicine could not.[60]

The arguments in favor of mathematics became so prevalent in the mid-sixteenth century that Aristotelian philosophers felt the need to defend their discipline against the rise of mathematics. Presenting their position, Alessandro Piccolomini's 1547 *Commentarium de certitudine mathematicarum disciplinarum* cast mathematics in a strictly pedagogical role. According to Piccolomini, mathematics was useful as a pedagogical tool because, where natural phenomena required years of experiment and observation to understand, mathematical demonstrations could be understood quickly, and mathematical causes could be identified on inspection. However, he argued that mathematics could not be more than a pedagogical tool because it could never address all four of the Aristotelian causes - formal, material, final, and efficient.[61] Drawing

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[59] Rose, *Italian Renaissance,* 185.
[60] Ibid., 188
[61] Alessandro Piccolomini, *Commentarium de Certitudine Mathematicarum Disciplinarum: in quo de Resolutione, Diffinitione et Demonstratione; nec non de materia et fine logicae facultatis, quamplura continentur at rem ipsam, tum mathematicam tum logicam, maxime pertinentia,* (Rome: 1547), 107-109; For a clear analysis of the four Aristotelian causes, see Peter Dear, *Revolutionizing the Sciences: European Knowledge and Its Ambitions, 1500-1700,* Second Edition (Princeton: Princeton University Press, 2009), 13-14. He explains the causes as follows. The formal cause is the nature or form of the entity in question. It relies on an understanding of the world as made up of categories of entities. Dear's example is that Socrates is mortal because that is the nature of his form – that of a man. The material cause addresses the material that composes an entity. Dear's example here is that a chair burns

from Aristotle's *Posterior Analytics*, Piccolomini identified only those demonstrations that could identify the Aristotelian causes as the "most powerful demonstrations" which could reveal the truth about the workings of nature.[62]

According to Piccolomini, mathematics could not have a "most powerful demonstration" because it was unable to deal with phenomena, a shortcoming that he attributed to mathematics being inherently nominalist. That is, he believed, that mathematical objects were never more than their names, which were abstract universals. Therefore, mathematical definitions could not be applied to physical realities; mathematicians could not consider phenomena, and mathematical demonstrations could never discuss causes, either proximate or immediate, of any natural phenomena.[63] Even worse, as Piccolomini noted, mathematical demonstrations of the same effect did not always proceed in the same way, meaning that the one effect could have different causes, a circumstance that he claimed left mathematics unable to distinguish between causes and effects. Since the final and efficient causes of phenomena always remained obscure to mathematicians, its internal validity (which Piccolomini did not challenge) could not be extended to external validity, and mathematics could never be more than a tool for philosophers, nor could

because it is made of wood. The final cause is "for the sake of which" something occurs. It explains why a phenomenon happens or an entity exists the way it does. Dear gives the examples of a sapling growing because it is striving to become a full grown tree or teeth being arranged the way they are to facilitate chewing. The efficient cause is the action by which an effect is brought about. Dear's example is that the pulling of a trigger is the efficient cause for the firing of a gun.

[62]Piccolomini, *Commentarium,* 87-89. Piccolomini uses the term "demonstratio postissima." The literal translation seems to be the best way to convey the superiority of such demonstrations over all others. Piccolomini does observe that the final and efficient causes are the same when dealing with natural phenomena.

[63] Ibid., 86.

it offer its own claims about the world.[64] A quick survey of mathematical demonstrations sufficed for Piccolomini to claim that mathematics did not possess any demonstrations of the most powerful sort.

Piccolomini's attack on the certitude of mathemtaics received a direct response from Francesco Barozzi, an Italian mathematician known for his 1560 translation of Proclus's commentary on the first book of *The Elements.* In 1560 Barozzi published a collection of works titled *Opusculum,* which contained a lecture he had given in Padua in 1559 promoting mathematics as well as two brief treatises on the questions of both the certitude and the mediate nature of mathematics. In the former treatise, he explicitly refuted all of Piccolomini's arguments.[65] Much of his defense of mathematics depended on the distinction between the universal and the particular with regard to mathematical demonstrations. He accused Piccolomini of making a hasty generalization about the universal traits of mathematical demonstrations from just a few particular examples. Barozzi was willing to grant that the demonstrations Piccolomini provided were indeed not among the most powerful. However, he reminded the reader (and Piccolomini) that all forms of knowledge rely on many kinds of demonstrations, not all of which are among the most powerful.[66] He provided his

[64] Mathematical objects were seen as immaterial, so the material cause made little sense in the context of mathematics. Piccolomini allowed that the formal cause could sometimes be addressed by mathematics as the form of a diagram could be used to as the cause of whatever result was claimed by the proposition.

[65] Francesco Barozzi, *Opusculum in quo uno Oratio et duae Quaestiones altera de certitudine, et altera de meditate mathematicarum continentur*, (Padua: E.G.P,1560). Although Barozzi is clearly directly engaging with Piccolomini's work, he does not ever name the earlier author. Instead his references throughout the text to "the most illustrious man" and "the more recent man" suggest that he saw himself as engaging primarily with Aristotle and Averroes.

[66] Ibid., 25-26. In particular he cited demonstrations of the properties of quadrangles, circles, spheres and other shapes as demonstrations of the most powerful sort.

own examples to show that some mathematical demonstrations could be classified as "most powerful."[67]

In addition to arguing for the plurality of demonstrations in any discipline, Barozzi further promoted mathematics using Proclus's arguments that mathematical demonstrations gained perfect certainty from the perfect nature of mathematical entities. Barozzi's versions of these arguments are reminiscent of Regiomontanus's, and seem to have influenced many sixteenth-century mathematicians, including Clavius. In order to convince his reader, Barozzi began by refuting Piccolomini's charge of nominalism. He argued that, a close examination of mathematical entities revealed essential forms, which, taking a Platonic stance, he claimed were real and, as explained by Plato in the *Timaeus*, pertained to part of the soul.[68] Thus, mathematical knowledge, far from being a self-contained body of knowledge about of abstract concepts without any connection to the world, was closely related to divine knowledge and the plan of Creation. Furthermore, he argued that because mathematics dealt with essences, its entities were perfect and stable, and, therefore, mathematical demonstrations could be perfectly certain. Neither perfection nor complete stability could be found in the natural world, making it impossible for natural philosophy to offer certainty comparable to that of mathematics.[69]

In his treatise on the mediate nature of mathematics, Barozzi, again drawing on Proclus, further claimed that the perfection and the stability of mathematical entities

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[67] Ibid., 24-27.
[68] Ibid., 13-18.
[69] Ibid., 15.

gave mathematics more than the ability to create certain demonstrations; they also gave it an intermediate position between natural philosophy and divine knowledge that gave mathematicians insight into divine knowledge. Mathematics gained its mediate nature, in his view, because it shared one quality, its accessibility to reason, with the natural sciences, and another, its immaterial and unchanging subject, with the divine sciences. This meant that theorems and concepts could be demonstrated through the use of arithmetic and geometry, just as the claims of the natural sciences could be demonstrated through logic.[70] But, unlike the demonstrations of natural philosophy, those in mathematics were always applicable and remained certain because, like perfect divine entities, mathematical entities were immaterial and unchanging.[71]

While Barozzi took it upon himself to address Piccolomini's arguments directly, he was far from the only sixteenth-century mathematician to reflect the positions Regiomontanus took in his oration. A brief comparison of the prefaces to Commandino's commentary on Euclid's *Elements,* published in 1572, and Clavius's commentary on the same text, published in 1574, shows that both sixteenth century mathematicians were engaged in Regiomontanus's project of restoring to mathematics the nobility, and consequently the status, it gained from its antiquity, its certainty, and its subject matter. Both Commandino and Clavius also embraced Barozzi's project of defending mathematics as a source of certain knowledge. Because Euclid's *Elements*

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[70] Mathematics today includes a branch called logic, but, in the sixteenth century, logic was a branch of the trivium. It was based on the development of syllogistic arguments. Geometry and arithmetic did not use the syllogistic form, which separated their demonstrations from the logic-based definitions of philosophy. However, as will be discussed in chapter 3, Clavius did show that mathematical proofs could be written as sequences of syllogisms.

[71] Barozzi, *Opusculum* 33.

was the first mathematics text in a typical Latin curriculum, it offered its commentators an invaluable opportunity to provide a broad discussion of the merits of mathematical study as part of the introduction. Indeed, Clavius's preface did such a thorough job of introducing the discipline that, at the end of his life, he used it to preface his collected works.[72] However, where Commandino emphasized the restorative rhetoric of Regiomontanus and Barozzi claiming to revive the "polish and splendor" of mathematics by showing its close links to the study of the divine, Clavius focused on the versatility of mathematics as a branch of philosophy whose certainty allowed it to bridge the physical and divine worlds.[73]

Following Regiomontanus's appeal to history, both Clavius and Commandino appealed to mathematics' ancient past as part of their arguments for its nobility. Commandino's approach of inserting mathematics into Biblical history, a common practice for Renaissance scholars attempting to Christianize Platonic philosophy, emphasized the divinity of mathematics instead of its certainty.[74] According to his narrative, antediluvian prophets, recognizing the nobility of mathematics, erected two columns, one of brick and one of stone, inscribed with mathematical knowledge in

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[72] Clavius's collected mathematical texts were published over the years 1611 and 1612 under the title *Opera Mathematica*. It was a five volume compendium of all of his textbooks and his work related to calendar reform. Christopher Clavius, *Opera Mathematica V tomis distributa ab auctore denuo correcta, et plurimis locis aucta* (Mainz: Antonius Hierat and Reinhardus Eltz, 1612).
[73] Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), *2*r*; "... non possum non vehementer dolere temporum nostrum conditionem, qua nobilis discipline cultus, & splendor squalore immenso, ac tenebris penitus contabescit…"
[74] D.P. Walker provides a discussion of how Renaissance scholars attempted to unite Platonism and Christianity, including the positioning of Moses as the source of Gentile knowledge in his *The Ancient Theology*. D.P. Walker, *The Ancient Theology: Studies in Christian Platonism from the Fifteenth to the Eighteenth Century*, (Ithaca: Cornell University Press, 1972).

order to preserve it through predicted disasters.[75] After the flood, Abraham, whom Commandino called "nearly divine," began anew the study of mathematics. He brought knowledge of it with him to Egypt where the subject was improved and eventually discovered by the Greeks. Commandino then gave a list of those Greeks whose work, he claimed, created a golden age for mathematics. This history of mathematics demonstrated the disciplines' nobility.[76] After all, what could be nobler than knowledge given to man by God?

Clavius's arguments stayed closer to Regiomontanus's claims for mathematics, emphasizing mathematics' continual progress through multiple cultures, rather than its divinity. His section on the history of mathematics, titled "Inventors of the mathematical disciplines," opened with the claim that mathematics "progresses little by little from the imperfect to the more perfect."[77] To illustrate mathematics' progressive development across various cultures in antiquity, Clavius gave narratives for the invention of each of the four branches of the quadrivium in the ancient world, just as Regiomontanus had done in his Oration.[78] According to Clavius, arithmetic was

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[75] Clavius provided the same account in the preface to his commentary on Sacrobosco's *Sphere*. He credited it to Flavius Josephus. See Christopher Clavius, *In Sphaeram Ioannis de Sacro Bosco Commentarius*, 3-4. Regiomontanus also gave the same narrative for astronomy, but not for mathematics as a whole. See Byrne, "A Humanist History," 51. It is not clear where Commandino drew his narrative from. His comment on the value of mathematics reads, "Quare nec primis illis temporibus, quae tam inculta creduntur, nobile matheseos studium incultum iacuit." (Commandino, *Euclidis Elementorum,* *3*v*).

[76] Commandino, *Euclidis Elementorum,* *3*v*. "Hoc post terrarum eluuionem apud chaldaeos summo praesertim Abrhami divini prope hominis studio ornatum, & acutum viguit. … Verum de his hactenus, neque enim historiam hic contexere propositum est. Sed haec pauca attigimus, ut antiquam huius studii nobilitatem obiter quasi digito ostenderemus."

[77] Clavius, *Euclidis Elementorum*, b*r*. "Immo vero singulas nequaquam summam adeptas esse perfeectionem statim ab initio, sed paulatim eas ab imperfectis ad perfectiora processisse, memoriae quoque proditum est."

[78] For summaries of Regiomontanus's histories see Byrne, "A Humanist History," 47-53. The various histories can be found in *Rudimenta Astronomica Algranagi*, α4*v* – β2*r*. While the narratives found in

Phoenician; geometry was Egyptian. In the Greek world, music and astronomy stemmed from myths about Mercury and Atlas, respectively. Like Regiomontanus, Clavius also observed that there were multiple narratives for the origins of astronomy.[79] By mentioning a variety of histories, Clavius illustrated the universality of his discipline that allowed it to flourish through various civilizations. In addition to mathematics' universal presence in ancient cultures, Clavius noted that mathematics spanned time as well. He observed that mathematicians from Pythagoras to Proclus built on each other's work, especially geometry, thereby ever-increasing the perfection and scope of the discipline.[80]

Clavius's use of the history of mathematics to demonstrate its universality ties into arguments for the certainty of the discipline, which was one of the key components of arguments for the nobility of mathematics. Clavius argued that mathematics had survived with little to no change from antiquity to the modern era

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Clavius's texts are similar to those in Regiomontanus's lecture, they are not identical. Clavius claims to have drawn his narratives for geometry and arithmetic from Proclus. (See Proclus, *A Commentary on the First Book of Euclid's Elements,* trans. Glenn Morrow, pp. 51-52.) The similarities are most striking in the narratives for astronomy. It is possible that Regiomontanus was Clavius's source. Byrne suggests that Regiomontanus may have used Isidore of Seville's *Etymologies*.

[79] Clavius, b*v*. "Astronomiam denique non pauci ab Atlante primum inventam esse autumant: Unde ob eximiam, qua primus inter mortals praeditus erat, Astronomia cognitionem, exortam esse volunt fabulam, illum suis humeris caelum sustinere; Alii putant, Chaldaeos diuturna observatione (quod etiam Cicero affirmat in libro de Divinatione) syderum scientiam adinvenisse. Alii Aegyptios primos huius scientiae faciunt inventores: Alii Assyrios: Alii denique gloriam hanc, & laudem Babyloniis esse deferendam, censent.

[80] Ibid., b*r-v*. "Caeterum paulatim deinde Geometria capta est exploriri, & non contenta suis finibus, se se ad corpora etiam coelestia dimetienda convertit, tradiditque principia universae Astronomiae, Perspectivae, Cosmographiae, & aliis disciplinis quam plurimis, quae ex ispa veluti radices dependent. Hanc Thales Milesius ex Aegypto in Graciam primus transtulisse fertur: Deinde eam insignes Philosophi, ac Mathematici plurimis, acutissimisque demonstrationibus locupletarunt, atque exornarunt: Inter quos hi sunt praecipui ex ueteribus; Pythagoras, Anaxagoras Clazomenius, Hippocrates Chius, Plato, Oenopides, Zenodorus, Brito, Antipho, Theodorus, Theatetus, Aristarchus, Eratosthenes, Architas Tarentinus, Euclides, Serenus, Hypsicles Alexandrinus, Archimedes Syracusius, Apollonius Pergaeus, Theodosius Tripolita, Mileus Romanus, qui & Menelaus, Theon Alexandrinus, Ptolemaeus, Eutocius Ascalonita, Pappus, Proclus, & alii pene innumeri, quos omnes longum esset recensere."

and could continuously serve as a foundation for further knowledge because its method, which was to reject any claim unless it was either so clear that it was beyond doubt or had already had its truth demonstrated without relying on any outside assumptions, created perfectly certain knowledge.[81] According to Clavius, mathematics was the only discipline to build knowledge by relying solely on previously demonstrated principles, making it the only reason-based discipline that could provide certain knowledge. Like Regiomontanus, he pointed out that while mathematics was widely accepted in much the same form as it had been written by the ancients, other fields, especially Aristotelian philosophy, were replete with disagreement because their methods allowed false assumptions to serve as foundations for arguments.[82] Thus, certainty earned mathematics a place among the "disciplines" rather than the lower arts, and, at least when it came to certainty, that place was above, not just alongside, natural philosophy within the hierarchy of disciplines.[83]

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[81] Clavius, *Euclidis Elementorum,* b2*r,* "Quod quam longe a Mathematicis demonstrationibus absit, neminem latere existimo. Theoremata enim Euclidis, caeterorumque Mathematicorum, eandem hodie, quam ante tot annos, in scholis retinent veritatis, puritatem, rerum certitudinem, demonstrationum robur, ac firmitatem. … Cum igitur disciplinae Mathematicae veritatem adeo expectant, adamant, excolantque, ut non solum nihil, quod sit falsum, verum etiam nihil, quod tantum probabile existat, nihil denique admittant, quod certissimis demonstrationibus non confirment, corroborentque, dubium esse non potest quin eis primus locus inter alias scientia omnes sit concedendus."

[82] Clavius never stated that it would impossible for philosophers to attain the same degree of certainty, but it is quite likely that he believed that the nature of the material studied would make eliminating all but the perfectly certain propositions from philosophy impractical if not impossible. He did recognize that philosophical disagreements arose from the practice building entire systems of knowledge on uncertain assumptions. If those assumptions were removed, everything built on them would collapse. In order for philosophy to achieve the same level of certainty found in mathematics, philosophers would have to start from scratch.

[83] Clavius, *Euclidis Elementorum,* a7*r,* "Aliis autem placet, ideo has artes praecaeteris nomen scientia, & doctrina sibi vendicare, quod solae modum, rationemque scientiae retineant. Procedunt enim semper ex praecognitis quibusdam principiis ad conclusiones demonstrandas, quod proprium est munus, atque officium doctrinae, sive disciplinae, ut & Aristoteles testatur; neque unquam aliquid non proabtum assumunt Mathematici, sed quandocunque aliquid docere volunt, si quid ad eam rem pertinet eorum, quae ante docuerunt, id summunt proconcesso, & probato: illud vero modo explicant, de quo ante nihil scriptum est. Quod quidem alias artes, disciplinasve non semper observare videmus, cum plerunque in

Commandino also cited the stability of mathematics from antiquity to the sixteenth century as evidence of its certainty. As Regiomontanus had once done for his listeners, Commandino reminded his reader that Euclid's *Elements* had been written nearly two thousand years earlier, and yet even after so much time, the enemies of Euclid and mathematics had not been able to find significant errors in the work.[84] However, in Commandino's argument, certainty is only the by-product of the true source of mathematics' nobility: the immateriality of mathematical objects which eliminated the inevitability of human error. According to Commandino, because no one could study material objects without introducing misconceptions, natural philosophy could never be as accurate as mathematics. Furthermore, Commandino claimed that the nature of mathematical entities gave the discipline a nobility beneath only that of a direct study of the divine. He argued that because mathematical concepts could be separated from the material changeability of the natural world, mathematics fell between the natural, whose subject matter is base and mutable, and the divine, whose subject matter is unchanging and perfect. Since each of the three branches of philosophy - natural, mathematical, and divine - was as noble as the

confirmationem eorum, quae ostendere volunt, ea, quae nondum sunt explicate, demonstratave, adducant." This quotation appears in the section titled "Mathematicae disciplinae cur sic dictae sint" or "Why the mathematical disciplines are so called." In this passage it appears that Clavius believed that through mathematical learning, one could discover the real nature of the universe. James Lattis came to a similar conclusion about Clavius's beliefs based on his study of the commentary on Sacrobosco's *Sphere.* (See Lattis, *Between Copernicus and Galileo,* 218.)

84 Commandino, *Euclidis Elementorum,* *5*v*, "Iam duo fere annorum millia abierunt, ex quo Euclides inter vivos conumeratus est. multos habuit adversarios, qui invidiae potius morbo, quam veritatis amore illius scripta omni studio labefactare sunt conati; nullam tamen adhuc in illis φευδογϛαφίαυ, nullum errorem, nullum paralogismum severi inquisitores deprehendere potuerunt." Regiomontanus had made a similar claim in his Oration. See Regiomontanus, *Oratio,* 1537, p. β4*r*. "Quod de nostris disciplinis nemo nisi insanus praedicare ausit quandoquidem neque aetas neque hominum mores sibi quicquidem detrahere possunt. Theoremata Euclidis eandem hodie quam ante mille annos habent certitudinem."

subject of its study, mathematical knowledge, as the step between natural and divine knowledge, possessed a dignity superior to that of natural philosophy but less than that of divine philosophy.[85]

Similarly, Clavius argued that, due to the abstract nature of its content, mathematics was clearly a branch of philosophy that lay between metaphysics, in which "material is separate from fact and reason," and physics, in which "fact and reason are united with sensible material."[86] While, like Regiomontanus, he allowed that the study of astronomy as a study of the connections between the perfect heavens and the flawed world, was a means through which the divine could be studied, his positioning of mathematics between physics and metaphysics was slightly more modest than Commandino's claim that mathematics was intermediate between natural and divine philosophy. Nevertheless, Clavius's view positioned mathematics as a bridge between studies of the imperfect concrete world, the subject of natural philosophy, and studies of the perfect abstract world of ideas, the subject of metaphysics, thereby endowing his discipline with a status above that of natural philosophy within the hierarchy of disciplines.

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[85] Commandino, *Euclidis Elementorum,* *3*v*, "Hinc triplex illud philosophiae genus, Divinum, quod quidem ut nomine, ita & re duo reliqua supra quam dici potest, antecellit; Naturale, quod tertium est, ac postremas ordine, ac dignitate habet partes; & medium, quod mathematicum appellatur: quoniam solum vere disci, ac sciri potest, ob summam rei subiectae constantiam, & certam demonstrandi rationem: Hoc quidem ut divinis substantiis inferius est (quid enim tam eximium, ut cum illis comparetur?) ita naturalibus praestat, atque sperius est; quae materiae funditus immersa, variam & mutabilem eius sequuntur naturam."

[86] Clavius, b*v*. "Metaphysices etenim subiectum ab omni est materia seiunctum & re, & ratione: Physices vero subiectum & re, & ratione materiae sensibili est coniunctum" In the sentence in which Clavius introduces the intermediate status of mathematics he actually says that it is intermediate between metaphysics (metaphysicam) and natural knowledge (naturalem scientiam). In his description of the fields he replaces "natural knowledge" with "physics" (physices), which were interchangeable terms at the time.

The Question of the Certitude of Mathematics within the Society of Jesus

Within the Society of Jesus, despite early rigorous programs of mathematics, Aristotelian philosophers seeking to defend the status of their discipline against the rise of mathematics found a voice in Benedict Pereira (1535-1610). He offered his view of the limited role of mathematics in his 1579 work, *De communibus omnium rerum naturalium principiis et affectionibus, libri quindecim*. There, he, like Piccolomini, placed the discipline in a supportive role to physics and metaphysics but denied it the ability to make its own knowledge claims about the physical world. He began his discussion with the observation that all sciences build on each other to reach moral philosophy and the science of the soul. These he claimed were the most highest forms of rational knowledge and were dependent on the knowledge found in natural philsophy.[87] Mathematics belonged to a lesser domain, speculative philosophy, which was composed of mathematics, physics, and metaphysics. Pereira believed that like physics - the study of the material products of divine thought – mathematics was based in matter, but, because it studied forms abstracted from matter, it like metaphysics – the study of the divine and immaterial – could be understood outside of material forms. Interestingly, Clavius had used the same traits of mathematics to argue that

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[87] Benedict Pereira, *De communibus omnium rerum naturalium principiis et affectionibus, libri quindecim*, (Paris: Micaelem Sonnium, 1579), 12-13. "Porro, scientia non dicitur unioce, de subalternate & subalternata, quia subalternata, cum accipiat principia a subalternante, pendet essentialiter ab illa, nec potest sine adminiculo eius, scientiae nomen & rationem obtinere, at omnes scientiae practicae subalternantur speculativis, ut omnes fere artes mechanicae pendent a scientiis Mathematicis; & Medicina pendet a Philosophia naturali, scientia moralis magna ex parte pendet a scientia animae: nam diviso habituum, & virtutum moralium pendet ex distinction potentiarum animae. Distinctio etiam passionum, & quae de foelicitate traduntur, supponunt multa ex Philosophia naturali."

mathematics was intermediate between the two philosophical disciplines. For Pereira, this placed it at the bottom of the hierarchy, thereby limiting it to a supportive role for both of the other disciplines. The properties mathematics shared with physics and metaphysics thus enabled it to train students who sought to understand the higher branches of philosophy, either material or immaterial, but were insufficient to allow it to make its own knowledge claims.[88] Indeed, Pereira dismissed mathematical knowledge as trivial because he believed that quantities, the subject of mathematics, were merely accidents. Thus, unlike the properties of matter studied by physics and the principles of nature studied by metaphysics, quantities could not be understood as essential to objects or phenomena, and mathematics could not be used to uncover any of the Aristotelian causes.[89]

Pereira furthered his arguments by using the stability of mathematical objects, the very trait that mathematicians claimed gave it certainty, to deny mathematics a foothold in reality. In his view, not only are quantities not essential to physical entities, but mathematical objects are also abstracted from change.[90] They, therefore, cannot cause change and so cannot explain causes – formal, material, efficient, or final. Although Pereira did not explicitly raise the charge of nominalism, the

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[88] Ibid., 14-15.
[89] Ibid., 38-39. Here Pereira reduced mathematics to quantity, which may not be intuitive to today's mathematicians, but fits with a Renaissance understanding of the quadrivium in which arithmetic and music study discrete quantity, and geometry and astronomy study continuous quantity.
[90] Ibid., 115. Pereira used the word "motu" which is usually translated by motion or movement. However, in this case, I believe it is better interpreted as change, of which motion or movement are specific cases. Since Pereira is discussing the inability of mathematics to address causal questions in general, a more general translation seems appropriate. "Res Mathematicae, abstractae sunt a motu, ergo ab omni genere causae: Antecedens per se clarum est; consequentia patet ex eo quia omnes causae sunt connexae aliquo modo cum motu, id quod aperte declarat definitio cuiusque."

contention that mathematical entities cannot be changed tied them to their definitions as effectively as Piccolomini's accusation. Thus, in agreement with Piccolomini, Pereira argued that mathematics could not supply the most powerful demonstrations, because, in arguing solely from its own internal logic, it had no claim to external validity. Mathematics, thus, remained a subordinate discipline, whose clear demonstrations could only provide training for youths who would go on to the more difficult and more meaningful demonstrations of physics and metaphysics.

While Pereira raised a strong voice against the status of mathematics, Clavius positioned himself as the discipline's defender within the Society of Jesus. As discussed above, he provided arguments in support of the certainty of mathematics in his preface to *The Elements* analogous to those presented by Regiomontanus and other sixteenth-century mathematicians, including Barozzi. Indeed, the latter was one of Clavius's most cited sources. The two mathematicians corresponded, and Clavius had certainly read Barozzi's translation of Proclus's fifth-century commentary on the first book of *The Elements* – and possibly the *Opusculum* – when he wrote his own commentary on Euclid.[91] Barozzi's claims for the intermediate nature of mathematics appear fundamentally unchanged in Clavius's text. However, while the Italian's arguments were largely devoted to the study of geometry, which he used to represent all of mathematics, Clavius endeavored to establish a complete curriculum and so argued for the value of studying a variety of branches of mathematics. Nowhere were

[91] Ugo Baldini's collection of Clavius's letters shows five letters between Barozzi and Clavius in the years 1585-1587. *Christoph Clavius: Corrispondenza* ed. U. Baldini and P.D. Napolitani (Pisa: Universtia di Pisa, 1992).

those arguments more pronounced than in the preface to his 1570 commentary on Sacrobosco's *Sphere,* the introductory text to astronomy and the only text besides *The Elements* for which Clavius wrote a preface longer than a few pages. There, Clavius (again echoing Regiomontanus) argued that, due to the divine nature of the celestial bodies and the certain foundations of geometry and arithmetic, astronomy allowed mathematics to bridge the natural and divine worlds. Because astronomy was dependent on the prinicples of the other branches of mathematics, it served to demonstrate the power of the entire discipline – including abstract geometry and arithmetic – to contribute to knowledge about the universe. It was through mathematics that an accurate description of the cosmos could be developed from the physical assumptions made about the basic structure of the universe.[92] Thus, Clavius brought the arguments for the importance of the certitude of mathematics to fruition in astronomy, the branch of mathematics he believed to be most noble.

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[92] In 1543 Copernicus famously cautioned that those who were not mathematicians could not judge the merit of his *De Revolutionibus.* While Clavius was a staunch defender of the geostatic system (see Lattis, *Between Copernicus and Galileo* – Clavius did agree that the Ptolemaic system could not survive as it was presented in the *Almagest,* but he never abandoned the geostatic worldview.), Copernicus's assertion that only mathematicians could judge his claims about the universe likely appealed to Clavius since asserting the truth of the structures of the universe outlined by astronomy (either helio- or geo-static) required that the arguments for the certainty and nobility of mathematics be accepted as proof of the ability of mathematics to make claims about the world. However, because astronomy was a branch of mixed mathematics, it also relied on physical assumptions. While mathematics could create an accurate picture of the universe, choosing which mathematical description was accurate required a reliance on probable arguments. Based on scripture and Aristotelian philosophy, Clavius argued that the Ptolemaic system was far more probable than the Copernican. See Lattis, p. 141. Nevertheless, the reliance on probable arguments to choose between mathematical descriptions of the universe does not detract from the certainty of mathematics. Lattis also points out that deciding between the Copernican and Ptolemaic systems required physical assumptions only because both met the mathematical criteria of "economy of explanation… and practical utility" (p. 141). Whichever system was chosen, it was only through mathematics that the physical assumption of the position or movement of the planets could be completely explained.

Indeed, Clavius's preface to *The Sphere* offered the same arguments for the status of mathematics that we have seen in all the sixteenth-century texts studied here: the history and the certainty of mathematics endowed the discipline with nobility. He opened with a discussion of the history of astronomy based primarily on Flavius Josephus's *Antiquities of the Jews,* but recounted the same narrative for the history of mathematics that Commandino would later provide in his commentary on *The Elements.* That account viewed Abraham as the father of astronomy, which, as a gift from God to the patriarchs, is a divine science.[93] The certainty of mathematics clearly emerges as the story continues. Abraham brought astronomy to the Egyptians, who, in turn, gave it to the Greeks. The rediscovery of Greek texts, thus allowed modern mathematicians to continue building on a divine past knowledge.[94]

In a section titled "De Praestantia Astronomiae" or "On the Excellence of Astronomy," Clavius argued that astronomy excelled in the two standards on which the nobility of a subject as a whole was judged: the nobility of the object of study and the certainty of demonstrations. To prove that celestial bodies were noble, Clavius relied primarily on Aristotle's arguments for their incorruptibility and their position as the cause of all inferior phenomena (i.e., all terrestrial phenomena). These arguments made the celestial bodies the most noble subject possible in the study of natural philosophy. Moreover, he placed mathematics above natural philosophy in the hierarchy of disciplines because he believed that by studying the workings of the celestial sphere, an astronomer could contribute to theology. Following the arguments

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[93] Clavius, *Sphaeram*, 3.
[94] Ibid., 3-4.

of unnamed theologians, Clavius asserted that, while God exists in all places, He is most easily understood to belong in the heavens, the place where His omnipotence and goodness are most obvious.[95] Thus, Clavius argued that the celestial bodies, as incorruptible entities, were closer to God than the imperfect terrestrial world, and, therefore, they acted as the mediators and connections between the superior and inferior, the divine and the human.[96] In this way, astronomy, and with it the rest of mathematics, not only had a place alongside the disciplines of natural philosophy, but its inherent nobility gave it a status above natural philosophy.

As for the certainty of demonstrations, here too, Clavius argued that astronomy exceeded natural philosophy. In his view, astronomical demonstrations rested securely on the demonstrations of geometry and arithmetic, which were both developed by denying the truth of any claim that was not demonstrably true. This secured for astronomy the highest degree of certainty in its own demonstrations.[97] Thus, according to Clavius, through both its innately noble subject matter and its certainty, astronomy, and so mathematics, supplied the bridge between uncertain human knowledge of the imperfect world and certain divine knowledge of God's plan.

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[95] Ibid., 8-9. "Quamuis enim Deus non huic vel illi loco sit alligatus, sed ubivs locorum (quod nullis aliis convenit rebus) existat; ponitur tamen in coelo, tanque nobiliori mundi parte, ubi maxima suam omnipotentiam, & bonitatem manifestat, ut Theologi asserunt."

[96] Ibid., 8 "Tertio, quam corpora coelestia sunt propinquiora nobilissimo ac primo enti, puta, Deo glorioso; Immo secundum Averroem corpus coeleste est mediator, ac ligamentum superiorum cum inferioribus, & locus aeternorum, ac diviniorum, omnes etenim philosophi, ac nationes etiam quantumuis barbarae, in coelo Deum tamque in sede collocant propria."

[97] Ibid., 9. "Quod si modum demonstrandi, quo utitur Astronomia, consideremus, non solum omnes naturales disciplinas haec scientia longe superabit, sed nec inter Mathematicas scientias infima existimanda erit. Adhibet in ad ea confimanda, de quibus agit, demonstrationes efficacissimas, Geometricas nimirum, & Arithmeticas, quae ex sententia omnium philosophorum primum certitudinis gradum obtinent. Quare non sine ratione ex utorque capite, nempe nobilitate subiecti, & certitudine deomonstrandi, voluit Ptolemaeus ad initum Almagesti, Astronomiam simpliciter inter reliquas scientias esse primam."

Clavius's Ideal

The nobility and certainty of mathematics established for it a privileged place in Clavius's curricular thinking. These features made it a necessary foundation for the abstract studies of philosophy and theology and the more concrete tasks of public administration. Using the works of ancient philosophers and theologians as evidence, Clavius laid out his argument in a section entitled "The various uses of mathematical disciplines" in his preface to *The Elements.*[98] He first treated metaphysics and theology, reprising his earlier arguments about the intermediate status of mathematics between physics and metaphysics, and coupling them with a discussion about human cognition. Crediting Proclus (whose work he had most likely read in Barozzi's translation), Clavius contended that human intellect cannot immediately grasp abstract concepts. It must first seek to understand the concrete. Only from there, through the intermediate mathematical concepts, could it reach the abstract.[99] Thus, no study of metaphysics or theology could be fruitfully conducted without the prior study of mathematics. Clavius extended this argument from metaphysics to theology by citing several church fathers, including Augustine and Jerome, on the use of various branches of mathematics in theology. He never explained exactly how mathematics is used within the study of theology, but for each topic listed, from numerology to music,

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[98] Clavius, *Euclidis Elementorum*, b2-b3*r*. The section title is "Utilitates variae mathematicarum disciplinarum."
[99] Clavius, *Euclidis Elementorum,* b2*v* "Quam ob rem, antequam a rebus physicis quae materiae sensibus obnoxiae sunt coniunctae, ad res metaphysicas, quae sunt ab eadem maxime auulsae, intellectus ascendat, necesse est, ne harum claritate offundatur, prius eum assuefieri rebus minus abstractis, quales a Matheamaticis considerantur, ut facilius illas possit comprehendere."

geography, and astrology, he cited a sainted theologian as his source.[100] If the saints who had studied and developed theology believed that mathematics was necessary, who could say otherwise? Clavius remained equally vague in his arguments for the necessity of mathematics to philosophy and public administration, preferring to cite the texts of ancient philosophers, especially Plato and Aristotle, as evidence. Public administration is treated the most briefly with only a reference to Plato's *Republic* and *Timaeus.*[101]

Clavius continued these general themes in his introduction to *The Sphere,* the final and longest section of which was devoted to the utility of astronomy. As the culmination of the study of mathematics, astronomy was thus the discipline best able to establish mathematics' utility. Indeed, the section was designed to leave the reader with the impression that there was no area of human knowledge which was not supported by astronomy (and thus mathematics). It named "natural theology," metaphysics, natural philosophy, medicine, and poetry as disciplines dependent upon astronomy. Nautical arts, ecclesiastical computations, cosmography, and politics were also identified as completely dependent upon astronomy.[102] For most of the applications of astronomy, Clavius gave only brief references to ancient authors as evidence. However, relative to theology, he provided an extensive discussion and

[100] Clavius, *Euclidis Elementorum,* b2*v.* In addition to Augustine and Jerome, to whom Clavius credited the arguments that numerology is indispensable to the study of sacred scripture and music is necessary to theology, Clavius cited Basil and Gregory of Nazianzus, also known as Gregory the Theologian, to support the claims that astrology, geometry, geography are also indispensable to theology.
[101] Further discussion of Clavius's thoughts on the public utility of mathematics can be found in the next chapter.
[102] Clavius, *Sphaeram,* 10-11. Clavius argued that poets need astronomy because the only excellent poems include reference of some kind to the motions of the stars. Therefore, no poet who hoped to succeed in his endeavors could ignore astronomy.

cited a variety of biblical passages to support the claim that astronomy aids the study of the divine as a form of natural theology.[103] In the first of these, the Letter to the Romans (1:20), Paul reminded his readers that God can be perceived and understood through the study of Creation. Clavius used to remind his own readers that celestial bodies are the most divine parts of Creation, and therefore the best sources through which to attain an understanding of the invisible God.[104] Psalms 8 and 19 and chapter 13 of the Wisdom of Solomon also served as evidence for the value of astronomy in studying God.[105] All of these verses indicated that the splendor of Creation, especially the heavens, could be seen as a mirror for the splendor of God the creator. Astronomy thus provided a direct link between the study of the world and the study of the divine.

These arguments in Clavius's prefaces expressed the principle that guided the creation of his ideal program of study, the most rigorous of the three curricula discussed above: The study of a breadth of mathematical disciplines could enable students to pursue whatever philosophical or theological study they chose. To facilitate this goal, the curriculum covered geometry, arithmetic (both practical and speculative), algebra, geography, and music. Since Clavius aimed to secure a place

[103] Ibid., 10. "Ex quo factum est, ut Astronomia, quae de praestantissimis istis corporibus disputat, a plerisque Theologia naturalis vocetur."
[104] Ibid., 9. In the New Oxford Annotated Bible (New Revised Standard Version), this verse (Romans 1:20) is rendered as "Ever since the creation of the world, his eternal power and divine nature, invisible though they are, have been understood and seen through the things He has made."
[105]Ibid., 9. The passages Clavius indicated are as follows in the New Oxford Annotated Bible (New Revised Standard Version): Psalm 8:3-4: "When I look at your heavens, the work of your fingers, the moon and the stars that you have established; what are human beings that you are mindful of them, mortals that you care for them?" Psalm 19: "The heavens are telling the glory of god and the firmament proclaims his handiwork." Wisdom of Solomon 13 is about using nature to study God. It tells the reader to know how much more beautiful God, the creator of celestial bodies, is than the bodies themselves.

for mathematics among the higher disciplines by illustrating its importance to theology, he focused his curriculum on astronomy. Six of the twenty-two topics outlined were explicitly identified as part of astronomy, and several more, such as the study of sines and the study of cylindrical sections and ellipses, were necessary to the pursuit of that branch of mathematics.[106]

However, while Clavius's position as the mathematics professor in Rome gave him some influence over the Jesuit mathematics curriculum, and while his textbooks provided Jesuit mathematics teachers with the arguments to promote mathematics, his was just one voice in the debate. He could not guarantee the content of the mathematics curriculum or the status of his discipline in all Jesuit schools. Even in Rome, Pereira challenged Clavius's claims for the elevated status of mathematics. And, as Ignatius had insisted in the *Constitutions,* how a subject was taught was at least as important as what was taught. Thus, for Clavius it was not sufficient to argue for the status of mathematics in the prefaces to his textbooks. He also had to ensure that the structure of the schools and the actions of their teachers and administrators guaranteed mathematics its place among the higher disciplines. Therefore, he supplemented the arguments he provided in his mathematics texts with a brief work on the ways in which the Society could promote mathematics in its schools. There, Clavius justified a high status for mathematical sciences and offered practical

[106] Clavius, "Ordo servandus," 110- 113. Those six topics are the study of the sphere (using his commentary on Sacrobosco), the use of geometry from astronomical instruments, the structure of the astrolabe, horology, problems of astronomy, motion of the planets and spheres. Algebra and conics are also explicitly included for their use to astronomy.

guidelines for ensuring that students and teachers would esteem mathematics as worthy of that status.[107]

Clavius's first suggestion for the promotion of mathematics was to ensure that the mathematics teacher be erudite and of good reputation, so that his colleagues and students would respect him. He argued that only if the mathematics professor was held in high esteem and treated as an equal by the philosophy professors would students be able to see that mathematics and philosophy were equal in status and closely related to one another. To enable the mathematics professor to secure such authority, Clavius warned that he should not be given too many other duties. He needed time to further his own learning. Clavius also insisted that as a sign of respect from his colleagues in philosophy, the mathematics professor needed to be invited to solemn occasions such as the granting of degrees and disputations, and he must have a part in the examination of students advancing to their degrees.[108]

Clavius's second suggestion was that students needed to be made aware that mathematics was useful and necessary to the rest of philosophy so that they would not disregard their studies in the field.[109] Indeed, he believed that the two disciplines were

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[107] Christopher Clavius, "Modus quo disciplinae mathematicae in scholis Societatis possent promoveri (1582)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 115-117; Christopher Clavius, "De re mathematica instructio (Ad annum 1593)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 117-118.

[108] Clavius, "Modus quo disciplinae mathematicae," 115. "Primum, deligendus erit magister eruditione atque auctoritate non vulgari. Alterutra enim si absit, discipuli, ut experientia docet, non videntur ad disciplinas mathematicas allici posse. Ut autem maiorem apud discipulos auctoritatem habeat magister, et disciplinae ipsae mathemtaicae maiori in pretio sint, ac discipuli earum utilitatem necessitatemque intelligant, invitandus erit magister ad actus solemniores, quibus doctores creantur et disputationes publicae instituuntur."

[109] Ibid., 116. "Secundo ergo loco, necesse est, ut discipuli intelligant, has scientias esse utiles et necessarias ad reliquam philosophiam recte intelligendam, et simul magno eas ornamento esse omnibus

so closely related that "natural philosophy is maimed without the mathematical disciplines," but he feared that that reality was hidden from students because mathematics had historically been looked down upon and belittled by philosophers whose own knowledge of the subject was limited.[110] As a result, he claimed that students suffered under the tutelage of philosophers who derided mathematics and, "on account of their ignorance of these [mathematical disciplines] very often committed many errors, and those most grave" in their work.[111] To create the appropriate view of mathematics, Clavius suggested that students of physics simultaneously study mathematics and be exposed to the numerous topics covered by mathematicians. Furthermore, he insisted that teachers should encourage the study of mathematics, in both the general course and at more advanced levels in private academies, because mathematics was a great ornament in the perfection of the students' erudition, and because teaching mathematics as a branch of philosophy could prevent disgrace to the Society's reputation should Jesuit authors err for want of mathematical knowledge.[112] In order to cement the disciplines' elevated status and to inspire students to the study of mathematics, Clavius wanted to add mathematics to the

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aliis artibus, ut perfectam eruditionem quis acquirat. Immo vero, tantam inter se habere affinitatem hasce scientias et philosophiam naturalem, ut nisi se mutuo iuvent, tueri dignitatem suam nullo modo possint."

[110] Ibid., 116. "Omitto philosophiam naturalem sine disciplinis mathematicis mancam esse et imperfectam, ut paulo infra docebimus."

[111] Ibid., 116. "Immo, propter earum ignorationem nonnulli philosophiae professores saepissime multos errores, eosque gravissimos, commiserunt, et (quod peius est) scriptis etiam mandarunt; quorum aliquos in medium proferre non esset difficle."

[112] Ibid., 116. "Pari ratione oporteret praeceptores philosophiae callere disciplinas mathematicas, saltem mediocriter, ne in similes scopulos magna famae, quam Societas in litteris habet, iactua et dedecore incurrerent."

list of classes in which gifted students could give monthly presentations to all of their colleagues. Naturally, the best presentations were to be publically praised.[113]

Of course, even if properly motivated to the study of mathematics, Jesuit students would need good mathematics teachers if they were to succeed in earning the Society a good reputation for its members' mathematical knowledge. Clavius thus recommended that schools create private academies of ten to twelve students who had shown extraordinary promise in mathematics.[114] Clavius himself implemented such an academy in Rome in 1593. It functioned as an advanced seminar in which he was able to achieve his ideal for the study of mathematics, albeit only locally. His students were treated to Clavius's most extensive curriculum covering everything from geometry to algebra and horology. Indeed, his advanced textbooks were likely designed for this academy rather than for the Jesuits' public course of study.[115]

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[113] Ibid., 117. "Praeterea ad haece studia maxime incitabuntur scholastici, si singulis mensibus omnes philosophi in unum aliquem locum convenirent, ubi unus discipulorum habeat brevem commendationem disciplinarum mathematiciarum … Ubi laudari possent ii, qui melius problema propositum solvissent, vel pauciores paralogismos, qui non raro occurrunt, commisissent in novis demonstrationibus inveniendis. Ita enim fieret, ut non parum inflammarentur ad haec studia cum viderent sibi propositam esse hanc gloriam; et simul intelligerent eorumdem praestantiam, maioresque in illis hac exercitatione facerent progressus."

[114] Ibid., 115-116. "Ut autem Societas semper habere possit idoneos harum scientiarum professores, eligi deberent aliquot ad hoc munus obeudeum apti 12 idonei, qui in privata academia instituerentur in variis rebus mathematicis; aloquin non videtur posse fieri, ut haec studia in Societate diu permaneant, nedum promoveantur; vel et magnum Societati afferent ornamentum, ut frequentissime in colloquiis et conventibus principum virorum de illis sermo habeatur, ubi intelligunt nostros mathematicarum rerum non esse ignaros.

[115] Ugo Baldini, "The Academy of Mathematics" in *Jesuit Science and the Republic of Letters* ed. Mordechai Feingold, (Cambridge, MA: The MIT Press, 2003), 47- 98. Baldini explains that before 1593 the academy existed as an informal group of students who pursued mathematics in addition to their other responsibilities. When it was formalized in 1593, students of the academy were exempted from teaching grammar during the year between the study of philosophy and theology. The textbooks for the academy likely included the *Algebra,* the *Astrolabe*, and the treatises on horology. Indeed, based on the prefaces of Clavius's texts, it seems probable that only the commentaries on Euclid and Sacrobosco, the *Epitome arithmetica* and the *Geometria practica* were intended for the general course of study. Of those, only the commentary on Euclid was ultimately required in the *Ratio Studiorum.*

The academy's activities make it clear that Clavius's goals for mathematics went beyond training future philosophers and theologians. Clavius justified the academy's creation as a means to train mathematically capable teachers who could staff the Jesuits' European schools and mathematically capable missionaries who could succeed in building new missions without access to specialists. Yet, the academy also functioned as a research group in which Clavius and his students pursued new developments in mathematics, including the development of new theorems and instruments.[116] In this way, Clavius sought to demonstrate how mathematics could enable students to study the divine and bring honor to the Society, and it was through the academy that Jesuit mathematicians developed and published their results, including their 1611 judgment on Galileo's telescopic observations.[117] The contemporary fame of members of the academy – including Christopher Grienberger and Orazio Grassi, both of whom were active participants in the Jesuits' debates with Galileo, attests to Clavius's success on the second goal. Indeed, in the first years of the seventeenth century, it was nearly obligatory for mathematicians staying in Rome to meet with Clavius or his colleagues. Such famous mathematicians as Giovanni Magini (1555-1617), Galileo, Johann Schreck (1576-1630), and Marino Ghetaldi (1568-1626) were among those who visited the academy.[118]

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[116] Ibid., 55-58.

[117] Ibid., 55. Baldini notes that publishing seems to have been up to the head of the academy, not the individual authors.

[118] Ibid., 53-57. On page 68 Baldini notes that in 1612, the year Clavius died, the Collegio Romano was "second to no other European scientific institution." Baldini's classification of the academy as a "scientific institution" illustrates the significance of the academy as a formal group of scholars devoted to the pursuit of natural knowledge at the start of the Scientific Revolution. While the term is anachronistic, it clearly designates Clavius's academy as recognizably similar to modern scientific institutions in composition (a small group of scholars) and aims (uncovering new knowledge and

Conclusion

Clavius devoted most of his work to establishing his ideal mathematics curriculum. It was crafted on arguments presented by mathematically inclined humanists, notably Regiomontanus, Francesco Barozzi, and Federico Commandino, for a restoration of the ancient authority of the discipline and its elevation to at least the status of natural philosophy within the hierarchy of disciplines. Like these mathematicians, Clavius viewed mathematics as a discipline that linked the physical and the divine and the only way through which human reason could generate certain knowledge. His curriculum reflected that view, especially in its emphasis on astronomy, the branch of mathematics that he believed was best able to facilitate the study of the divine and thus was best suited to advancing the Jesuits' mission of saving souls.

Not everyone shared such a positive view of mathematics, however. Aristotelian philosophers sought to defend the status of their discipline by arguing for mathematics' role as nothing more than a tool. Thus, while Clavius's ideal and his curriculum aligned with arguments of sixteenth-century humanists, it was not clear that the Jesuit schools would embrace the mathematician's vision in their planned curriculum, even on as small a scale as the Roman academy.

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training new students in their discipline and training). However, the academy was short-lived, dissolving shortly after Clavius's death. It is not clear why exactly the academy failed to survive, but Baldini argues that it is possible that it was a victim of the Jesuits' dispute with Galileo and their defense of a scholastic cosmology that was central to Clavius's vision of mathematics but was increasingly discarded by seventeenth century mathematical scholars.

When Clavius was named to the chair of mathematics at the Collegio Romano in 1563, the process of writing the curriculum for Jesuit schools had barely begun, and the status of mathematics was disputed. Ignatius's own discussion of mathematics was vague on the question of the discipline's status; he only required that what was taught advance the Jesuit mission of saving souls. In the first few years of the Jesuit schools, Clavius's predecessors suggested curricula that covered several branches of mathematics from pure geometry and arithmetic to a variety of mixed branches like geography, a mathematical study that could have helped bring students to the early Jesuit schools. The breadth of their suggested curricula paved the way for Clavius to offer an extensive program of mathematical study, but he still had to contend with Jesuit philosophers who did not believe that such a curriculum was necessary or valuable to Jesuit schools. Traces of that internal debate are visible in Clavius's arguments in support of mathematics in the prefaces to his textbooks, especially the commentaries on Euclid's *Elements* and Sacrobosco's *Sphere,* and in his ideal curriculum, both in his suggested order of topics and his suggestions for how mathematics should be taught. Ultimately, as the existence of his academy of mathematics attests, Clavius met with enough success to teach his ideal curriculum to a few students from 1593 until his death in 1612, precisely the window of time in which Kepler and Galileo, respectively at Tübingen and Padaua, were developing their mathematical proposals in natural philosophy.

Chapter Two

Christopher Clavius and the Jesuit Mathematics Curriculum: Practical Priorities

"The mathematical disciplines should be counted as not only useful, but truly also as completely necessary not only to perfectly learning other arts, but, indeed, also to rightly instituting and managing public affairs."[1]

Christopher Clavius, 1574

Over the course of his teaching career as the professor of mathematics at the Jesuit Collegio Romano from 1563 until 1612, Christopher Clavius carried out an extensive pedagogical project. In the early 1570s he published his first two textbooks, a commentary on the *Sphere* of Sacrobosco (1570) and a commentary on Euclid's *Elements* (1574). In 1581 he suggested three mathematics curricula of varying degrees of rigor. He then went on to write textbooks to accompany most of the topics his curricula covered. While Clavius emphasized the noble status that he thought mathematics deserved because it informed both natural philosophy and theology, his pedagogical work included significant segments on practical mathematics as a tool that princes and those close to them could use to the benefit of their people. Because

[1] Christopher Clavius, *Euclidis Elementorum Libri XV Accessit XVI de solidorum Regularium comparatione* (Rome: Vincentium Accoltum, 1574), b2*r*. "Non solum utiles, verumetiam necessariae admodum censeri debent disciplinae Mathematicae cum ad alias artes perfecte perdiscendas, tum ad rem etiam publicam recte instituendam, & administrandam."

his tenure in Rome covered the almost twenty-year period from 1581 to 1599 when the Jesuits were writing their general curriculum, the *Ratio Studiorum,* Clavius's pedagogical work became the example of mathematics education on which the authors of the curriculum based their initial assessment of the role of mathematics within Jesuit schools. Ultimately, driven by the logistical concerns of securing the patronage necessary to their schools and of staffing them fully, the curriculum writers used Clavius's arguments for practical mathematics to define the discipline. When the 1599 *Ratio Studiorum* was distributed to the Jesuit schools, the priests assigned to teach mathematics found themselves with a brief and open-ended curriculum. For two months, the students would learn exclusively from Euclid's *Elements*. After that the instructor would "add some geography or astronomy or similar matter which the students enjoy hearing about."[2] As it turned out, despite Clavius's arguments for mathematics', most especially astronomy's, noble status as a discipline that could inform theology, Jesuit schools outside of the Roman center were more likely to teach geography than astronomy and clearly favored practical branches of mathematics over the theoretical questions found in astronomy.[3]

Clavius's attention to the potential uses of his discipline was hardly unique. Evidence of a broad interest in practical mathematics during the fifteenth and sixteenth centuries can be found in the quantity of texts published on various applications of the

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[2] Allan P. Farrell, trans. *The Jesuit Ratio Studiorum of 1599* (Washington DC: Conference of Major Superiors of Jesuits, 1970), 46.

[3] Antonella Romano, *La Contre-Réforme Mathématique: Constitution et Diffusion d'une Culture Mathématique Jésuite à la Renaissance.* (Rome: École Française de Rome, 1999), 3. Romano observes that the schools would teach the mathematics their patrons desired, which was often not astronomy.

field. His *Epitome arithmeticae practicae* (1583) and *Geometria practica* (1604), promising knowledge to facilitate most mundane activities of any society, were part of a broad discourse that explored the practical utility of abstract mathematics. Other authors in that discourse included Orontius Finé, who published his *Arithmetica practica* in 1544 and his *De re et praxi geometrica* in 1556, and Albert Dürer who wrote on the construction and measurement of various shapes. Clavius was likely familiar with those works.[4] English authors like Robert Recorde, who published *The Ground of Arts* (1543) on arithmetic and *Pathway to Knowledge* (1551) on geometry, and Leonard and Thomas Digges, who published *A Geometrical Practise Named Pantometria* (1571), worked in the same vein. Even texts on pure mathematics could be part of the discourse on the applications of the discipline. When Federico Commandino, the Urbino humanist mathematician, translated his commentary on *The Elements* into Italian, his son-in-law, Valerio Spacciuoli, added a letter of dedication explaining that the motivation for the translation was repeated requests from those who used mathematics but did not read Latin and wanted a complete and accurate Italian translation of *The Elements* for the geometry's applicability to everyday tasks.[5]

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[4] Clavius used Dürer's work as a source for some of his figures in the solid geometry books of *The Elements*. See Christopher Clavius, *Euclidis Posteriores libri sex a X ad XV. Accessit XVI de solidorum regularium comparatione* (Rome: Vicentium Accoltum, 1574), 209*v* – 219*v*. In his preface to his *Geometria practica*, Clavius named Orontius Finé as one of the many learned men who had written on mathematics. Christopher Clavius, *Geometria practica* (Rome: Aloisyius Zannettus, 1604), 1. "Quamobrem & multos, & eruditos viros habuit, qui partes illius omnes accurata, & diligenti scriptione persecuti sunt: Inter quos, ut Leonardus Pisanus, Frater Lucas Pacciolus, Nicolaus Tartalea, Orontius, Cardanus, aliique praecipuas obtinuerunt."

[5] Valerio Spacciuoli, "All'Illusstrissimo et eccelentissimo Signore Il Sig. Francesco Maria II. Feltrio della Rouuere Duca VI. d'Urbino," in Federico Commandino, *De gli Elementi d'Euclide* (Urbino: Domenico Frisolino, 1575). "Ma, perche tal lingua non è intesa da tutti quelli, che si servono delle mathematiche; essendo venuto all'orecchie del Commandino, che l'Italia desiderava (poiche ha quasi nel suo idioma libri di tutte le scienze) godere ancora le fatiche fatte da lui intorno a questo libro, non contentandosi affatto di quelle, c'ha fin hora hauute."

When the first English edition of *The Elements* was published in 1570, its translator, Sir Henry Billingsley, made clear in his letter to the reader that his purpose was to enable non-Latinate artisans to study mathematics as a means to develop "inventions of straunge and wonderfull thinges."[6] There were also numerous texts that built on a tradition of technical writing that had begun in the fifteenth century and focused on specific branches of mathematics and their potential, often military, applications.[7] Among such texts, are the Venetian Niccolo Tartaglia's *La Nova Scientia* (1537), a treatise on the use of mathematics in artillery, and the Spanish Martin Cortes de Albacar's *Arte de Navegar* (1551), a practical guide to using mathematics for navigation.[8]

However, among Latinate authors, the mundane applications of mathematics were often frowned upon for their attachment to profit. Indeed, Commandino included a lament in his preface to his commentary on Euclid's *Elements*, that his contemporaries' focus on material profits required him to list practical uses of mathematics to convince readers to study Euclid, even though he believed the nobility of the discipline should be sufficient motivation.[9] Despite Spacciuoli's insistence that

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[6] Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), ij.

[7] Pamela O. Long, *Openness, Secrecy, Authorship: Technical Arts and the Culture of Knowledge from Antiquity to the Rennaisance* (Baltimore: The Johns Hopkins University Press, 2001), 102-103. Long argues that the increase in technical texts in the fifteenth and sixteenth centuries was tied to a political culture in which legitimacy was intertwined with constructive arts, creating a demand for technical treatises on painting, sculpture, architecture, fortifications, artillery etc. One such author was Leon Battista Alberti (1404-1472). Among her earliest examples, Long names the physicians Conrad Kyser (1366-1405) and Giovanni Fontana (1395-1455), both of whom wrote texts on the instruments of warfare.

[8] Cortes's work was translated into English by Richard Eden in 1561.

[9] Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), *4*v*. "Sed quoniam plerique his praesertim temporibus sola utilitate ad optimarum artium studia excitantur, liberalseque, colunt disciplinas, videamus obsecro, an mathematicae nullius sint commodi ad

the translation was for those who wished to apply mathematics to mundane tasks, that disparaging comment remained intact in the Italian translation. One of Commandino's students, Guidobaldo del Monte (1545-1607), took a different approach to distancing himself from profit-seekers. When his 1577 treatise on machines, *Mechanicorum liber*, was translated into Italian under the supervision of the Venetian military engineer Count Giulio Savorgnano, who hoped that it could be put to use in military applications, del Monte requested that his name be left out of the project.[10] Clavius was not immune to such concerns, but, as I will show in this chapter, instead of distancing himself from the applications of mathematics, he designed his arguments for practical mathematics to show that the discipline belonged in the hands of princes and those close to them.

In this chapter I will examine the place of practical mathematics in Clavius's work to show that his vision of practical mathematics as a tool for princes and those close to them guided the authors of the Jesuit curriculum in their considerations of the place of mathematics in the Order's schools. I will begin by studying how Clavius included practical mathematics in his own curricular project as it is represented by his corpus of textbooks and their prefaces. I will then contrast the vision of the utility of mathematics Clavius presented in the prefatory material to his commentary on

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iuuandos humanae vitae usus, uti caeca quorundam turpissimi lucri cupiditas falsa iam praedicatione divulgavit, ita ut qui hanc amplectuntur facultatem ab imperitis, vel a io studio occupatis hominibus palam derideantur, tamquam in re inutili, atque uana oleum, & operem perdant. Agamus igitur pingui, quod aiunt, Minerva, quando nobis negocium est cum iis, qui sola quaestus ratione persuaderi possunt, & inuramus hanc notam ingenuae, ac nobili disciplinae, ut lucrum & divitias pollicendo huiusmodi hominum sibi studia, & gratiam comparet."

[10] M. Henninger-Voss, "Working Machines and Noble Mechanics: Guidobaldo del Monte and the Translation of Knowledge," *Isis*, 91 No. 2 (June 2000), 233-259. Del Monte was content to allow his work to be translated as long as he was not seen to be a part of the process.

Euclid's *Elements* with that presented by Billingsley and his collaborator John Dee in their respective letter to the reader and preface to the English commentary on the same text. In this discussion it becomes clear that, unlike his English contemporaries, Clavius did not see mathematics as a tool for craftsmen but instead, in keeping with the Jesuits' desire to become educators of the elite, treated it as a tool for princes. Finally, I will trace the development of the mathematics portion of the *Ratio Studiorum* from the first draft of 1586 to the final draft of 1599 to show how Clavius's vision of practical mathematics guided the authors of the curriculum and defined the place of mathematics in the *Ratio Studiorum*.

Practical Mathematics in Clavius's Curriculum

For Clavius there were three reasons to study mathematics: nobility, utility, and pleasure. While the last of these reasons was a matter of personal taste and unlikely to be convincing to students who did not wish to study mathematics in the first place, the other two reasons could be used to justify the study of the discipline to even the most reluctant of students. As discussed in the previous chapter, the arguments for the nobility of mathematics were part of an extensive sixteenth-century debate over the status of mathematics based on the certainty of the discipline and its ability to contribute to philosophy and inform theology. Turning to more mundane concerns, Clavius's arguments for the utility of mathematics centered on illustrating that the subject was essential to the welfare of any society and, consequently, was essential to those in leadership positions, an argument that appealed to the Jesuits planning the curriculum who were searching for ways to best enable the Jesuit schools to train good Catholic rulers for European societies.[11]

The simplest argument for the utility of mathematics to society was just a list of the practical applications of the discipline. Thus, Clavius's suggested curricula (written in 1581), in which he included an extensive collection of practical topics, and the practical subjects covered by his textbooks, are the first testament to the value he placed on the applications of his discipline.[12] Even in the least rigorous of the three curricula he suggested, Clavius required practical arithmetic, ecclesiastical computation, horology, perspective, geography, the astrolabe, and the measurement of plane and solid figures. In the more rigorous programs of study, he added gnomonics (the study of sundials) and mechanics. To ensure that students had the means to develop their mathematical skills, he planned to write a textbook on each of these topics. Although Clavius did not succeed in this ambitious goal, the textbooks he did write presented the foundation for a course of study in a variety of branches of practical mathematics and a well-developed program in the use of astronomy for calendrical calculations, Clavius's own primary activity outside of teaching. In order of publication, the textbooks are *Gnomonices* (1581), *Epitome arithmeticae practicae* (1583), *Fabrica et usus instrumenti horologiorum* (1586), *Astrolabium* (1593), *Horologiorum nova descriptio* (1599), the *Compendium brevissimum describendorum horologiorum horizontalium ac declinatium* (1603), and the *Geometria practica*

[11] See chapter 1 for a discussion of the goals of the Jesuit schools.
[12] For a more in depth discussion of Clavius's three curricula, see chapter 1.

(1604).[13] The emphasis on the uses of astronomy for timekeeping, a product of Clavius's own efforts to fill the Church's longstanding need for a more accurate calendar, foregrounded the calendrical component of ecclesiastical computation and, thus, made mathematics a tool to be used by theologians (including the pope) for the benefit of all Christendom.

However, while the prefaces to all of the practical texts could have been used to justify the study of mathematics based on its utility, Clavius only took advantage of that opportunity in the prefaces to the two practical textbooks removed from the specialized study of astronomy, the *Epitome arithmeticae practicae* and the *Geometria practica*. He seems to have thought that while the benefits of astronomy (especially in preparing an accurate calendar) may have applied to everyone, only a specialist, who did not need further justification to study mathematics, would have had the ability to make and understand the measurements of celestial bodies.[14] Indeed, in the prefaces

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[13] Christopher Clavius, *Geometria practica*, 1-2. Since Clavius thought that he could not create a single textbook that perfectly united the many works in the field of practical geometry, he focused on the measurement of lines, surfaces, and solids as the foundation for all applications of the discipline in his *Geometria practica.* This approach is clearly meant to be compared to that of Ioannes Antonio Magino, who Clavius claimed was the greatest mathematician to contribute to the field even though "he only taught the measuring out of lines." (pp. 1-2) "Primas tamen adiudicarim Io. Antonio Magino praestanti Mathematico; qui tametsi tantum linearum dimensiones docuit, ea tamen copia, doctriana, perspicacitate cuncta tradidit, ut locum non modo iis, qui ante scripserunt, sed spem posteris aequalis gloriae, ne dum maioris, ademisse videatur."

[14] Clavius never used a term like "specialist." Even when he described other mathematical scholars he usually used a phrase like "eruditos viros," or "learned men." (See the above note.) This choice emphasizes that Clavius saw mathematics as part of a complete education. He did use the word "Mathematicus" to describe Johannes Sacrobosco in the preface to his 1570 commentary on the English scholar's text. However, that was preceded by the moniker "Philosophus," suggesting that mathematics was an addition to the general study of philosophy, or a specialization therein. (Christopher Clavius, *In Sphaeram Ioannis de Sacro Bosco Commentarius* (Rome: Victorium Helianum, 1570), 2. "Ideo Ioannes de Sacrobosco natione Anglus egregius sua tempestate Philosophus, ac Mathematicius, qui floruit circa annum Domini 1232.") Thus, the modern term "specialist" seems to fit Clavius's views because an educated scholar able to read his texts could aspire to both of the two descriptions Clavius gave to Sacrobosco – philosopher and mathematician – such that the latter moniker expressed an extension of the former in a particular field of study.

to the texts on the applications of astronomy, he simply gave the necessary mathematical background for the topic at hand and brief outlines of the sections in the book. In contrast, the mundane uses of mathematics found in the *Epitome arithmeticae practicae* and the *Geometria practica,* such as accounting and measuring distances, would have been employable by students who did not specialize in mathematical study, let alone in astronomy.[15] Thus, those texts needed to be accessible to all students, including those inclined to ask the perennial question of "when will I need this?" Indeed, in his preface to the *Epitome arithmetica*e *practicae,* Clavius expressed his hope that the text would be read by all Jesuit students. He claimed that important men requested that his book be communicated not only with those who eagerly sought it out but also with those who attended Jesuit schools, because it was such a useful text.[16] Because Clavius did justify the study of practical mathematics in his *Epitome arithmeticae practicae* and *Geometria practica*, those two texts seem to have been intended for most, if not all, Jesuit students. Thus, a brief analysis of their prefaces will serve to illustrate Clavius's outlook on the importance of the practical branches of mathematics.

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[15] Paul Klein, who lived a little over a century after Clavius, comes to mind. He was the Jesuit who drew the first map of the Palau islands, now known as the Carolines. For a discussion of that map, see Ulrike Strasser, "Die Kartierung der Palaosinslen: Geographisce Imagnination und Wissenstransfer zwischen europäischen Jesuiten unt mikronesischen Insulanern um 1700," *Geschichte un Gesellschaft,* 36, 2010, pp. 197-230.

[16] Clavius, *Epitome arithmeticae practicae* (Rome: Dominici Basae, 1583)*,* 4. "Is libellus cum imprudenti mihi excidisset, & in manus hominum venisset, summis precibus contenderunt a me viri graves, ut cum plurimis communicarem quod fore dicerent, ut is utilissimus accideret cum caeteris studiosis, tum vero iis, qui nostras scholas frequentant: quorum utilitati nolle consultum, non esse eius, qui se suaque omnia Dei gloriae, omniumque commodis consecrasset." Clavius did not name a specific patron. In fact, the *Epitome arithmeticae practicae* lacks a dedication letter entirely.

In the first sentence of his preface to the *Epitome arithmeticae practicae,* Clavius expressed the value of practical arithmetic as the glue that holds society together. He claimed that "without arithmetic, at least as I think, no science, as Plato does say, nor society of man can exist."[17] He then began a discussion of the social utility of arithmetic, pointing out the obvious need for arithmetic in accounting for any type of business. If the reader should have stubbornly persisted in saying that business could be done without arithmetic, Clavius reminded him that "it is equally shameful and destructive to defraud and be defrauded" in the course of business transactions. Thus, even an honest businessman who had not learned arithmetic, as a likely victim of fraud, was a social liability, while his knowledgeable counterpart could strengthen society by using the tools arithmetic provided to run his business optimally.[18] It was not just business that benefitted from arithmetic. According to Clavius, other, unnamed, disciplines may have been less obviously dependent on the field but nonetheless would collapse if arithmetic were doubted because small errors in accounting could have devastating effects on any assumed results. Even the "astrologer and geometer" needed numbers to have their theorems gain acceptance by the common man.[19] Clavius went so far as to argue that practical arithmetic separated

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[17] Ibid., 3. "Sed etiam, quod sine Arithmetica, ut ego quidem existimo, nulla Scientia, ut Plato audit dicere, neque ipsa hominum societatis posit consistere."

[18] Ibid., 3. "Plurima enim in mutuis commerciis, conventisque, quibus fere haec hominum coniunctio continetur, tempora incidunt, ut rationes accepti, … quibus in rebus circumvenire, & circumveniri, aeque turpe, & perniciosum est."

[19] Ibid., 3. Clavius did not specify what those other fields were, but he did note that even geometry and astronomy could not explain everything because only arithmetic could supply one with a complete understanding of numbers, without which, according to Clavius, sound judgment in many areas was not possible. "Iam vero caeterae disciplinae sic Arithmetica nituntur, ut haec non videatur concidere posse, quin illae casu eodem labefactatae corruant. Neque enim aut Astrologus, aut Geometra theoremata in vulgus probabit sua, ut non solum veritatem, sed etiam voluptatem habeant cum utilitate coniunctam,

civilized men from barbarians. Paraphrasing Plato, he claimed that "those who remove arithmetic from their way of life, to such an extent remove good sense and all of civilization from the world," not least because business would become corrupt without arithmetic.[20]

While Clavius emphasized the value of arithmetic to business, he offered a broad spectrum of uses for practical geometry, claiming that it was necessary to almost every activity of a society. However, unlike his argument for the necessity of arithmetic, Clavius's arguments for the value of geometry avoided sweeping claims about the social role of the discipline. Instead, Clavius sought only to show that the study of practical geometry was superior to the training without the abstract foundation of geometry craftsmen received in various tasks. And the tasks to which geometry could be applied were myriad. Indeed, Clavius promised that the certain methods he laid out for the measurement of distances, heights, and depths would be useful to "the foundation of buildings, cultivation of fields, treatment of arms, observation of the stars, and other arts."[21] However, instead of whetting his readers' appetites with detailed explanations of the uses of practical geometry, Clavius allowed the breadth of his list to speak for itself, keeping his focus on the way in which geometry improved on the knowledge of craftsmen. According to Clavius, the

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qui universam numerorum naturam animo penitus comprehensam non habuerit: quod si tantillum in rationibus putandis lapsus fuerit, iam caeterarum rerum ingentem ruinam videas."

[20] Ibid., 3. "Sed tamen vere dixit Plato, prudentiam, atque adeo humanitatem omnem e mundo eos tollere, qui Arithmeticam e vita tollant."

[21] Clavius, *Geometria Practica*., 1. "Etenim dum certa ratio traditur, qua_ camporum longitudines, altitudines monitum, vallium depressiones, locorum omnium inaequalitates inter se & interulla deprehendere metiendo debeamus: cuilibet liquet, ut arbitror, quantum commodi, utilitatisque substructioni aedificiorum, culti agrorum, armorum tractationi, contemplationi siderum, aliisque artibus, & disciplinis ex horum cognitione manare possit."

superiority he sought came from the certainty of geometry, which allowed that “any profit out of mathematics may be able to be secured to the conveniences of human life not by empty showing off but so that it is certain as the subject itself.”[22] Unlike a craftsman’s show of his skill through the production of some desired object, the geometer’s methods could be easily reproduced by any student of mathematics when the need for a similar project arose. However, lest his reader come to the conclusion that practical geometry was just a tool for craftsmen, Clavius argued that while craftsmen used mathematics, it was learned mathematicians, employed by kings and princes, who explained how the discipline could be used in “exact and careful writing” that made the discipline of practical geometry more than showing off a trade.[23] Those authors showed that the world could be understood geometrically, and, in theory, they could perfect any of the tasks done by craftsmen based on the rules of their discipline. Since building, agriculture, astrology, and warfare all involved the use of practical geometry, Clavius implicitly argued that knowledge of the discipline was essential to anyone who wished to govern, as each of those fields concerned the running of a sixteenth-century polity.

While the practical studies of the various branches of mixed mathematics were obvious sources for arguments for the utility of mathematics, even less mundane

[22] Ibid., 1. “...in hoc quicquid est laboris veniebam alacer, ut qui fructus e Mathematicis percipi possint ad humanae vitae commoda, non inani venditatione, sed re ipsa constaret.”
[23] Ibid., 1. “Haec enim una Mathematicarum rerum scientiae pars, sicut ab artificibus ob sui necessitatem auide semper est arrepta: ita ob insignes utilitates, quas in re tota militari suppeditat, in maximorum Principum, Regumque aulis omni tempestate versata est. Quamobrem & multos, & eruditos viros habuit, qui partes illius omnes accurata, & diligenti scriptione persecuti sunt.” Clavius lists Leonardo of Pisa, Fr. Luca Paccioli, Niccolo Tartaglia, Orontius Fine, and Girolamo Cardano.

branches of mathematics could be deemed useful. Indeed, in his commentaries on Sacrobosco's *Sphere* and Euclid's *Elements,* Clavius included arguments for the utility of all domains of mathematics, especially to the educated elite who might serve as advisers to princes (including Jesuits). For astronomy, which Clavius took to be the highest branch of mathematics, the applications he listed focused on the relationship between the study of the heavens and other areas of scholarship, rather than naming specific tasks to which astronomy could be applied. While, he began with theology, metaphysics, and natural philosophy, he also included the practical studies of medicine (for which he cited Galen's use of astrology (primarily lunar cycles) in timing the administration of remedies), ecclesiastical computations (for which he supplied the annual calendrical calculations for feast days, especially Easter, as evidence), nautical arts (for which he insisted that the dependence on astronomy was self-evident), and poetry (for which he noted that the beauty of the heavens and the perfection of celestial motion provided excellent subject matter). However, the majority of his discussion of the utility of astronomy drew the discipline outside the academy and focused on public affairs, or as he put it, "the administration of the public affairs, as in agriculture, warfare, and other such fields."[24] For these topics Clavius relied on historical anecdotes to give evidence for the value of mathematics. The heroes of these stories were rarely mathematicians, but instead were famed leaders, such as Sulpitus, Pericles, Dionysus the Areopagite, Hadrian, and Julius Caesar, whose uses of

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[24] Christopher Clavius, *Sphaeram*, 11. "Omitto, quod haec scientia summe est necessaria ad reipub. administrationem, ut ad agriculturam, ad bella gerenda, & alia huiusmodi." While these sciences were omitted from his original list of uses of mathematics, his discussion of them covers the final three pages of his preface.

mathematics were alleged to have saved their civilizations, their armies, themselves, or even their souls.[25] He even included more recent narratives, such as that of an unnamed Spanish duke who saved his troops from starvation by timing his demand that the native Jamaicans provide him with supplies to coincide with a lunar eclipse.[26]

Likewise, in his commentary on *The Elements*, Clavius also introduced his arguments for mathematics' utility by establishing its relationship to other areas of study, including theology and philosophy.[27] However, when he shifted from mathematics in general to the introduction of geometry, and specifically Euclid's geometry, he focused his discussion on the discipline's foundational role for various branches of mixed mathematics. He opened this portion of his discussion with an analogy between Euclid's *Elements* and the alphabet. Just as the first step to learning how to read is learning the alphabet, if one is to learn mathematics, one must start with *The Elements,* which he called an "abundant fountain" from which myriad uses flow.[28] Those uses included measuring any desired dimension of fields, mountains, or islands (i.e. geography), making instruments for the observation of the stars and measuring

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[25] Ibid., 12-14. Clavius gave the stories in varying degrees of completeness. In several cases, he simply alluded to anecdotes with claims that these famous leaders acknowledged the necessity of astronomy to their polities.

[26] Ibid., 12. According to the narrative, the duke promised to bring destruction to the natives if they did not bring the Spaniards supplies. They treated the lunar eclipse as representative of the duke's power and an omen of the promised destruction. Frightened by such a sign, they immediately welcomed the Spaniards and gave them all the supplies they needed.

[27] For more on these arguments, see chapter 1.

[28] Christopher Clavius, *Euclidis Elementorum*, b4*v*. "Quamobrem sicut is, qui legere vult, elementa literarum discit prius, & illis assidue repetitis utitur in vocibus omnibus exprimendis, sic qui alias disciplinas Mathematicas desiderat sibi reddere familiares, elementa haec Geometrica plene ac perfecte calleat prius, necesse est. Ex his etenim elementis, veluti fonte uberrimo omnis latitudinum, longitundinum, altitudinum, profunditatum, omnis agrorum, montium, insularum dimensio, atque divisio; omnis in caelo per instrumentae syderum observatio, omnis horologiorum sciotericorum composito, omnis machinarum vis, & ponderum ratio, omnis apparentiarum variarum, qualis cernitur in speculis, in picturis, in aquis, & in aere varie illumnato, diversitas manat."

celestial bodies and their motions (i.e. astronomy), measuring the passage of time (i.e. horology), and building machines of any kind (i.e. mechanics). As he emphasized in the *Geometria practica,* Clavius argued here that geometry was useful to kings and princes, not just to craftsmen. Once again, he turned to history, offering Archimedes as an exemplar of a mathematician whose skills made him uniquely valuable to his king, as proof of the discipline's necessity to political leaders. According to the legends cited by Clavius, because of his geometrical skill the Syracusan mathematician had been indispensable in small matters - devising a means to measure the amount of gold in the king's crown - and much larger matters - single-handedly holding a Roman invasion at bay with his machines.[29] Indeed, esteemed by all (even the Romans) for his knowledge of mathematics, Archimedes was the ideal practical mathematician for Jesuit students to emulate, and Clavius's curriculum, with its wealth of practical topics was designed to make such an emulation possible.

Approaches to Practical Mathematics: Clavius's Elements vs. the Billingsley-Dee Edition

Clavius's presentation of Archimedes as an ideal model emphasized the value of mathematics, including its practical branches, to the uppermost echelons of society. However, in the sixteenth century, the value of a rigorous mathematics education for artisans was also widely discussed. In fact, although craftsmen could learn their trades on the job without mathematical literacy, mathematics was increasingly called upon to

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[29] Ibid., b5*r*.

inform arts as disparate as architecture and gunnery. Vernacular treatises on mathematics, such as Niccolo Tartaglia's 1537 *La nova scientia* on gunnery, sought to place mathematics in the hands of the craftsmen who could apply it. Furthermore, as Deborah Harkness has shown, at least in London, sixteenth-century mathematical educators worked to convince the merchant and artisan class that an understanding of the fundamental theories of geometry and arithmetic could provide valuable problem-solving skills necessary for innovation.[30] Perhaps the most famous example of an attempt to provide non-Latinate craftsmen with a mathematical education is the first English translation of Euclid's *Elements* published in 1570 by Henry Billingsley, an English haberdasher, with a preface by John Dee, the famed philosopher and mathematician of the Tudor court. While it is tempting to see Clavius's inclusion of practical mathematics in his curriculum as part of a project to mathematize crafts, a comparison of Billingsley's and Dee's presentations of the utility of mathematics in the prefatory material to their edition of Euclid to Clavius's illustrates that there were two distinct approaches to the value of practical mathematics: one aimed to improve the mathematical literacy among craftsmen in hopes of future innovations that would improve the commonwealth, and the other sought to bring mathematics into courts as a tool for various facets of public administration.

The Billingsley-Dee edition of *The Elements* opened with a letter from Billingsley to his readers, in which he, the translator of the Greek text, promised that

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[30] Deborah Harkness, *The Jewel House: Elizabethan London and the Scientific Revolution* (New Haven: Yale University Press, 2007). 116-118. Harkness also notes that the arguments made in London for the utility of mathematics were part of a pattern of similar arguments across Europe. As evidence, she cites a number of technical authors from the continent, including Niccolo Tartaglia and Petrus Ramus.

the study of mathematics, beginning with Euclid, would benefit the commonwealth of England as well as "beautifie the mind" of its students.[31] And while the ability of mathematics to improve the mind and soul of man was Billingsley's opening gambit, the emphasis of his brief letter was on the utility of mathematics, through which he hoped to convince artisans to read his and other mathematical texts. Billingsley claimed that he had seen "many good wittes both of gentlemen and of others of all degrees" attempting to study the mathematical arts and failing in their endeavors because there was no English version of *The Elements* with which to begin their study. He feared that without such men being able to study mathematics, the English were failing to keep up with their continental counterparts amongst whom "flourishe[d] so many cunning and skilfull men, in the inventions of straunge and wonderfull thinges."[32] But he hoped that his translation of *The Elements* would spur the English into a deeper study of mathematics by providing a starting point for all English mathematics students and by encouraging other authors to translate Latin and Greek mathematical texts from the Continent.[33] He would consider his "paines and travaile" worthwhile if mathematics became a widely used tool to advance the ingenuity of

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[31] Billingsley, *The Elements of Geometrie,* ii*r*.
[32] Ibid., ii*r*.
[33] Ibid., ii*r*. Billingsley's translation of *The Elements* from continental sources does seem to have served as an example of a method of bringing Greek and Latin works from the Continent into the hands of English readers. In his *Copernican Question,* Robert Westman notes that in the late sixteenth century English textbooks were often cobbled together from continental sources. He gives Thomas Blundville's 1594 *Exercises* and 1602 *Theoriques of the Seven Planets* as examples of such "collected" works. Robert Westman, *The Copernican Question: Prognostication, Skepticism, and the Celestial Order* (Berkeley: University of California Press, 2011), 434.

English "men of all degrees," just as he claimed it had done for continental Europeans.[34]

Billingsley's letter served as a brief plea to potential readers, but it was far too short to offer fully developed arguments for the value of mathematics to a non-Latinate audience. He abdicated the task of developing such arguments to a well-known mathematician, John Dee, whom he (or the printer, John Daye) requested write the preface to lend credibility to a mathematical text produced by a merchant.[35] Dee, eager to justify the study of mathematics at least in part to combat his reputation as a "Caller and Coniurer of wicked and damned Spirites," accepted the request and wrote the *Mathematicall Preface,* in which he attempted to make his discipline as intriguing as possible to a broad audience.[36] Indeed, he opened his preface by raising the very question that men who had learned their trades on the job were most likely to ask about mathematics: how can one use geometry? Instead of asking and answering that

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[34] Billingsley, *The Elements of Geometrie,* ii.

[35] During his lifetime, Dee was well-known by mathematicians in England and on the Continent. He had studied at Cambridge in the 1540s. In the 1550s and 1560s he made several trips to the Continent, including a visit to Urbino in 1563 when he met with Federico Commandino, whose own commentary on *The Elements* was briefly discussed in chapter 1. While in Paris, Dee was touted as the only Englishman who knew something of mathematics. Allen G. Debus, *John Dee: The Mathematicall Preface to the Element of Geometrie of Euclid of Megara (1570),* (New York: Science History Publications, 1975), 2-4; John Heilbron, "Introductory Essay" in *John Dee on Astronomy: Propaedeumata Aphorisitica (1558 & 1568)* ed. Wayne Shumaker (Berkeley: University of California Press, 1978), 1-49. Regarding Dee's role in the translation of Euclid, it should be noted that Dee assisted with the commentary by offering his own additions to and comments on the ancient text. These additions were most frequent in the later books on solid geometry. For the ambiguity regarding who asked Dee to write the preface, see Nicholas Clulee, *John Dee's Natural Philosophy: Between Science and Religion*. (London: Routledge, 1988), 146.

[36] John Dee, "Mathematicall Preface" in Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), Aij. Harkness notes that Dee wanted to make mathematics "alluring," which likely did not help him fight his reputation as a conjurer, even though he hoped to show that, as Harkness put it, "Mathematics could also help to explain phenomena that might otherwise be dismissed as strange and outside the natural order." (Harkness, *The Jewel House,* 112).

question directly, Dee brought it up through a discussion of the differences between Plato's and Aristotle's approaches to teaching mathematics. Plato, while "divine" and a "great Master," drove students away by never expressing the purpose of his teaching.[37] In contrast, Aristotle advised his readers of the topic and the purpose of each of his lessons before he began, something Dee believed was necessary to his own task of "bringyng into common handling, the *Artes Mathematicall*."[38] Thus, in order to satisfy the desires of Billingsley's intended merchant readers, Dee set out to follow the example of Aristotle to the best of his ability by explaining the uses of mathematics.[39] To accomplish this goal, he devoted his preface to presenting his "mighty, most pleasunt, and frutefull *Mathematicall Tree*" in both its "chief armes

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[37] Dee, "Mathematicall Preface," first page. Dee was heavily influenced by Neoplatonism, especially the idea that nature was wholly defined by numerical harmonies and was thus written in mathematics. (See Heilbron, "Introductory Essay" 4-5 and also I.R.F Calder, "John Dee Studied as an English Neoplatonist" (PhD diss., London University 1952), 7-14). He was not being facetious in his praise of Plato as "divine" and a "great Master." However, he recognized the utilitarian interests of his audience. He feared that they would be likely to follow Plato's disappointed listeners to more productive tasks if he did not first convince them of a worthwhile end to the study of mathematics. Dee's rather comical discussion of Plato's failure to entice Athenians to the study of mathematics allowed him to appeal to the intended audience of Billingsley's text, even as he justified the use of Platonic arguments for the status of mathematics in the preface to a text intended to make mathematics accessible to those who could best take advantage of its practical implications.

[38] Dee, "Mathematicall Preface," first page.

[39] Ibid., first and second pages. Dee also used his preface to attempt to attract his own audience of university-educated readers. His discussion of his intended readers is as follows. "Nor (Imitatyng Aristotle) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall,* I am able, either playnly to prescribe the materiall boundes: or precisely to express the chief purposes, and most wonderfull applications therof. And though I am sure, that such as did shrinke from *Plato* and his schole, after they had perceived his finall conclusion would in these thinges have ben his most diligente hearers (so infinitely mought their desires, in fine and at length, by our *Artes Mathematicall* be satisfied) yet, by this my Praeface & forewarnyng, Aswell all such, may (to their great behofe) the soner, hither be allured: as also the *Pythagoricall, and Platonicall* perfect scholer, and the constant profound Philosopher, with more ease and spede, may (like the Bee,) gather hereby, both wax and hony."

[geometry and arithmetic] and second (grifted) branches [various branches of mixed mathematics which had applications to the physical world]."[40]

Dee's "Mathematicall Tree" directly illustrated the uses of mathematics and its potential value to craftsmen, but he still felt the need to justify explicitly the publication of an English version of Euclid's *Elements.* He took on that task at the conclusion of his preface where he presented a clear and concise argument for the utility of mathematics to craftsmen and the Commonwealth. Despite protesting that it was unnecessary to explain "why, in our vulgare Speche, this part of the Principall Science of *Geometrie*, called *Euclides Geometricall Elementes*, is published, to your handlyng: being unlatined people, and not Universitie Scholers," Dee scolded any university-trained scholars who would deny the English-reading merchants and artisans access to mathematical knowledge. He said that such a man was without "charitie toward his brother" and "care and zeal for the bettering of the Common state of this Realme."[41] He argued that "common artificers" already used mathematics in their work and that such men "with their owne Skill and experience, already had, will be hable (by these good helpes and informations) to finde out, and devise, new workes, straunge Engines, and Instrumentes: for sundry purposes in the Common

[40] Ibid., second page.
[41] Ibid., Aiiii*r-v*. Debus, *John Dee,* 12. As Debus helpfully notes, Dee's presumed university-based adversaries feared a loss of status if universities were no longer the source of all theoretical knowledge. Harkness adds to that the argument that some university-trained scholars believed that mathematics was the language of God and was not suited to the untrained intellects of craftsmen, who would be too easily enticed by the presumed ability of mathematics to access dark magic. She offers Francis Bacon as an example of a scholar who warned against too much study of mathematics. (Harkness, *Jewel House,* 100).

Wealth[.] Or for private pleasure[.] And for the better maintaining of their own estate[.]"[42]

Like Billingsley and Dee, Clavius opened his text with a justification of the study of mathematics in general. But, unlike Billingsley's plea for an expanded English study of mathematics and Dee's appeal to non-Latinate readers, Clavius's arguments for the study of mathematics focused on making mathematics attractive to the nobility, in-keeping with the Jesuits' general desire to gain influence with ruling classes wherever they went. His arguments appeared in his dedication letter addressed to Emmanuel Philiberto, Duke of Savoy, a supporter of the mathematical arts.[43] As one would expect in a letter addressed to a nobleman, the justification offered for the study of mathematics was designed to show mathematics to be worthy of a man of that status, rather than showing it to be a tool. Indeed, Clavius made typical appeals to the value of mathematics based on its antiquity and its certainty, arguing, as discussed in

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[42] Dee, *Mathematicall Preface,* A.iiii*r.*

[43] In many ways, Clavius's dedication letter appears to be a *pro forma* appeal to a potential patron. In it he thanks the Duke for his support on behalf of himself and the Society of Jesus, which suggests that he was seeking to secure the existing patronage relationship between the Duke and the Jesuits. (For a brief description of that relationship see Paul F. Grendler, *Schooling in Renaissance Italy: Literacy and Learning 1300-1600,* (Baltimore: The Johns Hopkins University Press, 1989), 368.) For more on the culture of patronage and the role of dedication letters see Robert Westman, "Proof, Poetics, and Patronage: Copernicus's preface to *De revolutionibus*" in *Reappraisals of the Scientific Revolution* ed. David C. Lindberg and Robert S. Westman, 167-206; and Mario Biagioli, *Galileo, Courtier: The Practice of Science in the Culture of Absolutism,* (Chicago: The University of Chicago Press, 1993). It does not appear that Clavius himself had any kind of sustained relationship with the dukes of Savoy. The dedication letters to the 1574 and 1589 editions of his commentary on Euclid's *Elements* are the only letters to the royal family at Savoy. (In 1589 the duke and addressee of the letter was Charles Emmanuel, the son of Emmanual Philiberto). Evidence for Emmanuel Philiberto's support for mathematicians can be found in Giovanni Benedetti's dedication letter to his 1585 *Diversarum Speculationum Mathematicarum & Physicarum* in which the author, writing to Charles Emmanuel, recalled the late Emmanuel Philiberto's patronage of his mathematical work, including bringing him to Savoy from Parma to act as a court mathematician. (Giovanni Bendetti, *Diversarum Speculationum Mathematicarum & Physicarum* (Turin: Nicolai Bevilaquae, 1585), A2).

the previous chapter, that the excellence of mathematics was not to be lightly dismissed as it enabled one to understand the natural world by providing a means to create certain knowledge about worldly phenomena.[44] *The Elements*, as the foundation of all of mathematics, was valuable for providing entry into such a noble discipline. However, while the nobility of mathematics may have justified the study of the subject, it was not the reason Clavius published his commentary on Euclid. In this letter, Clavius claimed that he was driven to publish to allow the commentary that he had developed in his years of teaching "to emerge into the light and the hands of men" for "the public utility of these studies."[45] While he never specified what the public utility of mathematical studies was, by naming it as his reason for publishing, especially since he did so in a letter addressed to a prince, he created a promise that mathematics could somehow be used by the prince to improve public welfare in his domain.[46]

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[44] It should be noted that Dee, who could be grouped with the mathematical humanists discussed in the previous chapter, did not ignore these arguments. Before establishing his "Mathemticall Tree" he included a discussion of the nature of mathematics in which he argued that mathematics was intermediate between the supernatural and the natural giving it a "meruaylous newtralitie" and a "straunge participation betwene thinges supernaturall, imortall, intellectual, simple and indivisible: and thynges naturall, mortall, sensible, compounded and divisible," giving mathematics the ability to explain phenomena that appeared to be magical. While these arguments reflect Dee's own position as a university-educated mathematical humanist, they are not the means through which the *Mathematicall Preface* establishes arguments for the utility of mathematics and will not be discussed further in this chapter.

[45] Clavius, *Euclidis Elementorum*, a3*v*. "Quae cum ego multo annos partim publice docendo, partim privatim commentando, & cum alliis viris doctis communicando diligentius pertractassem, collegissemque (ut fere fit) in meum privatum usum nonnulla, quae ad eorum cognitionem facere viderentur; faciendum mihi necessario existimavi, praesertim auditorium, amicorumque meorum precibus fatigatus, praeterea Laurentii Castellani ciuis Romani liberalitate invitatus, qui omnes ad id necessarios sumpt benigne admodum suppeditavit, ad publicam studiosorum utilitatem, in lucem manusque hominum exire permitterem."

[46] Of course public welfare in the sixteenth century was not the same as public welfare in the twenty-first century. The public utility intended by Clavius could be described as applications of mathematics that could inspire pride in one's *patria.* These could include public works in architecture and art, improvements to agriculture through irrigation projects and even more accurate land surveys, and

Despite differing on whether mathematics could improve public welfare through the ingenuity of craftsmen or the knowledge of a prince, Billingsley, Dee, and Clavius all promised that mathematics could be used to improve the commonwealth but remained vague on the specifics of how that promise could be fulfilled. Those specifics emerge in Dee's and Clavius's discussions of the divisions of mathematics. As each author described the various branches he identified as mathematical arts, he necessarily explained how mathematics could be applied. Indeed, as mentioned above, in the *Mathematicall Preface,* Dee explicitly set out to create an outline of the discipline of mathematics that could explain its utility. While his arguments in his justification for the translation of Euclid's *Elements* into English had focused primarily on the potential mechanical applications of mathematics ("straunge Engines"), the "Mathematicall Tree" he established in his preface, for which he included a graphic representation (Figure 1), is a detailed description of thirty-four branches of mathematics and their myriad potential uses. Of the thirty-four branches described, Dee identified only two, namely, geometry, the study of the properties of magnitudes, and arithmetic, the study of the properties of numbers, as the "principall" branches of mathematics in which mathematical entities were studied independently of the physical world. However, even these branches, could be focused on the study of

improvements to military efforts, both in defensive fortifications and offensive war engines. *Patria* is perhaps best defined as "homeland." Its literal translation is "fatherland." In early modern Italy, *patria* usually indicated the locality, perhaps a city or a duchy, to which someone had sentimental connections and a sense of fealty. For a discussion of the connection between mathematics and *patria* in the sixteenth and seventeenth centuries see Alexander Marr, *Between Raphael and Galileo: Mutio Oddi and the Mathematical Culture of Late Renaissance Italy* (Chicgao: University of Chicago Press, 2011), 29-40.

Figure 1: John Dee's Groundplat of the Mathematical Tree

Source Huntington Library.

J. DEE.

Here haue you (according to my promiſſe) the Groundplat of my MATHEMATICALL Præface: annexed to *Euclide* (now firſt) publiſhed in our Engliſhe tounge. An. 1570. Febr. 3.

Sciences, and Artes Mathematicall, are, either

- *Principall, which are two, onely,*
  - Arithmetike.
    - *Simple, Which dealeth with Numbers onely: and demonſtrateth all their properties and appertenances: where, an Vnit, is Indiuiſible.*
    - *Mixt, Which with aide of Geometrie principall, demonſtrateth ſome Arithmeticall Concluſion, or Purpoſe.*
  - Geometrie.
    - *Simple, Which dealeth with Magnitudes, onely: and demonſtrateth all their properties, paſſions, and appertenances: whoſe Point, is Indiuiſible.*
    - *Mixt, Which with aide of Arithmetike principall, demonſtrateth ſome Geometricall purpoſe: as* EVCLIDES ELEMENTES.
  - *The vſe whereof, is either,*
    - *In thinges Supernaturall, æternall, & Diuine: By Application,* Aſcending.
    - *In thinges Mathematicall: without farther Application.*
    - *In thinges Naturall: both Subſtātiall, & Accidentall, Viſible, & Inuiſible. &c. By Application:* Deſcending.
  - *The like Vſes and Applications are, (though in a degree lower) in the* Artes Mathematicall Deriuatiue.
- *Deriuatiue frō the Principalls: of which, ſome haue*
  - *The names of the Principalls: as,*
    - Arithmetike, *vulgar: which conſidereth*
      - Arithmetike of moſt vſuall whole Numbers: And of Fractions to them appertaining.
      - Arithmetike of Proportions.
      - Arithmetike Circular.
      - Arithmetike of Radicall Nūbers: Simple, Compound, Mixt: And of their Fractions.
      - Arithmetike of Coſſike Nūbers: with their Fractions: And the great Arte of Algiebar.
    - Geometrie, *vulgar: which teacheth Meaſuring*
      - *At hand* —
        - All Lengthes. — Mecometrie.
        - All Plaines: As, Land, Borde, Glaſſe, &c. — Embadometrie.
        - All Solids: As, Timber, Stone, Veſſels, &c. — Stereometrie.
      - *With diſtāce from the thing Meaſured, as,*
        - *How farre, from the Meaſurer, any thing is: of him ſene, on Land or Water: called* Apomecometrie.
        - *How high or deepe, from the leuell of the Meaſurers ſtanding, any thing is: Seene of hym, on Land or Water: called* Hypſometrie.
        - *How broad, a thing is, which is in the Meaſurers vew: ſo it be ſituated on Land or Water: called* Platometrie.
        - *Of which are growen the Feates & Artes of*
          - Geodeſie: *more cunningly to Meaſure and Suruey Landes, Woods, Waters. &c.*
          - Geographie.
          - Chorographie.
          - Hydrographie.
          - Stratarithmetrie.
  - *Propre names as,*
    - Perſpectiue, — *Which demonſtrateth the maners and properties of all Radiations: Directe, Broken, and Reflected.*
    - Aſtronomie, — *Which demonſtrateth the Diſtances, Magnitudes, and all Naturall motions, Apparences, and Paſſions, proper to the Planets and fixed Starres: for any time, paſt, preſent, and to come: in reſpecte of a certaine Horizon, or without reſpecte of any Horizon.*
    - Muſike, — *Which demonſtrateth by reaſon, and teacheth by ſenſe, perfectly to iudge and order the diuerſitie of Soundes, hie or low.*
    - Coſmographie, — *Which, wholy and perfectly maketh deſcription of the Heauenly, and alſo Elementall part of the World: and of theſe partes, maketh homologall application, and mutuall collation neceſſary.*
    - Aſtrologie, — *Which reaſonably demonſtrateth the operations and effectes of the naturall beames of light, and ſecrete Influence of the Planets, and fixed Starres, in euery Element and Elementall body: at all times, in any Horizon aſſigned.*
    - Statike, — *Which demonſtrateth the cauſes of heauines and lightnes of all thinges: and of the motions and properties to heauines and lightnes belonging.*
    - Anthropographie, *Which deſcribeth the Nūber, Meaſure, Waight, Figure, Situation, and colour of euery diuers thing contained in the perfecte body of MAN: and geueth certaine knowledge of the Figure, Symmetrie, Waight, Characterization, & due Locall motion of any percell of the ſayd body aſſigned: and of numbers to the ſaid percell appertaining.*
    - Trochilike, — *Which demonſtrateth the properties of all Circular motions: Simple and Compound.*
    - Helicoſophie, — *Which demonſtrateth the deſigning of all Spirall lines: in Plaine, on Cylinder, Cone, Sphære, Conoïd, and Sphæroid: and their properties.*
    - Pneumatithmie, — *Which demonſtrateth by cloſe hollow Geometricall figures (Regular and Irregular) the ſtraunge properties (in motion or ſtay) of the Water, Ayre, Smoke, and Fire, in their Continuitie, and as they are ioyned to the Elementes next them.*
    - Menadrie, — *Which demonſtrateth, how, aboue Natures Vertue, and power ſimple: Vertue and force, may be multiplied: and ſo to directe, to lift, to pull to, and to put or caſt fro, any multiplied, or ſimple determined Vertue, Waight, or Force: naturally, not ſo, directible, or moueable.*
    - Hypogeiodie, — *Which demonſtrateth, how, vnder the Sphæricall Superficies of the Earth, at any depth, to any perpendicular line aſſigned (whoſe diſtance from the perpendicular of the entrance: and the Azimuth likewiſe, in reſpecte of the ſayd entrance, is knowen) certaine Way, may be preſcribed and gone, &c.*
    - Hydragogie, — *Which demonſtrateth the poſſible leading of Water by Natures law, and by artificiall helpe, from any head (being Spring, ſtanding, or running Water) to any other place aſſigned.*
    - Horometrie, — *Which demonſtrateth, how, at all times appointed, the preciſe, vſuall denomination of time, may be knowen, for any place aſſigned.*
    - Zographie, — *Which demonſtrateth and teacheth, how, the Interſection of all viſuall Pyramids, made by any plaine aſſigned (the Center, diſtance, and lightes being determined) may be, by lines, and proper colours repreſented.*
    - Architecture, — *Which is a Science garniſhed with many doctrines, and diuers Inſtructions: by whoſe iudgement, all workes by other workmen finiſhed, are iudged.*
    - Nauigation, — *Which demonſtrateth, how, by the Shorteſt good way, by the apteſt direction, and in the ſhorteſt time: a ſufficient Shippe, betwene any two places (in paſſage nauigable) aſſigned, may be conducted: and in all ſtormes and naturall diſturbances chauncing, how to vſe the beſt poſſible meanes, to recouer the place firſt aſſigned.*
    - Thaumaturgike, — *Which geueth certaine order to make ſtraunge workes, of the ſenſe to be perceiued: and of men greatly to be wondred at.*
    - Archemaſtrie, — *Which teacheth to bring to actuall experience ſenſible, all worthy concluſions, by all the Artes Mathematicall purpoſed: and by true Naturall philoſophie, concluded: And both addeth to them a farder Scope, in the termes of the ſame Artes: and alſo, by his proper Method, and in peculiar termes, procedeth, with helpe of the forſayd Artes, to the performance of complete Experiences: which, of no particular Arte, are hable (Formally) to be challenged.*

nted by *John Day.*
.1570. Feb.25.

natural entities, in which case they were counted among the remaining thirty-two branches, which Dee called "derivative" because they were developed from one or both of the principal branches.[47]

The thirty-two derivative branches of mathematics were broken into two groups, "vulgar" geometry and arithmetic, and those branches of mathematics that have their own names. As the descriptor "vulgar" suggests, the first two derivative branches Dee described, vulgar arithmetic and vulgar geometry, are the mundane and common applications of the principal branches of mathematics, including computation with various kinds of numbers and the measurements of objects of one, two, or three dimensions. Vulgar arithmetic has five branches, one for each of five kinds of numbers.[48] According to Dee, these can be applied to business, mixing substances such as metals, establishing the organization of troops for military engagements, and determining and enforcing just laws.[49] The eight branches of vulgar geometry cover the measurement of objects, either close at hand (three branches) or at a distance (five branches). The three branches applied to measuring objects close at hand – "mecometrie" (the measurement of lines), "embadometrie" (the measurement of planes) and "stereometrie" (the measurement of solids) could all be used for gauging

[47] Dee, *Mathematicall Preface,* Groundplat. Dee divided both arithmetic and geometry into "simple" and "mixt." Simple arithmetic and geometry are the studies of pure number and pure magnitude, respectively. Mixed arithmetic is the demonstration of principles about number with the aid of magnitudes. Mixed geometry is the opposite. Dee classified Euclid's *Elements* as mixed geometry because it demonstrates geometrical principles with the occasional aid of arithmetical principles. According to Dee, all four of these branches could be used to study supernatural, mathematical, and natural things. However, when applied to natural things, Dee felt that they were more properly considered derived branches than principal.
[48] Ibid., Groundplat. The five kinds of numbers are whole numbers, proportions, circular numbers, radical numbers, and cossic numbers (i.e. the study of algebra).
[49] Ibid., iii*r*-a.i*r*.

the size of objects, whether the perimeter, area, or volume of any figure, a skill used often by merchants.[50] The five branches dedicated to the study of objects at a distance – called by Dee "geodesie," "geographie," "chorographie," "hydrographie," and "stratarithme" – all played a role in mapping terrain and determining to what use an area of land could be put. Geodesy was the surveying of land. Its use of geometrical principles allowed its students to measure land at a distance by using geometrical figures to approximate the size of geographical features. Geography allowed the positions of various features, natural or manmade, to be determined relative to one another and in absolute terms on the globe. Chorography was the detailed mapping of small areas, such as cities, without regard to their global surroundings. Hydrography was the study of water features. It included mapping bodies of water, measuring tides, measuring the variation of a compass, and other tasks that could aid navigation. Finally, stratarithmetry is the study of how many men can be placed into any plane figure, a skill that could aid the tactician in determining how many men a military company could use in any given place and could aid in estimating the size of advancing enemy armies.[51]

While Dee's list of applications for vulgar arithmetic and geometry is extensive, the truly wide reach of mathematical utility did not emerge from Dee's discussion until he reached the nineteen branches of derivative mathematics with their own names. These branches are, using Dee's Elizabethan spellings, perspective,

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[50] Ibid., aiii*v*. For the importance of gauging see Michael Baxandall, *Painting and Experience in Fifteenth-Century Italy, Second Edition,* (Oxford: Oxford University Press, 1988), 86-93.
[51] Ibid., a.iii*v* – b.i*r*.

astronomie, musike, cosmographie, astrologie, statike, anthropographie, trochilike, helicosophie, pneumatithmie, menadrie, hypogeiodie, hydragogie, horometrie, zographie, architecture, navigation, thaumaturgicke, and archemastrie.[52] In his descriptions of each branch, Dee named various practical tasks to which the branches could be applied, allowing the list of the derivative branches to serve as a catalog of practical mathematical topics, some of which would have been of interest to philosophers who sought to use mathematics to make sense of the physical world, and some of which would have been of interest to craftsmen.[53] In the first category, Dee described perspective, or the study of radiations, including those that were refracted or reflected, as the study that allowed one to make sense of whatever was seen. It was necessary to any study of astronomy or astrology and to the practical art of catoptrics, the making of mirrors and lenses. Dee defined astronomy as the study of the motions of heavenly bodies. It could be classified as a practical art because it allowed the measurement of time in various forms (seasons, lunar cycles, days, years, zodiacal cycles, etc.). Astrology was the study of the influence of heavenly bodies on the terrestrial world. It could be applied to meteorology and the prognostication of natural events. It was essential to natural philosophy and the primary path through which man

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[52] Ibid., Groundplat. I have left Dee's spellings intact because some of his names are no longer in use, and it seemed odd to modernize some but not all when they were listed together. For the ease of my reader, as I treat each one individually, I modernize the spellings for all those that are readily recognized.

[53] For several of the branches Dee also included discussions, sometimes quite lengthy, of the utility of mathematics to further study, not just practical tasks. Dee's own interest in such intellectual utility is most evident in the first five branches he named, each of which he claimed was necessary to the study of other branches of mathematics and he argued allowed mathematics to study the supernatural as well as the natural. After the fifth discipline (statics) was discussed, Dee claimed that pressure from the printer forced him to keep the remainder of his explanations brief. Several of the remaining discussions simply name uses of the disciplines and give example of ancient practitioners.

could study the supernatural. Music was used to train the mind to judge sounds "by sense and reason" so that the "minde may be preferred, before the eare." Dee claimed that it could be applied to "marvailous" effect in curing both mental and physical disease. Cosmography, the description of the structure of the universe, included the study of the positions of heavenly bodies and could be applied to navigation, medicine and timekeeping.

Most of Dee's remaining branches may have been more appealing to artisans and craftsmen. Dee suggested that through "experimentes of the balance," statics, the study of weights and the motions that could be produced using weights, could yield profits arts like gunnery and shipbuilding.[54] Miners, architects (should they need to design secret passages), and surveyors attempting to determine to whom underground minerals belonged could use "hypogeiodie" was the study of mapping and tunneling underground. Anthropographie is the use of number to study man, which could be applied to art and sculpture, architecture, and anatomy. "Trochilike" was the study of circular motions. It was essential to the building of mills or any other type of wheel work, which Dee claimed was present in various ways in mining. "Helicosophie" taught how to produce and understand spiral lines on various surfaces, which was essential to the development of any machine that used a screw. Pneumatics studied the relationships of the four elements in hollow geometrical spaces. It was essential to the development of pumps and bellows and other hydraulic inventions. Dee did not

[54] Ibid. b.iiii*r* – c.iii*r*. Statics was a topic that held much intellectual appeal. Here, Dee showed how mathematical objects could be studied independently of physical uses, by offering mechanical solutions (based on statics) to the classical unsolvable problem of doubling the cube.

name Ctesibus and Hero of Alexandria here, but it seems he was thinking of their inventions, including hydraulic organs – which he did name. "Menadrie" was the study of the multiplication of forces through levers and cranes. It was essential to any machine based on a lever and was seen as especially useful to war machines because it was through the use of such machines that Archimedes supposedly held the Roman siege of Syracuse at bay. "Hydragogie" was the study of the motion of water. It was intended to allow the routing of water to any desired point through the use of pipes or other means of redirecting the natural flow of a spring or river. "Horometrie" was the measurement of time. Dee claimed that it allowed the precise measurement of time from any location, which Dee asserted was essential to "Man's affaires." "Zographie" was the study of drawing and painting, specifically the study of how to create illustrations that looked realistic.

The last four branches, architecture, navigation, "thaumaturgike," and "archemastrie," drew heavily on the previously named branches of mathematics as they made physical the theories described by those branches. Architecture, the study of buildings, was essential to creating public and private living spaces, fortifications, and shipbuilding. Navigation allowed the piloting of a ship from one point to another. "Thaumaturgike" was the use of mathematics to create mechanical wonders, including objects that could imitate the motions of living things.[55] The last branch named,

[55] Deborah Harkness claims that Dee's reputation as a conjurer began with the construction of a mechanical beetle during his years at Cambridge. (See Harkness, *Jewel House,* 103). Thaumaturgike was thus a means for Dee to argue that even things that appeared to be magical were simply constructions of mathematics that could be fully understood through reason. Indeed, Dee did devote part of his argument to precisely the point that such constructions were signs of ingenuity, not dark magic.

"archemastrie" was Dee's "Experimentall Science" through which the conclusions of all other mathematical arts were shown to be true in experience. It served as the link between mathematical theory and practice.[56] On the simplest level, archemastrie required that the feats claimed to be possible in other branches of mathematics, such as the ability of a machine to mimic lifelike motion (thaumaturgike) or a river to be rerouted (hydragogie), be accomplished, not just illustrated in theory. Thus, at the end of his preface, Dee placed mathematics firmly in the hands of craftsmen who could use the various branches outlined to achieve the experiential goal of archemastrie. It must be noted that Dee was not advocating experience as a direct precursor to modern experiment. While the experiential component of archemastrie could involve experience of the applications of mathematics, Dee's understanding seems to be in line with Peter Dear's notion of early modern experience as a general sense of how nature works, not a particular sense of how something happened in a specific time and place. Thus, rerouting a river could be experienced through the consideration of water flowing around boulder or through a pipe, but the archemaster would not need to design an experiment that controlled for variables to make general claims about fluid dynamics.[57] However, the field was not just the physical rendering of mathematical theory; Dee also believed that further knowledge could be gained through

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[56] Ibid., A.iii*v*. Dee references Nicholas Cusanus and Roger Bacon in this section. His experimental science should be understood as more aligned with their desires to observe phenomena than with a modern version of controlled experiments.
[57] See Peter Dear, *Discipline and Experience: The Mathematical Way in the Scientific Revolution*, (Chicago: The University of Chicago Press, 1995), 4.

archemastrie's experiential method, meaning craftsmen could produce new knowledge as they produced new inventions.[58]

Clavius's outline of the disciplines of mathematics was far shorter than Dee's, and his list of uses was much less dizzying. In order to focus the utility of mathematics on the potential interests of a prince, Clavius selected categories of potential applications such as art and warfare, rather than specific feats of engineering that would interest craftsmen, such as the construction of a mill or a ship. In his preface, Clavius included a section entitled "Divisions of the Mathematical Disciplines" in which he described two possible outlines of mathematics.[59] The first, which he ascribed to the Pythagoreans, was the traditional division of mathematics into the quadrivium: geometry, arithmetic, astronomy and music. These branches of mathematics were the ones Clavius used when he discussed the nobility of the discipline that made it worthy of the time of a prince because they establish mathematics as the study of abstract quantity, which Clavius used later in his preface to argue that mathematics could bridge physics and metaphysics. However, Clavius noted that these four branches of mathematics were not the whole discipline. They

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[58] For a thorough discussion of Dee's archemastrie see Nicholas H. Clulee, *John Dee's Natural Philosophy: Between Science and Religion* (London: Routledge Library Editions, 1988), 170-176. Clulee notes that archemastrie did include occult and magical elements. However, he also notes (p. 175) that the artisans for whom the *Mathematicall Preface* was intended would have been unlikely to see such elements.

[59] Clavius, *Euclidis Elementorum,* a7*v*-a8*v*. The section title is "Disciplinarum Mathematicarum divisio." Clavius's source for the divisions of mathematics was Barozzi's translation of Proclus's commentary on the first book of Euclid's *Elements*. While he only noted that source when he introduced the second division of mathematics, the entire content of his section, including the descriptions of the named branches of mathematics and the ancient sources to whom the two divisions are attributed, appears in Proclus's text. See Proclus, *A Commentary on the First Book of Euclid's Elements*. trans. Glenn R. Morrow. (Princeton: Princeton University Press, 1970), 29-35.

were instead the foundation for "all other mathematical sciences that treat quantity in any way whatsoever such as perspective, geography, and the like, [which] can be easily reduced to these four mathematical sciences."[60]

It was the second division of mathematics that Clavius claimed "elegantly and fully" showed the extent of the mathematical disciplines, including the suggestion of applications to public welfare.[61] In this division, which Clavius attributed to Geminus of Rhodes, mathematics was split into intelligible and sensible branches. The intelligible branches, arithmetic and geometry, treated topics devoid of materiality. The six branches of sensible mathematics - astrology, perspective, geodesy, canonics or music, calculation, and mechanics - were those which studied topics based in physical materiality.[62] For each of the intelligible branches of mathematics, Clavius offered a brief description which served to illustrate the potential uses of mathematics to a polity. Astrology was used to measure the motions of and distances between stars and planets. It also made possible the measurement of hours, and, as Clavius claimed Hippocrates taught, was essential to good medical practice.[63] Perspective was used to

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[60] Clavius, *Euclidis Elementorum*., a7*v*. "Ad has autem quatuor scientas Mathematicas, quarum Arithmetica, & Geometria purae, Musica vero, & Astronomia mixtae dicunt, omnes aliae quovis modo de quantitate agentes, qualis est perspectiva, Geographia, et cetera huiusmodi, vel facile, ut a capita, a quibus dependent, reduci possunt."

[61] Ibid., a7*v*. "Quam quidem divisionem quoniam eleganter, copiesque docet, ad quenam sese extendant Mathematicae disciplinae..."

[62] Ibid., a7*v*-a8*r*. "Prioris generis statuunt duas longe primas, praecipuasque; scientias, Arithmeticam & Geometriam: In posteriori vero genere consituunt sex, Astrologiam, Perspectivam, Geodaesiam, Canonicam, sive Musicam, Supputatricem, atque Mechanicam."

[63] In his 1570 preface to his commentary on Sacrobosco's *Sphere*, Clavius noted that he used the terms "astronomia" and "astrologia" interchangeably (Clavius, *Sphaeram,* 6). In practice, he seems to have used astronomy to denote what he had described in his *Sphere* as theoretical astronomy dealing with the motions of the spheres to make claims about the nature of the universe, or as he put it the "universam mundi machinam." Astrology then denoted "practical astronomy" which was the application of astronomy to human life through its ability to measure time and through prognostication. However, sixteen years before the condemnation of astrological predictions in a 1586 papal bull, he explicitly

understand vision through the use of geometry. It included explanations for differences between appearances and reality and enabled its students to create optical illusions, including those necessary to sixteenth-century paintings. Clavius emphasized the practical nature of this branch of mathematics by choosing to use the word "perspectiva" instead of "optica." While either word seems to have been acceptable, "optica," which fully conveys the use of mathematics to study vision, lacks the practical connotations that "perspectiva" gains from its use in art.[64] Geodesy applied the principles of geometry to physical objects, often approximating geographical or manmade features with geometrical shapes, such as cones and cylinders, to measure objects found in the world, such as mountains and wells. It was essential to military undertakings, for which knowledge of the terrain was necessary. Likewise, Clavius argued that history was not possible without geodesy because it required knowledge of historical geography. Canonics, or music, studied audible harmonies, connecting the auditory senses with an intellectual understanding of relationships between notes. Such knowledge could be applied to the pursuit of musical arts. Calculation was another term for practical arithmetic, and it taught

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noted that prognostication had been taken too far and entered the realm of superstition. Therefore, it was (in his view, rightly) condemned by the Church. Instead prognostication was intended for meteorological and medical applications. See Clavius, *Sphaeram,* 7. For a discussion of Clavius's views on astrology see Robert Westman, *The Copernican Question*, 207-208.

[64] Clavius followed his source Barozzi in the use of the word "perspectiva" when naming and describing this branch of mathematics. However, he was also familiar with Federico Commandino's commentary on *The Elements,* and the Urbino mathematician used the word "optica" in a similar passage in his preface (Commandino, *Euclidis Elementorum,* *4*r*). It is possible, in fact probable, that Commandino also based his discussion of the branches of mathematics on Proclus, but he could well have translated the names of the disciplines directly from the Greek. Thus, it seems that in sixteenth-century Latin usage "perspectiva" and "optica" were somewhat synonymous. Since Clavius had read both Barozzi's and Commandino's work, he certainly was aware of the choice between the two terms.

students how to use number in connection to concrete objects. Clavius noted that it was useful in military affairs by making it possible to optimally distribute limited human resources. Finally, mechanics, the study of machines, could be used to build a wide variety of instruments for an equally wide variety of purposes. As examples of the fruits of mechanical study, Clavius offered Archimedes' war machines, which protected Syracuse from the Romans for a time, his armillary sphere, and the varied machines attributed to Hero and Ctesibius of Alexandria, including clocks and theatrical machines that could imitate motion of living creatures.[65] Where the quadrivium allowed Clavius to argue for the nobility of mathematics, this division of mathematical disciplines allowed Clavius to show that his field had a role in nearly every facet of a city's life, including various forms of art, timekeeping, mapping, medicine, and warfare. Truly, a discipline with such public applications was worthy of the study of princes.

The Utility of Mathematics in the Ratio Studiorum

Clavius's vision of the utility of mathematics as a tool for rulers to use in public projects was part and parcel with the Jesuits' belief that their schools were a

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[65] Clavius, *Euclidis Elementorum*, a8*v*. Hero of Alexandria and Ctesibius are both known for their inventions. Ctesibius is remembered as the "father of pneumatics" for his work developing pumps. He also developed a water clock which remained the most accurate clock until the seventeenth century. Hero developed a stand-alone fountain, a holy water "vending machine," and an engine that used falling weights to open doors, among other things. As Clavius noted, his work was largely based on equilibrium. "Quaedam mirabilium prorsus rerum effectirx … quippe quae aliae quidem spiritibus maximo cum artificio construit, quemadmodum etiam Ctesibius, atque Heron operantur; alia autem ponderibus, quorum motus quidem inaequilbrium, status vero aequilibirum esse causam censendum est, ut Timaeus etiam determinavit; alia vero nervis, spartisque animatas convolutiones, ac motus imitantiubs."

missionary activity. On missions outside of Europe mathematics could help the Jesuits secure their position and even convert non-Christians. The Jesuits' early missionary to China, Matteo Ricci, credited his ability to gain favor among the upper classes of the Chinese to his mathematics education under Clavius. He believed that the Chinese granted the Jesuits entry to their empire because they were impressed by the gifts of mathematical objects, such as prisms, and clocks, that Ricci and his predecessors presented to local leaders and even to the emperor. At one point when the Jesuits were at risk of being expelled from China or, worse, killed, Ricci believed that the respect he gained for a world map he had drawn in 1584 allowed them to stay. Furthermore, due to his talent for mathematics, especially astronomy, Ricci attracted the attention of wealthy Chinese families who requested that he teach that subject to their sons as they prepared for the civil service exam. Once he had the boys' attention, he could use his lessons as an opportunity to have spiritual conversations. If a student converted, the Jesuits were then free to teach him theology in a more rigorous manner.[66]

Even in Europe, the argument that mathematics was a discipline worthy of princes provided Jesuits a means to gain or retain favor with noble patrons, allowing them to establish the schools through which they intended to save souls. Antonella Romano has shown that, as part of an agreement that allowed the Jesuits to operate schools in France, seventeenth-century French Jesuits devoted their mathematical

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[66] Peter Engelfriet, *Euclid in China: The Genesis of the First Chinese Translation of Euclid's Elements, Books I-VI (Jihe Yuanben, Beijing, 1607) and Its Reception Up to 1723* (Leiden: Brill Academic Publisher, 1998), 58-68.

work and teaching to technical projects in practical mathematics topics in response to the needs of the French monarchy.[67] In fact, even though the mathematics curriculum in the final draft of the *Ratio Studiorum* was limited in scope, it left open the opportunity for Jesuit schools to teach any branch of practical mathematics desired by their local patrons. After teaching the first six books of Euclid's *Elements,* the teacher was instructed to "add some geography or astronomy or similar matter which the students enjoy hearing about."[68]

The final version of the mathematics curriculum is evidence that it was Clavius's presentation of the utility of mathematics in his suggested curricula and textbooks that ultimately convinced the Jesuits to include mathematics as a higher discipline in their curriculum. In this section, I will trace the development of the mathematics portion of the *Ratio Studiorum* from the first draft of 1586 to the final draft of 1599, to explain how that came about even though, as discussed in the previous chapter, much of Clavius's pedagogical project was devoted to establishing the place of mathematics within the hierarchy of disciplines for the discipline's nobility. Even when Clavius did discuss the utility of mathematics, he always first mentioned its utility to the study of philosophy and theology as a study of an immaterial and perfect subject matter and its certain method of producing results. Thus, it is clear that he saw the arguments for utility as secondary to those for nobility. Yet, it was the former that prevailed with the authors of the *Ratio Studiorum.*

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[67] Romano, *La Contre-Réforme Mathématique,* 3.
[68] Farrell, *The Jesuit Ratio Studiorum,* 46. Of course, patrons may well have dictated what it was that students would enjoy. That seems to have been the case in France.

Clavius's efforts to secure a place for mathematics started as soon as the Jesuit Order began to formalize its curriculum. In 1581, Claude Aquaviva, the newly elected General of the Order, faced with both a growing demand for colleges and tremendous success in several existing colleges, appointed a committee to draft a general curriculum for all Jesuit colleges. In that same year Clavius wrote three possible mathematics curricula. Only one year later, he wrote a brief statement on the promotion of mathematics in Jesuit schools.[69] In so doing, he ensured that his voice would be heard by the first curricular committee, whose twelve priests spent three years, codifying curricular material from the preceding thirty-five years of Jesuit teaching experience and pedagogical philosophy, including Clavius's recent work on mathematics education.[70] In those documents, Clavius clearly promoted the value of mathematics for its nobility. His course of study treated astronomy as the goal of all mathematical study because of its proximity to the divine. His suggestions for the promotion of mathematics centered on the treatment of the mathematics professor as an equal of the philosophy professor and the embarrassment that would be brought upon Jesuit schools if they did not include a rigorous mathematics curriculum.[71]

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[69] Christopher Clavius, "Ordo servandus in addiscendis disciplinis mathematicis (1581)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 110-115; Christopher Clavius, "Modus quo disciplinae mathematicae in scholis Societatis possent promoveri (1582)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 115-117; Christopher Clavius, "De re mathematica instructio (Ad annum 1593)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 117-118. The last document dates in the period of revision between the second (1591) draft of the *Ratio* and the final (1599) draft.

[70] Farrell, *The Jesuit Code of Liberal Education: Development and Scope of the Ratio Studiorum*¸ (Milwaukee: The Bruce Publishing Company, 1938), 219-223.

[71] Clavius, "Ordo servandus" 111-115; "Modus quo disciplinae mathematicae" 115-117. Clavius did not specify who exactly would cause the Jesuits to be embarrassed if they were ignorant of

As it turned out, Clavius's arguments for the nobility of mathematics were well-suited to the early efforts of curriculum writing. In 1584, the original committee was replaced by a new committee of six members who were chosen for their doctrinal expertise in theology and philosophy.[72] That committee was charged with the task of creating a curriculum that would lead to the salvation of souls through proper doctrine. Practical concerns were secondary to the value of the curriculum for theological studies. Relying on the previous committee's collected documents, the six theologians began to write the *Ratio Studiorum,* producing the first printed draft in 1586. Their emphasis on the salvatory quality of content is clear from the structure of the document. After a brief introduction explaining that the *Ratio Studiorum* was to fulfill St. Ignatius's demand for a curriculum to be used in all Jesuit colleges, the 1586 draft offers sections on each discipline. The sections are discursive and provide guidelines on what was to be taught and, to a lesser extent, the method to be used. Each section offered a brief explanation of the discipline and, in some cases, included an apology for the subject at hand.

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mathematics, but it is likely that he was thinking of Protestant scholars like Michael Maestlin and his students. However, he also could have been thinking of non-Jesuit Catholic mathematicians like Petrus Ramus. After all, if the Jesuits were to succeed in their efforts to become educators to Catholic elites, they would have been competing with other Catholic scholars for such influence.

[72] Farrell, *The Jesuit Code,* 223-227, Farrell indicates that there is no documentary explanation for changing the committees. It is possible that members of the original committee received other assignments, and the committee was gradually eroded until there was need for a new committee. Or, as Farrell suggests, the number of committee members (twelve) proved unmanageable, and a new smaller committee was seen as necessary.

The section on mathematics is profoundly apologetic and spends a great deal of time justifying the study and status of mathematical disciplines.[73] Clavius's influence on this apology is immediately obvious. Many of the arguments offered by the authors of the *Ratio Studiorum* are nearly identical to those put forward by Clavius. After a reference to the *Constitutions* and Ignatius's inclusion therein of a provision to teach mathematics as part of the philosophy curriculum, the authors, as Clavius had done in 1582, insisted that without mathematics the Society's schools would be lacking their "great ornament."[74] Throughout the first third of the section on mathematics, they continued with a defense of the discipline based on its value to Jesuit academies. As Clavius had argued in his defense of mathematics, the authors of the *Ratio Studiorum* claimed that mathematics was the greatest ornament to the academies because all celebrated academies taught the discipline.[75] Furthermore, as

[73] It was not just mathematics that received an apology. Farrell argues that the 1586 *Ratio Studiorum* also vindicates the humanities professors, defending their status in relationship to the teachers of the higher faculties, Ibid., 228.

[74] *Ratio Atque Institutio Studiourm,1586* in ed. Ladislaus Lukacs, *Monumenta Pedaegogica Societatis Iesu Vol. V: Ratio Atque Institutio Studiorum Societatis Iesu*, (Rome: Institutum Historicum Societatis Iesu, 1986), 109. "Constitutiones (4 Pr. Cap. 12 C) 'Tractabitur, inquiunt, logica, physica, metaphysica, moralis scientia et etiam mathematicae, quatenus tamen ad finem nobis propositum conveniunt.' Convenire autem videntur non parum, non solum quia sine mathematicis academiae nostrae magno carerent ornamento, quin et mutilate forent, cum nulla sit fere paulo celebrior academia, in qua suus non sit, et quidem non ultimus locus mathematicis disciplinis; sed multo etiam magis, quia illarum praesidio caeterae quoque scientiae indigent admodum." ("The Constitutions (Part 4, chapter 12) say, 'Logic, physics, metaphysics, and moral sciences should be treated, and also mathematics to the extent that it is appropriate to our ends. Moreover, what is appropriate is not small, not only because without mathematics our academies would be lacking the greatest ornament, and without it they would be maimed…"). In 1582 in his brief on promoting mathematics, Clavius had claimed that mathematics was a great ornament to students' erudition and that natural philosophy was maimed without mathematics. He had also expressed concern that without mathematics, Jesuits would lose the esteem of other scholars. See Chapter 1 for a discussion of Clavius's arguments in this brief. Christopher Clavius, "Modus quo disciplinae mathematicae," 115-117.

[75] Ibid., 109. See quotation in the previous note. Since the *Ratio* does not explicitly name any institutions that taught mathematics, it is not clear which specific academies, if any they had in mind. It is possible they were thinking of the University of Paris where the program of study followed by the founding Jesuits had included mathematics.

Clavius had done in the prefaces to his commentaries on Sacrobosco and Euclid, the *Ratio Studiorum* lauded mathematics as useful to all other disciplines, which should be understood as philosophy and theology, and claimed its study would produce men who were better able to serve the Church in a variety of ways. And while their emphasis, like Clavius's, was on the benefits of mathematics to the study of theology, the authors of the first *Ratio* did note the practical values for the discipline. They commended it for its applicability to diverse public matters, and gave the usual examples of navigation, agriculture, and the treating of disease.[76]

The remaining two paragraphs of the section on mathematics detail what branches of mathematics were to be taught and to whom. In these two paragraphs, Clavius's influence remained apparent, even though the value of practical mathematics became more evident. While Clavius's curricula were not explicitly named, the division into three levels was maintained. However, the lowest level of mathematics covered much less material than Clavius had suggested in any of his curricula. This class, required for all students, was restricted to three daily lectures for a year and a half. The authors suggested that two months be spent on the first few books of Euclid, while making clear the relationships between geometry and other fields, especially geography and the sphere. The remainder of the course should be divided between Euclid, the use of the quadrant, and the sphere or some other branch of mixed

[76] Ibid., 109. "Ut praetereantur interea, quae ex mathematicorum labore redundant in rempublicam utilitates in morborum curationibus, in navigationibus, in agricolarum studio. Conandum igitur est, ut sicut facultates caeterae, ita et mathematicae in nostris gymnasiis floreant, ut hinc etiam nostri fiant magis idonei ad variis Ecclesiae commodis inserviendum; cum praesertim non parum indecore careamus professoribus, qui rerum mathematicarum lectionem tam multis, tam praeclaris urbibus exoptatam habere possint."

mathematics more welcome to the students. Thus, while geometry (Euclid) and astronomy (the sphere) remained the primary focus, practical mathematics, in the form of geography and the open-ended suggestion for an appealing branch of mathematics, gained a foothold in the minimum required curriculum.

For more advanced students, more time spent on mathematics meant that more of Clavius's curriculum was covered, allowing the authors to incorporate both noble and practical mathematics into the first version of the curriculum. The second level of mathematics education in the 1586 *Ratio* was provided to physics students who were to continue their study of mathematics the following year with an hour lecture each morning covering the remaining branches of mathematics, as defined by Clavius. It is unclear how many subjects were considered branches of mathematics by the authors, but it seems likely that Clavius's shortest was the intended source.[77] Finally, the 1586 *Ratio Studiorum* provided for the private academy Clavius had suggested in 1582. It stated, "One professor, who now could be Father Clavius, can be established to bring together the full doctrine of mathematical matters, and to explain them in private to eight to ten of our men who are of at least average talent, not strangers to mathematics,

[77] Ibid., 110, "Altero postmodum anno iisdem auditoriubs, qui physici tunc erunt, prima hora scholarum a prandio reliqua pars compendii mathematici, a P Clavio conficiendi, explicabitur." The editor of the *Ratio Studiorum* draft, Ladislaus Lukacs, indicates that the authors are referring to Clavius's *Epitome arithmetica practicae.* I think this is too narrow of an interpretation of "reliqua pars compendii mathematici, a P. Clavio conficiendi." Certainly, practical arithmetic would have been part of the remaining parts of mathematics, but Clavius's curricula indicated much more. Since those curricula would have been available to the authors of the *Ratio Studiorum,* I see no reason to assume that they would intend the "remaining branches" to refer to a specific text on a single subject named in Clavius's curriculum rather than the entire curriculum. And, since Clavius' shortest curriculum was designed to cover two years, it could well be the intended curriculum for the physics students extending their studies beyond the original year and a half.

and have studied philosophy."[78] The *Ratio* left the curriculum of the academy up to Clavius, allowing him to teach his most rigorous curriculum. Since his students were supposed to go on to teach mathematics in other Jesuit schools, the academy, as Clavius had envisioned it, served as the primary means through which mathematics could be promoted in the Jesuit system by providing many Jesuit schools with highly qualified mathematics teachers.

Immediately after printing, the 1586 *Ratio Studiorum* was distributed to the various Provinces for review, and the process of revision got underway in Rome. This draft was not intended to be used in the colleges but was a step in the process towards a useable curriculum.[79] In fact, before 1586 came to a close and long before all of the responses to the curriculum were received, a revised version was produced. The apology for mathematics remained intact, but the mathematics course was edited.[80] Through the elimination of the provision for physics students to continue a more rigorous curriculum than the general student body, the three strands of mathematics study were reduced to two, and Clavius's academy was halved in size. The mathematics professor was combined with the physics professor, and mathematics was only taught for one year. In a single daily forty-five minute class the professor was to

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[78] Ibid.,110. "Professor, alter, qui modo P. Clavius esse posset, constituatur, rerum mathematicarum pleniorem doctrinam conferat in triennium, explicetque privatim nostris octo circiter aut decem, qui mediocri saltem sint ingenio, nec a mathematicis alieno, et philosophiam audierint."

[79] Farrell, *The Jesuit Code*, 230.

[80] *Ratio Studiourm S.I. Anni 1586 - Retractata* in ed. Ladislaus Lukacs, *Monumenta Pedaegogica Societatis Iesu Vol. V: Ratio Atque Institutio Studiorum Societatis Iesu*, (Rome: Institutum Historicum Societatis Iesu, 1986), 177. These edits were not minor changes, but seem to have stemmed from philosophical disagreements on the curriculum. Notably, the section on natural philosophy was left unchanged and the mathematics section was greatly diminished, suggesting that the committee was not persuaded by Clavius's arguments that mathematics possessed a nobility comparable to that of natural philosophy.

follow the curriculum that had previously been outlined for all students.[81] The reduction in the mathematics curriculum indicates a compromise with those philosophers who did not believe that mathematics was a viable path to salvation and, therefore, did not think it essential to the study of natural philosophy or theology, but it does not suggest that mathematics was substantially reduced in its status relative to philosophy. A small academy for advanced mathematics students was still promised, and the utility of mathematics was still praised.[82] Furthermore, mathematics was able to maintain its claim to nobility through new provisions. Notably, the revised curriculum adopted Clavius's suggestion that mathematics students should be more involved in the philosophy curriculum by requiring that every month a mathematics student should demonstrate some mathematical problem for the philosophy and theology students.[83] In addition, while the second tier of the curriculum had been officially eliminated, students who were absolved from the six-month period of

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[81] Ibid., 177. "Professor itaque mathematicae physicis omnibus explicet in schola a prandio tribus circiter horae quadrantibus Euclidis elementa; in quibus, postquam per duos menses aliquantisper versati fuerint, ita dividat praelectionis tempus, ut aliquid Euclidi, aliquid vero geographiae vel sphaerae, aliisve, quae libenter audiri solent, tribuatur."

[82] Ibid., 177. "Praeter hanc privatam academiam P. Clavius operam suam libenter collocaret in privatam item aliam academiam ex quattuor aut quinque de nostris, quibus, si ii philosophi iam sint, et medicori saltem ingenio, nec a mathematicis alieno, et ab aliis studiis biennio tantum vacarent, explicaret ille primarias mathematicae facultatis partes cum spe eximii fructus."

[83] Ibid., 177. "Semel aut iterum in mense auditorium aliquis in magno philosophorum theologorumque conventu illustre aliquod problema mathematicum enarret, prius a magistro, sicut oportet, edoctus. In cuiusque etiam mensis sabbato uno, praelectionis loco, praecipua, quae per eum mensem explicata fuerint, publice repetantur, non perpetua oratione, sed se mutuo percunctantibus auditoribus hoc fere modo: Repete illam propositionem. Quomodo demonstratur? Potestne aliter demonstrari? Quem usum habet in artibus et in reliqa vitae communis praxi? Nam et haec quoque indicanda essent a magistro inter praelegendum, quo magis auditores alliciat."; Clavius, "Modo que…," 117. "Praeterea, ad haec studia maxime incitabuntur scholastici, si singulis mensibus omnes philosophi in unum aliquem locum convenirent, ubi unus discipulorum habeat brevem commendationem diciplinarum mathematicarum; deinde cum uno aut altero explicet problema aliquod geometricum vel astronomicum, quod et iucundum esset auditoribus, et utile rebus humanis; qualia problemata plurima repiriri poterunt; vel declaret locum aliquem mathematicum ex Aristotele vel Platone, qualia loca apud ipsos non pauca sunt; vel etiam afferat novas demonstrationes quarumdam propositionem Euclidis a se excogitatas."

repetitions in philosophy were permitted to use that time to study mathematics privately.[84] Both of these provisions suggest that the authors of the curriculum believed mathematics could be beneficial, even if it was not essential, to the study of theology.

Over the course of the last few years of the 1580s, each Province sent its comments on the first 1586 draft back to Rome. The comments on the section of mathematics were short, often only a few sentences and varied widely in their sentiments on the mathematics curriculum. While some comments were positive, the critiques of mathematics reflected both philosophical concerns, arguing that mathematics did not possess the ability to inform the study of theology, and practical concerns, worrying about the logistics of teaching mathematics. The response from the Collegio Romano was generally positive but argued for more time for the mathematics curriculum and, whenever possible, two mathematics teachers, although the author recognized that the shortage of teachers meant such a set-up could not always be possible.[85] The Portuguese Province suggested lengthening the mathematics curriculum to three years, which is unsurprising given that Coimbra was a center of

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[84] Ibid., 177. "Ex iis philosophis, qui philosophiae repetitionem per sex menses absolverint, altero eiudem anni semestri spatio domi fiat per eundem sive per alterum professorem academia rerum mathematicarum, quas in hunc usum P. Clavius in compendium quoddam redegerit."

[85] "Iudicia patrum, in provinciis deputatorum, ad examinandum Rationis studiorum (1586) tractatum, qui inscribitur 'De mathematicis disciplinis,'" in ed. Ladislaus Lukacs, *Monumenta Pedaegogica Societatis Iesu, Vol. VI: Collectanea de Ratione Studiorum Societatis Iesu*, (Rome: Institutum Historicum Societatis Iesu, 1992), 293. "Satis videtur annus unus ut nunc fit Romae. Nam onerare ingenia logicorum, tempore praesertim aestivo, quo levantur alii ipsis vero est diligentius incumbendum in logicam, est res valde gravis et molesta, atque perniciosa. Deinde, sive praeceptor unus sit, qui ambas praelegant lectiones, sive duo, utrumque maxima habet incommoda. Circa 3.m – De altero professore mathematicae etc. Optandum esset vel maxime, tam pro mathematicis, quam pro linguis, quod patres praescribunt; sed quoniam multis iisque gravissimis premimur difficultatibus, quae consderanti facile occurrent, nulla ratione in praesens videtur id posse fieri."

mathematical study in the sixteenth century. In fact, their response was largely a description of their existing mathematics program.[86] The Province of Lyon echoed a letter written by Orotinus Finè regarding the mathematics program in Paris that said that all teachers of the arts curriculum had to be familiar with the first six books of Euclid and further suggested that all students be required to study practical arithmetic and Sacroboso's *Sphere*.[87] The Sicilian Province accepted everything with a brief "Nothing is repugnant to the fathers."[88] Other provinces were mildly critical of the proposed curriculum. The Milanese approved of Clavius's academy but believed that for the general mathematics curriculum, a single year of instruction was sufficient.[89] The Aragonese were concerned that some students were simply not capable of studying mathematics, and cautioned that the rector should be lenient where mathematics requirements were concerned.[90] The Venetian and Upper German Province expressed profound indifference towards Clavius's academy since they

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[86] Ibid., p. 294. "Mathematicarum lectio in hac provincia sic videtur distribuenda: Habebitur imprimis pomeridiano tempore per unam horam, quae prima erit a communi ingressu in gymnasia. Singuilis annis post Pascaha priori horae dimidio tradentur principia mathematicae, quibus primi cursus auditores interesse cogentur. ...Ita tamen, ut nullum triennium praeteretur sine Sphaerae praelectione, cui, exceptis dialecticae tyronibus, intersint quicunque eam e philosophorum schola nondum audierint."

[87] Ibid., p. 296. "Expediret saltem servare quod Orontius in quadam epistola scribit, fuisse decretum in academia parisiensi, ut scilicet nemo crearetur magister atrium, quin audivisset sex Prima Elementa Euclidis. Lovanii certe coguntur philosophi omnes audire arithmeticam practicam et Sphaeram Ioannes de Sacro Bosco."

[88]Ibid., 294. "Nihil patres repugnarunt."

[89] Ibid, p. 293. "Videtur nobis sufficere secundus annus philosophiae ad ea audienda de mathamticis et tradenda, quae necessaria sunt, ut sunt tres libri priores Elementorum Euclidis, Sphera, Astrolabium, aritmetica."

[90] Ibid., p. 294. "Aliqui de logicorum numero interdum reperiuntur prorsus inepti studio mathematicorum. Alii item sustinere non poterunt onus trium lectionum. Ideo consonum esset, rectorem, audito praefecto ac professore logicae, nominare, quos expediat mathematicis indulgere; praeter quos nec cogatur quispiam, nec excludatur."

believed that a good mathematics professor in any province should suffice.[91] Still others offered sharper critiques. The Rhenish Province praised the place of mathematics but shortened the curriculum to one year and, like the Venetian and Upper German Provinces, questioned the need for Clavius's academy.[92] The Toledan Province denied the need for two mathematics teachers, deemed the academy unnecessary, and expressed concern that requiring even a single mathematics teacher who did not also teach philosophy would remove an able priest from a more necessary teaching position.[93] The French Province questioned the need for mathematics at all in more than a few great schools, removing the need for a mathematics teacher in most schools.[94] Only the dissenting opinion from the Collegio Romano, ascribed by the author of the response to two philosophy professors, Benedict Pereira and Pedro de Parra, questioned the utility of mathematical study altogether. Pereira, as we saw in

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[91] The Venetians pointed out that if a talented mathematics student could be identified, they would wish to benefit from his presence. Ibid., p. 294, "Cogendi non videntur provinciae, ut tertio quoque anno unum, qui mathmaticis disciplinis det operam, Romam mittant. Iniuria enim fieret. Eae enim, quae iam habent, qui profiteri possint, quorsum cogentur? Quae non habent, cum primum unum vel alterum habuerint, qui mathematicas disciplinas profiteri possint, neque compelli debent."; The Upper German Province said that a provision for a mathematics teacher at every school should be sufficient. Ibid., 295. "Non esset astringenda provincia, ut tertio quoque anno aliquem ad mathematicum studium Romam mitteret, si iam ei esset de mathematicis professore satis provisum."

[92] Ibid., 295. The Rhenish Province pointed out that Clavius himself was German, and if another such professor could be in their province, a special academy in Rome would not be necessary. "Quod ad eum attient, qui ut mathesin perdisceret, in eaque ad provinciae totius utilitatem excelleret, Romam mittendus esset, gratiae quidem merito, tanto pro studio et opportunitate a provinciis omnibus habendae et agendae sunt. Ea tamen partum sentntia est, si quis in Germania nostra Clavius, excellens, inquam, mathematicus existeret, et earum atrium cupidis triennio vel anno saltem integro, aliarum occupationum vacatio esse posset, Romam quenque, hoc nomine, mettere, necesse nihil foret."

[93] Ibid., 294. "Quod de duobus magistris mathematicarum dicitur, cum in nobilissimis unversitatibus unus tantum sit earum atrium magister, et nobis unus sufficere videtur; praesertim, cum multa sint alia magis necessaria, et quorum maior est usus, quae in Societate desiderantur."

[94] Ibid., 295-296. "Visum est omnibus, in magnis duntaxat collegiis oportere esse professorem mathematicae extraordinarium, qui biennio absolvat cursum mathematicae… In parvis autem collegiis professorem ordinarium cursus philosophici mathematicas disciplinas breviter perstringere debere, ut hactenus in Gallia factum est, et satis bene succedit."

the previous chapter, did not believe mathematics could be more than a pedagogical tool, and in his response to the 1586 mathematics curriculum he expressed his belief that it was impracticable and unlikely to produce the claimed results.[95] However, Pereira's focus was on the study of philosophy and theology, and even he did not dispute the potential practical applications of mathematics. Thus, in most responses, mathematics was not seen as a fruitless study, but it was also not as valuable to theology as was natural philosophy, meaning that it was the first to suffer cutbacks due to teacher shortages, which was a commonly expressed concern.

After collecting all of the critiques from the Provinces, the committee and the professors at the Collegio Romano began the process of creating a new *Ratio Studiorum*. This time, the goal was to create a document that could be used on the ground as a practical guide to running a Jesuit college.[96] As such, the structure of the 1591 *Ratio Studiorum* is drastically different from that of the 1586 draft. In keeping with the changed role of the 1591 draft of the *Ratio Studiorum*, the apologies for disciplines are gone.[97] In fact, this new draft contains no discursive essay on each of

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[95] Ibid., 293. "P. Parra et P. Pererius longe aliter sentient. Censent enim, ne illud quidem experiendum esse, etiamsi ad praxim reduce posset."

[96] *Ratio Atque Institutio Studiorum 1591,* in ed. Ladislaus Lukacs, *Monumenta Pedaegogica Societatis Iesu Vol. V: Ratio Atque Institutio Studiorum Societatis Iesu*, (Rome: Institum Historicum Societatis Iesu, 1986), 230. "Verum quoniam ea pars, quae opinionum delectum censuramque continent, nunc edi mittique non potuit (mittenda tamen propediem speratur), idcirco data opera est, ut altera saltem pars, quae studiorum ordinem ac praxim instituit, mitteretur in mores inducenda per omnem Societatem."

[97] Some have interpreted the lack of apology for mathematics as a significant reduction in the status of mathematics and the influence of Clavius on the *Ratio Studiorum*. This fails to take into account the general structural changes of the *Ratio Studiorum* between 1586 and 1591. See Jesus Luis Paradinas Fuentes, "Las Matematics en La Ratio Studiorum de los Jesuitas: Una Nueva Interpretacion," Fundacion Orotava de Historia de la Ciencia," *LLULL: Revista de la Sociedad Española de Historia de las Ciencias y de las Técnicas* 35, no. 75, (2012): 146, https://dialnet.unirioja.es/servlet/articulo?codigo=3943923.

the subjects to be taught in Jesuit schools, but instead lists rules for each of the actors in a Jesuit school, from provincials and rectors, to prefects, professors and students. The rules for the provincial are divided into each subject taught in the schools, and are sometimes repeated later in the sections for the professors of each subject. This structure suggests that, unlike the 1586 document, which was to be read in its entirety by the reviewers, the 1591 document was instead a manual from which each actor was expected to read and obey the parts relevant to his role.

Mathematics had a small place in the 1591 *Ratio Studiorum,* in which practical mathematics became more prevalent than abstract studies for theological purposes. The discipline was covered in five rules for the mathematics professor and four rules for the provincial. Since three of those rules are listed in both sections, the discipline had a total of only six rules. Clavius's influence on the curriculum remained apparent, especially in the three rules that insisted on the nobility of mathematics. The first of those rules, the last shared rule, established that two public lectures, which were explicitly to follow Clavius' curriculum, should be given daily, one for physics students and one for metaphysics students. If possible, they should be given by separate professors. However, for the second lecture it is stated that metaphysics students were neither required nor permitted to attend without permission from their superiors.[98] Even so, these daily lectures allowed Clavius's curriculum to continue to

[98]Ibid., 236 and 285. This is rule 4 for the mathematics professor and rule 43 for the provincial. "Ubi commode fieri poterit, vel diversis horis professor idem, vel eadem hora professores duo binas quotidie lectiones publicas habeant, quibus mathematicum quoddam curriculum a Patre Clavio scribendum explicent duobus annis; priori quidem physicis, posteriori autem metaphysicis, tametsi ad hunc potsteriorem nostri nec compellendi, nec admittendi sint, nisi quibus id postulantibus superiors concesserint."

be taught in Jesuit schools and provided interested students with a way to supplement the required one year curriculum.[99] Second, the final rule belonging to the provincial, closely mimicked a concern Clavius had expressed in his 1582 note on promoting mathematics. It required that "those who are in charge take the strictest precautions that philosophers do not make light of the dignity of mathematics or refute the teachings of mathematics, as with regard to epicycles, in their teaching or elsewhere. For this often happens, so that he who knows less greatly detracts from these sciences."[100] Through this rule, even though mathematics did not have as strong a foothold in the 1591 curriculum as it had in 1586, the status of mathematics remained equal to philosophy. And finally, the first rule unique to the professor of mathematics (rule 2) provided for monthly demonstrations of mathematical problems to philosophy and theology students.[101] Thus, it granted the mathematicians an intellectual status comparable to that of the natural philosophers who were the audience for the presenting students. The presentations themselves served as the mathematicians' equivalent of the disputations that philosophy and theology students were required to give.[102]

However, the other three rules made it clear that the criticisms of the 1586 draft were taken into consideration, as the place for mathematics was drastically

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99 It remains to be uncovered what exactly was taught in all of these lectures between 1591 and 1599. It was during that time, in 1593, that Clavius's academy was formally implemented in Rome, thereby giving his curriculum a strong foothold at the Jesuits' primary institution.

100 Ibid., 236. "Severissime caveant, qui praesunt, ne philosophi professores inter docendum aut alibi mathematicorum dignitatem elevent, neve eorum refellant sententias, ut de epicyclis; fit enim saepe, ut qui minus ista novit, his magis detrahat."

101 Ibid., 285.

102 Ibid., 282. The philosophy disputations were weekly, leaving mathematics lagging a little bit.

reduced and its utility was emphasized. The first of these rules reduced mathematics to a one year course, taught in the second year of the philosophy curriculum. In that year, a daily forty-five-minute lesson was devoted to Euclid (exclusively for two months) and another branch of mathematics, chosen according the students' desires. As had the 1586 draft, this draft of the curriculum suggested geography or the sphere for the second branch.[103] The second rule limiting mathematics reduced Clavius's academy to an optional six-month course for interested philosophy students who had been absolved from the half-year of repetitions in philosophy.[104] Thus, this draft made it possible for astronomy to be entirely replaced by another branch of mixed mathematics with practical applications to daily life and severely limited Clavius's ability to train specialists in the pursuit of mathematics for its nobility. Finally, the second rule unique to the mathematics professor required monthly reviews of mathematics material that were explicitly to be conducted in a way that emphasized the utility of mathematics to attract students. Even if the review covered abstract geometry, teachers were to ask how it could be applied.[105] Thus, it seems that by

[103] Ibid., 236, 284. This is rule 1 for the mathematics professor and rule 42 for the provincial. "Audiant et secundo philosophiae anno philosophi omnes in schola tribus circiter horae quadrantibus a prandio mathematicam praelectionem ex elementis Euclidis; in quibus postquam per duos menses aliquantisper versati fuerint, ita dividatur praelectionis tempus, ut aliquid Euclidi, aliquid vero Geographiae vel Sphaeae alissve, quae libenter audiri solent, tribuatur."

[104] Ibid., 236, 286. This is rule 5 for the mathematics professor and rule 41 for the provincial. "Altero eiusdem anni semestri spatio ex iisdem phiosophis domi fiat academia rerum mathematicarum, quas navus aliquis ac bene peritus professor bis quotidie explicabit nostris; quibus severe interdicendum, ne caeteris ullis tunc studiis implicentur, sed mathematicis audiendis, repetendis, disputandis se totots tradant. Fiat in his quanto amplior fieri potest progressus iuxta P. Clavii compendium; et qui magnopere profecerint, nec ab ea re alieno sint animo, dicentur huic studio tam privatis academiis frequenter amplificando, quam publice, quando opus fuerit, profitendo."

[105] Ibid., 285. This was rule 3 for the mathematics professor. "3. In cuiusque etiam mensis sabbato uno, praelectionis loco praecipua, quae per eum mensem explicata fuerint, publice repetantur, non perpetua oratione, sed se mutuo percunctantibus auditoribus; hoc fere modo: Repete illam propositionem. – Quomodo demonstrator? Potestne aliter demonstrari? – *Quem usum habet in artibus et in reliqua vitae*

1591, while mathematics had retained its status among the higher disciplines, it was for its practical utility, not its sublime nature, that the Jesuits deemed it worthy of study.

Jesuit schools taught under the *Ratio Studiorum* of 1591 for three years. Then, from 1594 until 1598 the Provinces sent critiques of the new draft back to Rome. Alan Farrell cites two categories of critiques: first, general critiques that the 1591 *Ratio Studiorum* was too detailed and repetitive and, second, critiques of particular rules, especially when they seemed to conflict with local customs.[106] In the case of mathematics teaching, there were still practical concerns with finding capable mathematics teachers who were not elsewhere more urgently needed.[107] As in 1586, the editors of the *Ratio Studiorum* took the criticisms to heart. The general structure of the final draft of the *Ratio Studiorum* in 1599 remained very close to that of the 1591 draft. It is again a series of rules working down the hierarchy from the provincial to the rules for students. However, the rules are greatly abbreviated and repetitions almost entirely eliminated. Each role received a relatively small number of rules. Even the provincial only had forty rules, compared to ninety-six rules in 1591. After almost twenty years of research, writing, critiquing, and refining, the definitive *Ratio Studiorum* published in 1599 was a concise but detailed curriculum outlining exactly how Jesuit colleges were to be run and how classes were to be taught. In this version, the provincial, the rector, and prefect of studies were all given administrative

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communis praxi? – Nam et haec quoque indicanda sunt a magistro inter praeelegendum, quo magis auditore alliciat." (Emphasis mine.)

[106] Farrell, *The Jesuit Code*, p. 303.

[107] Fuentes, Las Matematics en La Ratio Studiorum," 141.

rules discussing the timeline of the curriculum, the number of classes to be taught, mentoring of new teachers, observing professors, and examining textbooks. In rules common to all teachers of the higher faculties, humility in presenting debatable topics was advised, and teachers were cautioned against dictations. For the lower faculties, the *Ratio Studiorum* reminded teachers that they were examples of the religious life for their students. Rules for the individual professors provided explanation of content to be taught and public exercises to be assigned to the students.[108]

The position of mathematics in the definitive *Ratio Studiorum* was substantially reduced from its place in the drafts of 1586 and 1591. The mathematics professor received only three rules, and mathematics was not even mentioned in the rules for the provincial.[109] The rules for the mathematics professor lay out the curriculum to be taught and the public exercises to be held for mathematics students. They were not substantially changed between 1591 and 1599, though they were somewhat abbreviated. Mathematics was to be taught in the philosophy curriculum in conjunction with physics, which was to be drawn from Aristotle's work.[110] In the same year students studied physics, they would also study mathematics for three-quarters of an hour each day. This maintained Clavius's suggestion of closely linking physics and mathematics in order to demonstrate the utility of the latter to the former. The first two months were to be devoted to Euclid's *Elements*. After that, time was divided between Euclid and another branch of mathematics. As in 1586 and 1591,

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108 Farrell, *The Jesuit Ratio Studiorum of 1599*, xi.
109 Ibid., 46.
110 Ibid., 40.

geography or astronomy was suggested as the other branch, but it was left open to anything the students would enjoy. The next rule specified that once a month a mathematics student must solve some celebrated mathematics problem in the presence of philosophy and theology students. Finally, the last rule mandated a monthly review of material being taught. Unlike in 1591, this rule did not specify that the utility of mathematics be emphasized. However, that does not indicate a shift in the Jesuits' attitude towards a universal acceptance of the sublime nature of mathematics. Indeed, the only dignity clearly granted to mathematics was that it was classified with the higher faculties. The rules providing for lectures following Clavius's curriculum and Clavius's academy were gone. Furthermore, there was no rule reminding the philosophy professors of the importance of mathematics.[111] Since the curriculum itself only required Euclid and some branch of mixed mathematics, which did not have to be astronomy, the end goal of the Jesuit mathematics curriculum appears to have been the teaching of practical branches of the discipline.

Of course, it must be recognized that because the *Ratio Studiorum* was intended for all Jesuit schools, it is a representation of the minimum mathematics curriculum. Indeed, while mathematics was no longer explicitly protected in the rules of the provincial, there was no indication that mathematics should be subordinated to philosophy. Furthermore, the chronic difficulty in finding mathematics teachers, as evidenced by complaints from the provinces, suggests that much of the retreat from

[111] Ibid., 46. Three academies are mentioned at the end of the document, the academy of theologians and philosophers, the academy of students of rhetoric and humanities, and the academy of students of the grammar classes, but these are student run study groups to review and discuss material as well as practice disputations. These were not opportunities for private instruction. Ibid., 105-111.

the position of mathematics in the revised 1586 curriculum could have been driven by logistical concerns.[112] The continually growing school system (almost one hundred schools had been created during the twenty years of the drafting of the *Ratio Studiorum*) required an ever-increasing number of teachers for all subjects, especially grammar and rhetoric.[113] The academy, as Clavius envisioned it, would have required that some young Jesuits be exempted from requirements to teach the lower disciplines at the end of their philosophy studies in order to pursue mathematics.[114] The Society could ill afford those exemptions, especially since the academy could not have kept up with the demand for new mathematics teachers in the growing school system anyway.

These practical concerns did not prevent schools that did have capable mathematics teachers from teaching rigorous curricula. As discussed in the previous chapter, Clavius ran an academy for talented students in the philosophy curriculum until 1610, only two years before his death. And, when he published his *Opera Mathematica*, he acknowledged the excellent mathematics teaching done at the Jesuit University of Mainz.[115] Still, the mathematics programs in Rome and Mainz were exceptions. Faced with the logistical impossibility of training enough teachers for all

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112 As discussed above, many of the responses to the 1586 *Ratio Studiorum* expressed some concerns about a lack of qualified mathematics teachers. The response from the Collegio Romano requested that there be two mathematics teachers at each school but acknowledged that such a request might not be feasible. The Venetian Province expressed concerns about sending their talented mathematicians to Rome since that would deprive them of the benefit of having such a scholar. The Toledan Province's response explicitly stated that requiring a mathematics professor at every college would deprive the schools of staff for more essential positions. See "Iudicia patrum, in provinciis deputatorum, ad examinandum Rationis studiorum (1586)", 293-296.

113 Farrell, *The Jesuit Ratio Studiorum of 1599*, iii.

114 Clavius, "De re mathematica…", 117.

115 Christopher Clavius, *Opera Mathematica V tomis distributa ab auctore denuo correcta, et plurimis locis aucta* (Mainz: Antonius Hierat and Reinhardus Eltz, 1612),)(3*v*.

of their colleges to pursue both the noble and practical branches of the study, the Jesuits had to choose between the two. And, by 1599, the Jesuit mathematics curriculum had been reduced from Clavius's heady study of perfect forms and celestial bodies to a study of Euclid and whatever practical branch was best suited to students in any given school. Ricci's use of mathematics to secure patronage in China and French Jesuits' pursuit of projects required by the French kings suggest that Clavius's arguments for the value of mathematics to the upper echelons of society paved the way for that choice.

Conclusion

As the professor of mathematics at the Collegio Romano when the *Ratio Studiorum* was being drafted, Clavius had a unique opportunity to influence the direction of the mathematics curriculum. He seized that chance by presenting his suggestions for a rigorous program of study in which practical mathematics was one component. However, while the sixteenth century was a time in which practical mathematics was recognized and developed for its potential applications to a variety of crafts, Clavius never intended his development of the discipline as a tool for craftsmen. Instead, he argued that through its applications to timekeeping, art, architecture, business, geography, and warfare, practical mathematics was at the heart of every social endeavor. It was a tool that could be used by princes to improve their domains.

Ultimately, it was Clavius's arguments for the practical uses of mathematics to the upper echelons of society, i.e. those in the courts, that allowed his discipline to retain its place in the Jesuits' curriculum. While the first draft of the *Ratio Studiorum* embraced Clavius's vision of mathematics as a noble study necessary to the advancement of theology, each successive revision reduced the role of the discipline. During the revision process, some critiques stemmed from the view that mathematics was not necessary to the study of theology, but many more expressed the belief that it was not logistically possible for Jesuit colleges to supply enough teachers for a universally rigorous mathematics curriculum. However, the practical value of mathematics was not called into question. Indeed, in their negotiations with patrons (at least in China and France), the Jesuits were able to employ practical mathematics as a means to gain favor. And so, by 1599, only Euclid's *Elements* and a practical branch of each school's choice were to be taught. Forced by teacher shortages to choose between Clavius's noble mathematics and Clavius's practical mathematics, the Jesuits chose the latter. Nevertheless, through their practical concerns, Clavius gained a foothold for his pedagogical project, creating the opportunity for some schools to teach from his textbooks in which he presented his vision of mathematics as the bridge between the mundane and the divine, both useful and noble.

Chapter Three

Not All Euclids Were Created Equal

"Now almost two thousand years have passed since Euclid was counted among the living. He has had many adversaries, who, out of the vice of jealousy more than love of truth, have tried with all zeal to undermine his work. Yet, up to this point, the stern investigators have been able to reveal no false writing in that work, not any error nor any paralogism."[1]

Federico Commandino (1572)

According to Proclus, when Ptolemy I asked Euclid for a way shorter than *The Elements* to learn mathematics, the geometer told the king that there was no royal road to geometry.[2] While that conversation is most likely apocryphal, the idea that *The Elements* was a necessary foundation to the study of geometry persisted long past Proclus's time. By the time Christopher Clavius published the first edition of his commentary on *The Elements* in 1574, dozens of versions of the text had been produced by mathematicians whose efforts to elevate their discipline often included providing their own commentaries on the foundational text. Because *The Elements* was an introductory text, it provided its commentators with the opportunity to define

[1] Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), *5*v*. "Iam duo fere annorum millia abierunt, ex quo Euclides inter vivos conumeratus est. multos habuit adversarios, qui invidiae potius morbo, quam veritatis amore illius scripta omni studio labefactare sunt conati; nullam tamen adhuc in illis φευδογραφίαν, nullum errorem, nullum paralogismum severi inquisitores deprehendere potuerunt."
[2] Proclus: *A Commentary on the First book of Euclid's Elements,* trans. Glen Morrow (Princeton: Princeton University Press, 1970), 57.

their discipline and the value of its study, and differing visions of mathematics led to divergent versions of the book.

Although *The Elements,* which dates to the fourth century BCE, had been taught in medieval universities, the text received new life at the hands of sixteenth-century commentators who relied on both the thirteenth-century commentary by Campanus of Novara, the most common source of commentaries on *The Elements* from the thirteenth century until the sixteenth century, and recently rediscovered Greek versions of the text, namely the fourth-century CE commentary of Theon of Alexandria and the fifth-century CE commentary on the first book by Proclus. [3] Bartolomeo Zamberti published the first Latin translation of a Greek version of the text in 1505, sparking a debate over the relative merits of the Greek and medieval texts. In 1516 the Parisian printer Henrici Stephani placed that debate in the hands of individual readers by publishing an edition of *The Elements* that included Campanus's and Zamberti's versions of each proof side-by-side.[4] After that many authors based their commentaries on a combination of Theon's and Campanus's editions, especially after 1533 when Theon's edition was printed in Greek by the Basel-based printer Simon Gyraneaus. Some mathematicians even included their own demonstrations for the Euclidean propositions. For example, in his 1557 commentary on the first six

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[3] Prior to the sixteenth century, Western universities had been using Euclid as the source for geometry education, but their copies were based on Campanus of Novara's thirteenth-century translation from Arabic versions. See H.L.L. Busard, *Campanus of Novara and Euclid's* Elements. (Germany: Franz Steiner Verlag, 2005), 32. See the introduction for a brief discussion of the transmission of Euclid.

[4] Thomas Heath, *The Thirteen Books of Euclid's Elements, Introduction and Books 1 and 2*. (London: Cambridge University Press, 1908), 98-99. The 1516 edition containing Zamberti's text is Bartolomeo Zamberti, *Euclidis Geometricorum elementorum libri XV* (Paris: Henrici Stephani, 1516).

books, Jacques Peletier added “here and there new demonstrations to Euclid.”[5] In 1566 Francois Flussas Candalla added an entire sixteenth book to the Euclidean corpus.[6] Not all mathematicians felt that such changes to *The Elements* were positive. As part of his efforts to restore ancient mathematical knowledge Federico Commandino, a significant figure in Paul Rose’s arguments for the existence of mathematical humanism, based his 1572 commentary on the 1533 edition of Theon’s text, eschewing medieval and contemporary additions to the text. Others, including Clavius, presented elements of the ancient, medieval, and contemporary versions of the text. In this, his work was similar to some vernacular versions of the text including Henry Billingsley’s 1570 English commentary.

For Christopher Clavius, *The Elements* was the first mathematics textbook in the curriculum he sought to establish within the Jesuit schools. It was the foundation for all future mathematical study. Thus, his commentary on *The Elements* was an essential part of his efforts to secure the place of mathematics within the Jesuit curriculum. In it, he attempted to balance two visions of mathematics – one that placed mathematics between natural philosophy and theology within the hierarchy of disciplines for its ability to uncover certain, universal truths and one that saw mathematics as the key to solving practical and mundane problems. The first of these visions, as discussed in Chapter 1, claimed for mathematics a place among the higher

[5] Jacques Peletier, *In Euclidis Elementa Geometrica Demonstrationum Libri sex* (Lyon: Ioan. Tornaesium et Gul. Gazium, 1557), A2*r*. “Novas Demonstrationes passim ad Euclidem adiecimus.”

[6] Francois Flussas Candalla, *Euclidis Megarensis Mathematici Clarissimi Elementa Geometria, Libris XV* (Paris: Iannem Royerius, 1566).

disciplines. The second, as discussed in Chapter 2, appealed to Jesuit missionaries and their patrons.

The dual nature of Clavius's goals for mathematics can clearly be seen in a comparison of his commentary on *The Elements* to two closely contemporary versions, Sir Henry Federico Commandino's Latin commentary (1572) and Billingsley's English commentary (1570). Both Commandino and Billingsley wrote their commentaries with clearly defined goals and audiences in mind, and each emphasized one of the visions of mathematics combined in Clavius's text. As a tutor to nobles in Urbino, Commandino likely intended his commentary on *The Elements* to be used by his students as he trained them to uncover universal truths through mathematical study. As a result, his text emphasized the abstract nature of mathematics, and his interest in *The Elements* appears to be based on his admiration for what he took to be a complete and coherent system of knowledge. Indeed, in his prolegomenon Commandino claimed that nothing needed to be added to or removed from Euclid's text for the reader to understand each proposition.[7] For Billingsley, a London merchant, mathematics was an aid to practical innovation, and he intended his commentary to be read by enterprising English craftsmen for their own potential profit.[8] As a result, his commentary accentuated the concrete physicality of

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[7] Commandino, *Euclidis Elementorum,* ***v*. "Postremo admirabilem omnium dispositionem antecedentique, et consequentium ordinem, ac cohaerentiam, ut nihil prorsus addi, aut detrahi posse videatur."

[8] Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara* (London: John Daye, 1570), ii*v* – ij*r*.

mathematical concepts, and he treated the geometrical knowledge found in *The Elements* as a toolbox for English readers pursuing knowledge for concrete benefits.

By comparing Clavius's commentaries to those of Commandino and Billingsley, whose projects each focused on one aspect of mathematics, I hope to show how Clavius created a pedagogical text that treated mathematics as a versatile discipline. I will begin by examining differences among the authors through the structures of the texts, both on the large scale of what they chose to include and exclude and on the smaller scale of reading aids each author included. In these comparisons, Clavius's pedagogical goals become evident, as his text was clearly designed for use by novice students. Then I will compare the presentations of the foundational material in the first book, including definitions, axioms, postulates, and the authors' explanations of propositions to reveal how each author envisioned mathematics. In these sections, Clavius's combination of philosophical and physical components of mathematics becomes clear. Finally, to illustrate how each author's views were carried throughout their presentations of mathematical content, I will compare how they treated the Pythagorean Theorem, the most iconic proposition from *The Elements*.

The Big Picture: What Counts as Euclid?

Despite mathematicians' assertions that the content of Euclid's *Elements* had not changed in the two millennia since it had been written, by the sixteenth century, there were myriad versions of the text, and the content was anything but standardized.[9] In fact, Billingsley, Commandino, and Clavius all included numerous additions to the Greek text. Some addenda were simple concepts that could be demonstrated in a sentence or two. For example, in their commentaries on the definition of a circle's diameter, all three authors inserted the claim that a circle is bisected by a diameter. At other points the commentators included alternative proofs for the Euclidean propositions and complicated extensions of Euclid's work that are classified as lemmas or corollaries and receive their own lengthy proofs. All three authors credited many of their additions to ancient mathematicians, including Apollonius, Archimedes, and Thales, as well as to later scholars, such as Proclus and Pappus. (Clavius credited the demonstration for the above-named addition to both Thales and Proclus.[10]) In Billingsley's and Clavius's texts, there are frequently additions credited to Campanus of Novara, whose translation was the source for most versions of *The Elements* between the thirteenth and sixteenth centuries, when many changes were made because most copies of the text contained incomplete proofs.[11] Billingsley and Clavius also both frequently cited modern authors. Clavius used a wider breadth of

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[9] For a discussion of such assertions, see chapter 1. The epigraph to this chapter is a standard example of such a claim.

[10] The definition of a diameter (Book I, definition 17) is a straight line that goes through the center of a circle and connects two points on the circle's circumference, so it is an additional claim to say that a diameter bisects a circle. See Christopher Clavius, *Euclidis Elementorum Libri XV Accessit XVI de solidorum Regularium comparatione* (Rome: Vincentium Accoltum, 1574), 8*v*. The same demonstration can be found in Commandino's commentary on page 3*v* and in Billingsley's on page 3*v*.

[11] John E. Murdoch, "The evidence of marginalia in the medieval Euclides latinus" *Revue d'histoire des sciences,* Vol. 56, No.2 (Juilliet-Decembre 2003), 369-382. Murdoch's analysis of the marginalia shows that readers would fill in the proofs with comments or even add alternate proofs from other sources.

modern sources than Billingsley, and, moreover, he often included his own additions and made changes to the Euclidean text itself.

While the practice of adding to and changing Euclid's work was not uncommon at any point in the text's history, these three commentators' decisions on what emendations to include reflect the variety of early modern projects for mathematics. Commandino's additions, which were strictly drawn from ancient sources, demonstrate his goal to restore ancient mathematical knowledge. Billingsley's breadth of sources suggests that he sought to provide readers with the contemporary extent of mathematical knowledge and familiarity with current scholarship, illustrating his efforts to make mathematics a tool for innovation to improve the future of his countrymen. Indeed, Billingsley noted in his letter to the reader that he had added numerous scholia and lemmas from other scholars, ancient and modern, immediately before expressing his hope that his text would inspire the English to a further study of mathematics.[12] Clavius's extensive naming of other sources, combined with his own changes to the Euclidean text, allowed him to position his own work as a pedagogical text designed to make mathematics – both ancient and modern – accessible and to show his discipline to be an evolving study to which his

[12] Billingsley, *Elements of Geometrie,* ii*v*-ij*r*. "In which booke also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inventions: which I have gathered out of many of the most famous & chiefe Mathematiciens; both of old time, and in our age: as by diligent reading it in course ye shall well perceave. The fruite and gain which I require for these my paines and travaile, shall be nothing els, but onely that thou, gentle reader, will gratefully accept the same: and that thou mayest thereby receave some profite: and moreover to excite and stirre up others learned, to do the like."

readers could contribute, continuing the centuries-long project of building on ancient knowledge.

Clavius's pedagogical purpose sometimes might appear pedantic, especially in some of his additional demonstrations that could only have been intended for uninterested students. For example, in Book I, Clavius demonstrated that all right angles are equal to one another after the definition for a right angle. The same claim appears later as an axiom, so its inclusion with the definition could seem redundant.[13] Axioms, however, are supposed to be self-evident. And, while the claim that all right angles are equal follows quickly from the definition of a right angle, it is not immediately evident. The definition states, "When a straight line standing on another straight line makes the angles that are on either side equal to each other, either angle is right: And the straight line which stands is said to be perpendicular to the line upon which it stands."[14] The reader must realize that there is only one way in which to stand one line on another with equal angles on either side. Therefore, in this case, while the addition is pedantic in that it asserts a claim that appears later on in Euclid's text, it also served to prepare the reader for the axiom by making explicit the claim that all right angles are equal immediately following the definition based on which it becomes clear.

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[13] The claim that all right angles are equal to one another appears in the tenth axiom of Clavius's version of Book I. Clavius, *Euclidis Elementorum,* 17. In Commandino's and Billingsley's texts, it is the fourth postulate. (Commandino, *Euclidis Elementorum,* 6*r;* Billingsley, *Elements of Geometrie*, 6*r*.) The reason for the difference in classification will be discussed later in this chapter.

[14] Clavius, *Euclidis Elementorum,* 5*v*. "Cum vero recta linea super rectam consistens lineam eos, qui sunt deinceps, angulos aequales inter se fecerit, rectus est uterque aequalium angulorum: Et quae insistit recta linea, perpendicularis vocatur eius, cui insistit."

Not all of Clavius's additions were in the commentary, and those he included as Euclidean enunciations were usually intended to fulfill his pedagogical imperative of ensuring that no demonstration introduced an unproven claim or undefined terms, a convention of which he reminded his reader at the start of the fifth book.[15] Already in the first book, Clavius added two definitions in order to fulfill that purpose.[16] Those definitions were for terms used towards the end of the first book – "parallelogram" (def. 35) and "complement" and "residual" (both included in def. 36) in parallelograms cut around the diagonal.[17] (See Figure 2). In Billingsley's and Commandino's works a parallelogram is described, but not defined, in proposition 34. Complements are introduced in proposition 43.[18] Neither author provided the definitions, and the reader came to understand the relevant terms only by working

[15] A Euclidean enunciation is the statement of a definition, postulate, axiom or proposition without further commentary or demonstration. For example, the enunciation of the definition of a right angle is given above in note 13. Clavius, *Euclidis Elementorum,* 144*r*. "Ut igitur institutum suum servet, definit prius vocabula, quae ad demonstrationes proportionum adhibentur."

[16] In the first book, Clavius also left out one definition: that for a portion of a circle which corresponds to definition 19 in Commandino's (p. 3*v*) and Billingsley's (p. 3*v*) texts. He moved that definition to the third book, which is the first book in which it is necessary. He may have done so in order to avoid overwhelming beginning students with information that they would not need until later.

[17] Clavius, *Euclidis Elementorum,* 13*v*-14*r.* In Latin the terms are "paralellogramum," "complementa," and "reliqua."

[18] Both Commandino and Billingsley use the word "supplement" instead. (Commandino 25*r;* Billingsley, 53*r*). Clavius's use of complement instead of supplement and residual for the other parts of the parallelogram may have helped make his text more readable, but may not have helped the reader peruse other mathematician's texts. Barrozzi also uses the word "complement." (Francisco Barozzi, *Procli Diadochi Lycii Philosophi Platonici ac Mathematici Probatissimi in Primum Euclidis Elementorum librum Commentariorum* (Padua: Gratiosus Perchacinus, 1560), 262.) Billingsley notes that Peletier used both "supplement" and "complement," but describes their use such that "supplement" applies to Clavius's "complement," and "complement" applies to Clavius's "residual." (Billingsley, *Elements of Geometrie,* 53*v*; Jacques Peletier, *In Euclidis Elementa Geometrica,* 41.) It seems that there was no agreement on the terminology. Heath's early twentieth century translation describes the relevant parallelogram, but does not name it. A twenty-first century edition of Euclid based on Heath's translation uses the word "complement." (Ed. Dana Densmore, *Euclid's Elements all thirteen books complete in one volume: The Thomas L. Heath Translation*, Green Lion Press: Santa Fe, NM, 2002.) It seems that at least in English, Clavius's chosen terminology prevailed, so perhaps it was the more common of the two.

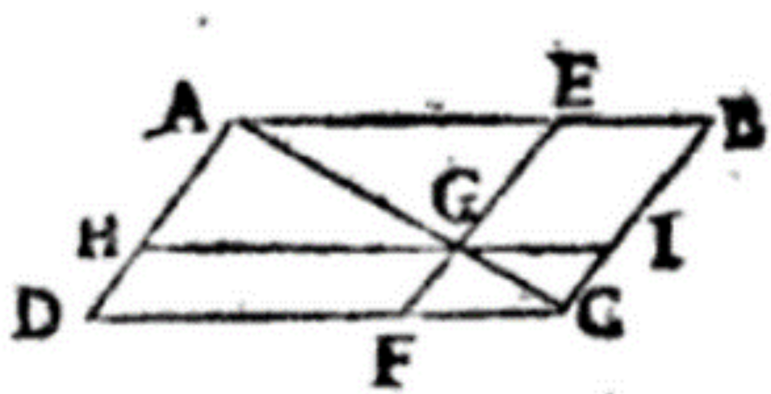

Figure 2: **Clavius's Diagram for Book One, Definition 36**

This diagram shows the residuals and complements of parallelogram ABCD cut around the diagonal AC. The residuals are AEGH and GICF. The complements are DHGF and GEBI.

through the proofs. The extra definitions in Clavius's text thus served to avoid the last minute introduction of new terms and the worse offense of propositions requiring the inference of an unstated definition.

Each author's project becomes even more apparent in his treatment of the additional books appended to *The Elements.* The first two books to be added to Euclid's original work, the fourteenth and fifteenth books in *The Elements*, were written by Hypsicles of Alexandria in the second century BCE, when Euclid's text was less than two hundred years old. These books extended the discussion of the Platonic solids that Euclid had begun in his last three books.[19] All three commentators considered here accepted Hypsicles' ancient additions as part of the Euclidean corpus. However, their varying treatments of a third supplemental book along with further additions to the fourteenth and fifteenth books and a treatise on non-Platonic solids, all written by François Flussas Candalla for his 1566 version, provide insight into each of their projects.[20] Commandino, whose text was based on the 1533 printed edition of Theon of Alexandria's fourth-century version of Euclid, simply ignored his French contemporary's work. This treatment is not surprising if we understand the

[19] The Platonic solids are the five solid figures that can be composed from equal, regular polygons. They are a tetrahedron, a cube, an octahedron, a dodecahedron, and an icosahedron.

[20] By his 1578 edition, Candalla had changed the treatise into two additional books written in the Euclidean style. Candalla's sixteenth book is the best known of these additions. (François Flussas Candalla, *Euclidis Megarensis mathematic clarissimi Elementa, libri XV* (Paris: Jacob du Puys, 1578), frontispiece.) Later authors, including Billingsley and Clavius, often included the sixteenth book which is often described as a direct continuation of the fourteenth and fifteenth books since all three treat the relationships of Platonic solids to one another. Billingsley also included the 1566 treatise as part of the sixteenth book. As far as I know, no one else included these texts as books within the Euclidean corpus. Since none of the original thirteen books treat non-Platonic solids, these books lack the topical continuity that the fourteenth through sixteenth books had with Euclid's work, which ends with the construction of the Platonic solids.

humanist's goal as a restorative project intended to revive, not build on, ancient mathematics.

In contrast, Billingsley's hope that craftsmen could derive profit from mathematics led him to include all of Candalla's additions. Because Billingsley could not predict precisely which piece of information would lead to profit, his own project was to supply his reader with as much material as possible, regardless of its temporal provenance. Indeed, when he introduced his translation of the Frenchman's version of the fourteenth book, Billingsley said that he chose to include it because it "containeth in it more Propositions than are found in Hypsicles, & also some of those propositions which Hypsicles hath, are by him somewhat otherwise demonstrated."[21] Since Candalla's additions more than doubled the number of propositions in both the fourteenth and the fifteenth books, by including them, Billingsley greatly expanded his readers' understanding of the relationships between the Platonic solids.[22] However, for the fourteenth book, Billingsley placed his translation of Candalla's version after his translation of Hypsicles's version as a second fourteenth book. By providing two distinct versions of the fourteenth book, Billingsley allowed Candalla's work to appear to be nothing more than a useful addition to the ancient text. However, when he moved on to the fifteenth book, he only included Candalla's version. A possible explanation for this arrangement is that he believed that the comparison between the

[21] Billingsley, *Elements of Geometrie,* 421*v*.
[22] Candalla has nineteen propositions in the fourteenth book and twenty-one in the fifteenth. (François Flussas Candalla, *Euclidis Megarensis mathematici clarissimi Elementa, libri XV* (Paris: Ioannes Royerius, 1566), 176*r* – 191*v*). As it was presented in Commmandino's translation, the Greek text, includes seven propositions in the fourteenth book and five in the fifteenth, far fewer than half of the number found in Candalla's text (Commandino, *Euclidis Elementorum,* 243*v* – 255*v*).

ancient and modern texts made possible by the doubled fourteenth book showed the modern version to be a more complete study of Platonic solids and absolved him from the need to supply the ancient version of the fifteenth book.

Billingsley's desire to present a comprehensive study of Platonic solids is even more pronounced in his defense for adding Candalla's sixteenth book to his own commentary. There he argued that "the sixteenth book [would] leave the reader wanting nothing conducing to the perfection of Euclides Elements."[23] In Billingsley's view, Candalla had completed what Euclid and Hypsicles had begun, thereby giving the reader a complete examination of Platonic solids. It is not clear if Billingsley's argument for the "perfection of Euclides Elements" included Candalla's treatise on non-Platonic solids, which he placed at the end of the sixteenth book without any further comment. It seems that he simply offered it to his readers because Candalla had included it in his own version and Billingsley was striving for completeness. Eager to provide his reader as many avenues to pursue the study of mathematics as possible, he had no reason to leave out any relevant and potentially useful study. Furthermore, since most solids encountered in the physical world are non-Platonic, a treatise that begins a study of such solids could aid those seeking practical applications of mathematics, and Billingsley might have felt it was a necessary component to the study of solid geometry begun by Euclid.

Clavius makes an interesting contrast. While he included many of Candalla's additions in his own work, he did not incorporate them into the text as fully as had

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[23] Billingsley, *Elements of Geometrie,* 445*v.*

Billingsley. Instead, he left clear distinctions between the ancient work and the modern additions allowing him to show mathematics to be an evolving discipline with ancient roots that, if restored, provided the potential for continued progress at the hands of new mathematicians.[24] Even the title of his second volume, *Euclidis posteriores libri sex X. ad XV. Accessit XVI de solidorum regularum comparatione,* specifically denotes the sixteenth book as an appendix to the Euclidean corpus rather than treating it as part of the main text. Furthermore, Clavius did not add any of his own commentary to the new book, indicating that, while he regarded it as a valuable extension of the Euclidean text, it was non-essential to a study of *The Elements*. The restorative component of Clavius's goal is even more pronounced in his exclusion of the treatise on the non-Platonic solids, which are not discussed in the ancient fifteen books. Even though he was willing to include the sixteenth book as a relevant supplement, he was not willing to extend his commentary beyond the scope of the ancient text with a discussion of non-Platonic solids, a subject that could be taught at another point of his curriculum, possibly the unit he included on the measurement of solid bodies.[25]

However, in his treatment of Candalla's additions to the fourteenth and fifteenth books, which he regarded as part of the Euclidean corpus, Clavius made the potential for forward progress of mathematics clear. In these books, he included

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[24] See Chapter 1 for a discussion of Clavius's vision of the history of mathematics as a continual evolution.
[25] Christopher Clavius, "Ordo servandus in addiscendis disciplinis mathematicis," in Ladislaus Lukacs (ed), *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 112.

Candalla's supplemental propositions as part of his own project to complete the comparisons between Platonic solids begun by Hypsicles. While Billingsley had simply translated Candalla's books, Clavius wrote his own versions in which he included the French mathematician's work and added numerous corollaries, lemmas, scholia, and even new propositions of his own. However, expanding the fourteenth and fifteenth books was not simply a means to provide his students with more information than the ancient texts contained; it was an opportunity to demonstrate how contemporary mathematicians could build on ancient knowledge. At the start of the fourteenth book, Clavius explained that Hypsicles only examined icosahedrons and dodecahedrons, but that he, like Campanus and Candalla before him, believed that a study comparing all of the regular solids to one another was necessary to complete the extension of Euclid's work that Hypsicles had begun.[26]

Clavius's presentation of the propositions in these books also draws the reader's attention to the temporal evolution of the Euclidean corpus and the cumulative nature of mathematical knowledge that he had established in his prolegomenon where he said that mathematics proceeded "little by little from imperfect to more perfect."[27] Within the fourteenth and fifteenth books, Clavius indicated the presence and order of each proposition in the Greek text and Campanus's

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[26] Clavius, *Euclidis Posteriores libri sex a X ad XV. Accessit XVI de solidorum regularium comparatione* (Rome: Vicentium Accoltum, 1574), 225*r*. "Qua in re maximo nobis adiumento fuisse Campanum & Franciscum Flussatem Candallam, qui diligentem operam, & sedulam in hoc negotio collocavit, non negamus."

[27] Clavius, *Euclidis Elementorum*, b*r*. "Immo vero singulas nequaquam summam adeptas esse perfeectionem statim ab initio, sed paulatim eas ab imperfectis ad perfectiora processisse, memoriae quoque proditum est."

text in the margin.[28] Thus, the reader could reconstruct Hypsicles' ancient text and Campanus's medieval text from subsets of Clavius's work. Candalla's text received no such recognition even though Clavius had acknowledged that the additions made by both Campanus and Candalla were tremendously useful in his own efforts in this project.[29] While these notations served the practical purpose of enabling readers to identify the same propositions in other versions of Euclid's *Elements,* whether they were based on Greek or medieval sources, the marginal indications of Greek and medieval works clearly illustrate that the project begun by ancient scholars, had been built upon by medieval scholars, and was now completed by Clavius.

Aids for the Reader: Presenting Additions to *The Elements*

Although only readers' comments could show how successful authors were in conveying their visions of mathematics, the presentation of the Euclidean content and additions to the ancient text is at least as revealing of authors' goals as is the content itself. One particularly interesting method of presentation is the index of additions that both Commandino and Clavius employ before the first book.[30] These indices were lists of demonstrations that the commentators had added to Euclid's text. By noting what material was added to the text by later mathematicians, they enabled the

[28] In the fourteenth book, Billingsley also uses marginal notations to indicate the place of propositions in Campanus's book. He does not continue this practice in the fifteenth book.
[29] See note 26.
[30] Clavius titles his "Index problematum ac theorematum, quae praeter ea, quae continentur in Euclidis propositionibus, in his elementorum libris demonstrantur." (Clavius, *Euclidis Elementorum,* c*r.*) Commandino conveys the same message with the shorter "Index eorum, quae in his libris demonstrantur praeter ea, quae Euclidis sunt." (Commandino, *Euclidis Elementorum,* **2*v.*) It should be noted that many of the additions in both texts were attributed to ancient mathematicians. Lemmas and corollaries were listed in both indices.

humanist authors to preserve the authenticity of the original work: everything that was not named in the index. In fact, precisely because the indices left out the original Euclidean enunciations (i.e. the statements of the definitions, postulates, axioms, and propositions), the original text is shown to be more valuable than the later additions. Euclid's enunciations were so important that they were to be learned in full, whereas the indexed additions could be studied "in parts" as the reader deemed them useful, which Ann Blair has argued was a common use for indices in the early modern period.[31] Billingsley, who was more interested in the future profit that could be derived from *The Elements* than in the ancient provenance of the text, did not include an index, leaving his reader without recourse to easily study the later additions more selectively than the original text.

However, even though both Commandino and Clavius sought to preserve the ancient work through providing indices of additions, a brief examination of the content of their indices illuminates the differences between their goals. From their presentation of the additions, it becomes clear that Clavius gave more value to the additions made to the original text than Commandino did. While Clavius always provided the full enunciation of an addition, Commandino frequently shortened his index entries by merely describing the new concept's relationship to an existing proposition. For example, both authors included the converse the proposition which states that the square drawn on the side of a triangle that subtends an acute angle is

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[31] Ann Blair, "Reading Strategies for Coping with Information Overload, ca. 1550-1700," *Journal of the History of Ideas* 64 (2003): 11-28.

less than the sum of the squares on the other two sides.[32] Commandino's index entry say, "The converse of proposition XIII."[33] Clavius's says, "If a square, which is described on one side of a triangle, is less than the [sum of the] squares that are described on the remaining sides of the triangles, then the angle comprehended between the two remaining sides of the triangles is acute."[34] By replacing the content of later additions with the relationship between the addition and the ancient text, Commandino emphasized the completeness of Euclid's original work. In contrast, Clavius, by presenting each addition as its own claim, showed the ancient text to be foundational to independent claims demonstrated by later mathematicians.

Of course, unless the reader was eager to flip back and forth from the body of the text to the index, the presence of an index at the beginning of a text would not be much use to a student needing to identify the shifts from translation to commentary as he worked through any given proposition. Commentators often relied on a variety of methods, including marginal notations, in-text citations, and changes in typeface, to indicate those shifts. In these notations, the commentators' various goals become clear. Billingsley's efforts to provide his reader with as much material relevant to the study of *The Elements* as he could are evident through his use of notes to indicate a breadth of sources from all time periods. His readers were the beneficiaries of his efforts to anthologize additions to Euclid's text. In contrast, Commandino's marginal

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[32] This proposition is Book II, proposition 13.
[33] Commandino, *Euclidis Elementorum,* **2*v*. "Propositionis XIII conversa."
[34] Clavius, *Euclidis Elementorum,* c4*r*. "Si quadratum, quod ab uno laterum trianguli describitur, minus sit eis, quae a reliquis trianguli lateribus describuntur quadratis: Angulus comprehensus sub reliquis duobus trianguli lateribus, acutus est."

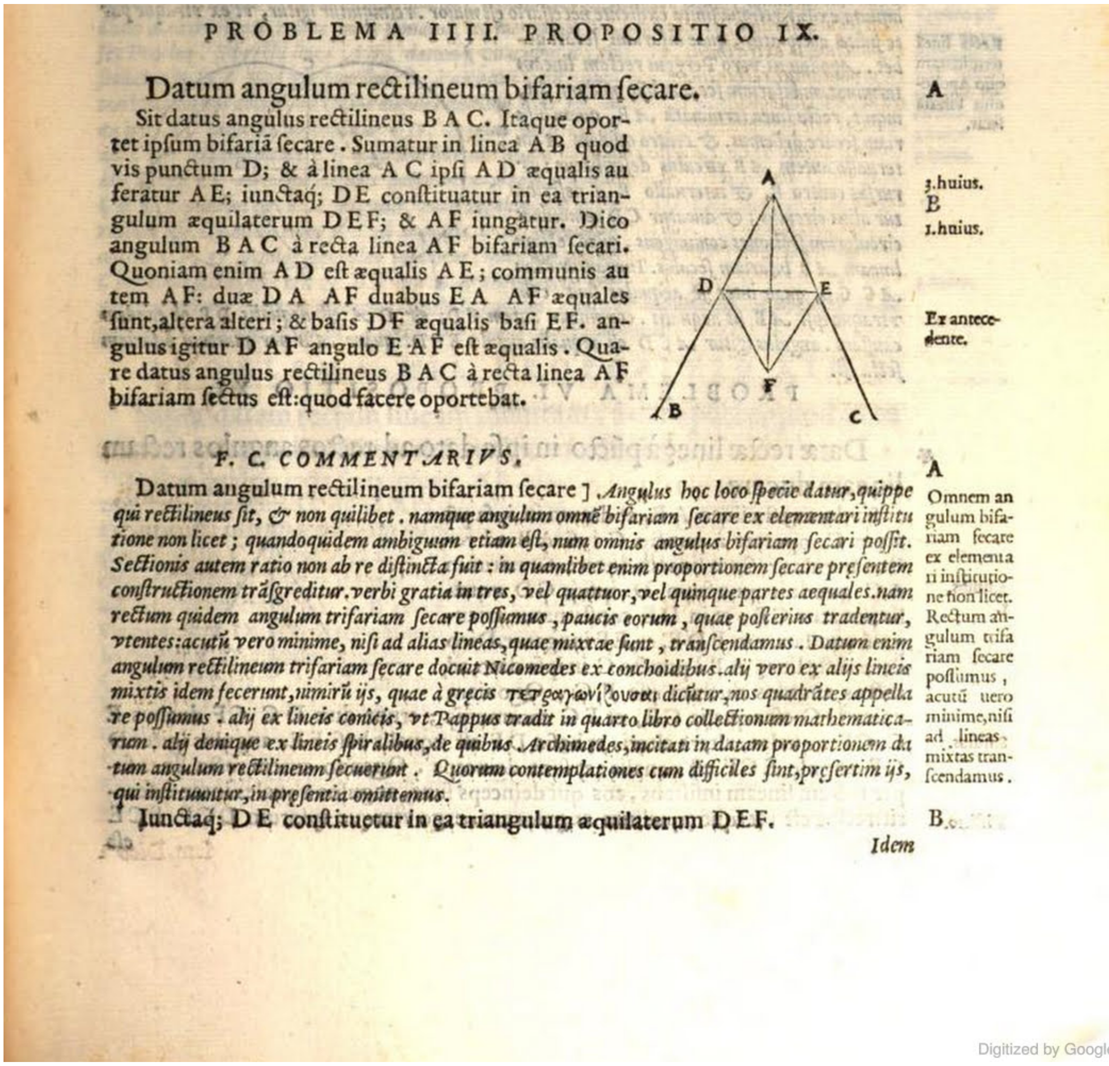
notations indicate that, with few exceptions, the addenda he included were attributed to ancient mathematicians, suggesting that his goal was the humanist one of restoring ancient knowledge. Clavius's notes were designed to illustrate the continual evolution of mathematics from ancient to modern times. His in-text citations reveal that most of the entries in his index, which lists more than double the addenda found in Commandino's, were attributed to medieval or modern authors, including himself.[35] Furthermore, he included marginal notes indicating Campanus's ordering of the propositions. On one level, these notes were clearly a pedagogical tool. Because there were versions of *The Elements* based on both Greek and Arabic sources available in the sixteenth century, references to Euclidean propositions in more advanced mathematical texts could be based on either Theon's or Campanus's ordering. Thus, because, as he expressed in his letter to the reader, he hoped his book would serve as a handbook for future study, Clavius provided the alternative numbering as a means to give his students the ability to recognize propositions cited from any source.[36] However, they also served to validate the medieval text as a source for sixteenth-century mathematicians. While Clavius himself had based his order of the propositions on Greek sources to create a more accurate representation of Euclid's work, including Campanus's numbering suggested that the medieval version had value

[35] Commandino's index lists 200 additional demonstrations. Clavius includes 585. The sources of the additions are not noted in the indices, but are clear within the texts.
[36] Clavius, *Euclidis Elementorum,*a5*v*. "Nam cum Euclides, propter singularem utilitatem, instar enchiridii, manibus semper debeat circumgestari, neque unquam deponi ab his, qui fructum aliquem serium ex hoc suavi Matheseos studio capere volunt, in eoque progredi; id vero in hunc diem exemplaribus omnibus maiore forma impressis, necdum factum videamus; hoc nostra editio certe, si nhil aliud, attulerit commodi, atque emolumenti."

as a resource for contemporary mathematicians, and made it possible for his readers to search for Campanus's changes and additions to various Euclidean propositions. Thus, Clavius's commentary was not just a translation of ancient mathematics, it was also an extension of medieval versions of *The Elements*.

Besides their role in identifying additions and changes made to the ancient text, marginal notes helped to clarify the Euclidean proofs. In Commandino's text such notations were designed to minimize the need he had to change or elaborate on the ancient demonstrations by flagging difficult passages for further commentary. The marginal notations are themselves just symbols, asterisks or letters that appear next to particularly difficult passages of Euclid's demonstrations. In his commentary following the demonstration, Commandino provided an explanation of each flagged passage with the same symbols reappearing in the margins alongside the relevant section of commentary. See Figure 3 for an example. Much like footnotes, these marginal notations allowed the reader to jump between the demonstration and the commentary in order to make the proof easier to understand. Commandino was thus able to improve the clarity of ancient demonstrations without damaging the integrity of the Greek text either by changing the ancient proof, as Clavius did, or by rewriting it completely in his commentary, as Billingsley did.

Where Commandino's notes furthered his restorative project by serving to preserve the ancient text, Billingsley's notes, in keeping with his desire to provide a text to the non-Latin reading merchants and craftsmen of England, served to



PROBLEMA IIII. PROPOSITIO IX.

Datum angulum rectilineum bifariam ſecare.

Sit datus angulus rectilineus B A C. Itaque oportet ipſum bifariã ſecare. Sumatur in linea A B quodvis punctum D; & à linea A C ipſi A D æqualis auferatur A E; iunctaq; D E conſtituatur in ea triangulum æquilaterum D E F; & A F iungatur. Dico angulum B A C à recta linea A F bifariam ſecari. Quoniam enim A D eſt æqualis A E; communis autem A F: duæ D A A F duabus E A A F æquales ſunt, altera alteri; & baſis D F æqualis baſi E F. angulus igitur D A F angulo E A F eſt æqualis. Quare datus angulus rectilineus B A C à recta linea A F bifariam ſectus eſt: quod facere oportebat.

A

3. huius.
B
1. huius.

Ex antecedente.

F. C. COMMENTARIVS.

Datum angulum rectilineum bifariam ſecare] *Angulus hoc loco ſpecie datur, quippe qui rectilineus ſit, & non quilibet. namque angulum omnẽ bifariam ſecare ex elementari inſtitutione non licet; quandoquidem ambiguum etiam eſt, num omnis angulus bifariam ſecari poſſit. Sectionis autem ratio non ab re diſtincta fuit: in quamlibet enim proportionem ſecare preſentem conſtructionem trãſgreditur. verbi gratia in tres, vel quattuor, vel quinque partes aequales. nam rectum quidem angulum trifariam ſecare poſſumus, paucis eorum, quae poſterius tradentur, vtentes: acutũ vero minime, niſi ad alias lineas, quae mixtae ſunt, tranſcendamus. Datum enim angulum rectilineum trifariam ſecare docuit Nicomedes ex conchoidibus. alij vero ex alijs lineis mixtis idem fecerunt, nimirũ ijs, quae à grecis τετραγωνίζουσαι dicuntur, nos quadrãtes appellare poſſumus. alij ex lineis conicis, vt Pappus tradit in quarto libro collectionum mathematicarum. alij denique ex lineis ſpiralibus, de quibus Archimedes, incitati in datam proportionem datum angulum rectilineum ſecuerunt. Quorum contemplationes cum difficiles ſint, preſertim ijs, qui inſtituuntur, in preſentia omittemus.*

A
Omnem angulum bifariam ſecare ex elementari inſtitutione non licet. Rectum angulum trifariam ſecare poſſumus, acutũ uero minime, niſi ad lineas mixtas tranſcendamus.

Iunctaq; D E conſtituetur in ea triangulum æquilaterum D E F.

B.

Idem

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Figure 3: Commandino's "Footnotes"

This is Commandino's ninth proposition in Book One. The capital letters "A" and "B" appear in the margin next to the demonstration for the proposition. These function as footnotes. The same letters reappear in the margins near the commentary, marking the beginning of a section. Each section begins with a line of text repeated from the demonstration. Commandino's commentary serves to clarify or expand on that line. The commentary continues on the next page for item "B."

emphasize the physical nature of pure geometry.[37] In the English text, marginal notations were used to identify the two distinct parts of a Euclidean proof: the construction, which builds whatever diagram is necessary for the proof, and the demonstration, which shows that the required claim is indeed true. Billingsley labeled the starts of both parts of Euclidean proofs for every proposition. In so doing, he drew the reader's attention to the constructions and the resulting diagrams as independent from the logic-based demonstration. Thus, the diagrams serve as the concrete objects of the demonstrations.

Clavius's annotations indicated the past propositions on which various statements within a proof depended. By calling attention to the cumulative and self-contained nature of Euclid's *Elements,* these clarifying notes served to show how mathematics was developed step-by-step from first principles to advanced theorems, which was precisely the certain method of mathematics that Clavius argued earned the discipline a status comparable to other philosophical studies.[38] For example, if part of a demonstration relied on the Pythagorean Theorem, Clavius included a note next to

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[37] For a discussion of Billingsley's intended audience, see chapter 2. In his letter to the reader he claimed to have translated *The Elements* for "good wittes" and men "of all degrees" who wanted to pursue the mathematical arts, as well as his hope that English readers would have the same access to ancient texts that enabled "the inventions of straunge and wonderfull thinges" already taking place on the Continent. Billingsley, *The Elements of Geometrie,* ii*v* – ij*r.*

[38] See chapter 1 for a discussion of Clavius's arguments about the certainty of mathematics. Clavius claimed that the certainty of mathematics arose from its method: never accepting something as true unless it could be proven from previously demonstrated claims. Clavius, *Euclidis Elementorum,* b2*r,* "Quod quam longe a Mathematicis demonstrationibus absit, neminem latere existimo. Theoremata enim Euclidis, caeterorumque Mathematicorum, eandem hodie, quam ante tot annos, in scholis retinent veritatis, puritatem, rerum certitudinem, demonstrationum robur, ac firmitatem. … Cum igitur disciplinae Mathematicae veritatem adeo expetant, adamant, excolantque, ut non solum nihil, quod sit falsum, verum etiam nihil, quod tantum probabile existat, nihil denique admittant, quod certissimis demonstrationibus non confirment, corroborentque, dubium esse non potest quin eis primus locus inter alias scientia omnes sit concedendus."

the claim that read "47.primi" to indicate the forty-seventh proposition in the first book, which is the Pythagorean Theorem. The reader then knew that the claim was valid, even if it was not obviously true, because it followed from the noted proposition. Should he have had any questions, he could have gone back and reviewed that proposition to make sense of the challenging claim. Thus, Clavius showed that Euclidean geometry was cumulative – the propositions only depended on prior enunciations – and that the study was self-contained. These were the only notes Clavius included in the margins of the proofs.[39] While the practice of referencing previous demonstrations from Euclid's text was *de rigueur* among commentators on *The Elements,* unlike other commentators, Clavius consistently placed these references in the margins, which created a visual emphasis on the certain method of geometry.[40] A reader could quickly glance through the margins and see how the proofs build on themselves as the number and variety of references increases as the text progresses.

Aids to the Reader: Understanding the Structure of *The Elements*

Each commentator studied here provided some sort of guideline to the structure of the Euclidean text. Like their guides to the additions, these guidelines,

[39] Clavius was able to avoid using other clarifying notes on the demonstrations because he rewrote the parts of the proof that he found to be unclear. The notes indicating Campanus's ordering of propositions accompanied the enunciations, not the proofs, and, as mentioned above, his citations to various sources for his additions were made in the text itself.

[40] Clavius explained that he placed the notes in the margin for the practical reason of avoiding interrupting the proof, which makes the visualization of the self-contained nature of *The Elements* a happy side-effect. Future research into readers of *The Elements* is needed to understand how readers reacted to having all of the citations in the margin or within the text, which is where Commandino and Billingsley often placed theirs. (Clavius, *Euclidis Elementorum,* 21*r*.) "Deinde ne cursus demonstrationum interrumperetur, citavimus principia, & propostiones Euclidis in margine quae quidem citationes intelligendae sunt modo infrascripto."

found in the authors' descriptions of the Euclidean books, offer insight into how each commentator perceived and wished to present the text. Both Clavius and Commandino supplied general outlines of *The Elements* in their prolegomenons. Clavius's outline clearly illustrates his pedagogical purpose. He did not treat each book individually and, instead, presented a four-part division of the text in connection with a similar division of the study of geometry. His division classifies the first six books as the study of plane geometry. (This division is further split into the study of individual planar figures and equality between them in the first four books, the study of proportions between magnitudes in the fifth book, and proportion between planar figures in the sixth book.) The next three books comprise the Euclidean examination of number theory, and the tenth book is its own unit on commensurability. The remaining five books are the study of solid geometry.[41] For pedagogical purposes, a clear explanation of the groupings of books was likely more useful than descriptions of individual books because it provided teachers with a way to divide *The Elements* in their curricula based on what kind of mathematics was going to be taught next. Indeed, in his own 1581 curricula, Clavius split up the units on Euclid along the lines of the division outlined in his prolegomenon, and each section preceded a branch of mathematics that could use that piece of Euclidean geometry. For example, the first four books preceded the study of Sacrobosco's *Sphere,* whose discussion of stellar positions requires plane geometry. Likewise, the study of number theory preceded the

[41] Clavius, *Euclidis Elementorum,* b5*v* – b6*r*.

study of algebra, in which cossic numbers served as placeholders for unknown quantities in numerical relationships.[42]

In contrast, Commandino's outline allowed the ancient text to speak for itself, emphasizing the inherent value of restoring ancient knowledge. In his prolegomenon, he simply described the structure of the work by listing the topics each of the fifteen books would cover.[43] For example, in his longest description he says:

> In the first book, one may clearly see that it treats rectilinear figures, triangles and parallelograms. And first it treats the creation and properties of triangles, then comparing them to each other by angles and then by sides. Next by introducing the properties of parallel lines, it goes to parallelograms, and declares their creation and demonstrates traits which are in them. Afterwards, it shows the fellowship of triangles and parallelograms, & by what means a parallelogram can be made equal to a triangle. Then the same is done for those squares which are described on the sides of right triangles, which study considers the proportion between that square described on the side that subtends the right angle and those which are described on the sides that comprehend the right angle.[44]

His matter-of-fact approach allowed the reader to know what to expect in each of Euclid's books, but, unlike Clavius, he gave little indication of how best the topics fit together or how to approach the study.

In addition to his prolegomenon's description of the books of *The Elements,* Commandino included scholia of one to two pages in length before the fifth, tenth and

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[42] Clavius, "Ordo servandus," 110-111.
[43] Commandino, *Euclidis Elementorum,* ***v* - **2*r*.
[44] Ibid., ***v*. "In primo igitur libro tractat de rectilineis figuris, videlicet de triangulis, ac parallelogrammis. Et primum triangulorum ortus, proprietatesque tradit, tum iuxta angulos, tum iuxta latera; ipsa inter se se comparans. Deinde parallelarum proprietates interiiciens ad parallelogramma transit, eorumque, ortum declarat, & symptomata, quae in ipsis sunt, demonstrat. Postea triangulorum, parallelogrammorumque, communicationem ostendit, & quo nam pacto parallelogrammum fiat aequale triangulo. Denique de iis, quae in triangulis rectangulis a lateribus describuntur, quadratis quam habeat proportionem quod a subtendente rectum angulum describitur ad ea, quae comprehendetiubs ipsum fiunt."

eleventh books to establish the central position of Euclid's work within the ancient study of mathematics. In these passages, he cited Eudoxus, Apollonius, and Pythagoras as the original authors of many of the Euclidean theorems, and he offered explanations of relevant mathematical ideas those mathematicians developed from their theorems. For example, in his scholion preceding the tenth book, the study of commensurable quantities, he outlined Apollonius's division of numbers into thirteen kinds and discussed the Pythagorean approach to commensurability.

The placement of the scholia is significant to such an image of Euclid's text because each of the three books following a scholion begins a new topic within geometry which suggested links between geometry and another mathematical topic. The fifth book treats proportionality, or the relationship between two quantities. As Commandino told his reader, proportionality is "common to geometry, arithmetic, music, and all simply mathematical disciplines."[45] The tenth book treats commensurability, which ties magnitude to number, i.e. tying geometry to arithmetic. The eleventh book begins the study of solid geometry, which Commandino notes was considered its own branch of study, called stereometry, by Plato.[46] However, it must be noted that these scholia were not intended to introduce new topics within *The Elements,* but instead to position geometry within mathematics. If the scholia had simply been meant to introduce new topics, Commandino surely would have included one before the seventh book, the first book on number theory. Instead, he included

[45] Ibid., 56*v*. "In quinto libro propositum est de analogiis tractare; hic enim liber communis est geometriae, arithmeticae, musicae, & omni simpliciter mathematicae disciplinae."
[46] Ibid., 188*v*. "Antiqui planorum cognitionem a scientia solidorum distinxerunt. Etenim illam geometriam appellarunt, ut etiam Plato ostendit in politicis; hanc autem stereometriam."

one for the tenth book, which studies commensurability in which magnitude and number are combined, allowing him to discuss the connection between geometry and number and to position *The Elements* as an ancient foundation to a complete understanding of number.

While Billingsley did not provide an outline of the entire text, at the start of each book he introduced the topic of study in a brief paragraph before the Euclidean text began. In these paragraphs, he repeatedly reminded his readers of the utility of the abstract concepts and demonstrations contained in *The Elements* to the mathematical arts. For example, the first book on the "grounds of Geometry" has the distinction of being the most general and necessary to all future study. In the second book on rectilinear figures, Billingsley was a little more specific about potential applications. He claimed that this could be used by the arithmetician to "gather many compendious rules of reckoning."[47] The description of the second book also promises two specific propositions, "one of an obtuse angled triangle, and the other of an acute: which with the ayde of the 47 proposition of the first book of Euclide, which is of a rectangle triangle, of how great force and profite they are in matters of astronomy, they knowe which have travayled in that arte."[48] These descriptions, although they lacked specific examples, gestured at Euclid's *Elements* as foundational to various branches of mixed mathematics and, thus, depicted the work as the key to developing concrete applications of mathematics.

[47] Billingsley, *Elements of Geometrie,* 1.
[48] Ibid., 60.

Like Billingsley, Clavius included introductory remarks to most of the Euclidean books. However, while Commandino's scholia focused on the value of *The Elements* as the foundation of ancient mathematical study and Billingsley's opening paragraphs emphasized the role of *The Elements* in creating future profit, Clavius's introductory remarks were focused on orienting the reader by alerting him to the beginning of a new topic of study. In the nine books (Books 1-5, 7, 10, 11, and 14) for which Clavius included brief remarks within the commentary on his first definition, he devoted only a sentence or two to naming and defining the topic of study about to be begun. Each of the books that has no such description is easily seen as a continuation of the previous book.[49] However, Clavius still offered some comments to encourage the reader to continue his study through promising the utility of geometry to further study. For example, in the description of the tenth book Clavius asserted that commensurability and incommensurability were necessary to an understanding of arithmetic and to algebra, particularly in the calculation of roots.[50] However, he neither tied further mathematical study to the ancient development of mathematics nor promised future profit through practical application of mathematics to contemporary

[49] The sixth book treats proportions of geometric figures, continuing the fifth book's more general study of proportion. The eighth and ninth books continue the seventh book's discussion of number theory. The twelfth and thirteenth books continue the study of solids introduced in the eleventh book, and the fifteenth book is the second addition of Hypsicles and continues to study the relationships of Platonic solids to one another. I have not included the sixteenth book in this discussion because Clavius treated it as an addition to rather than an integral part of Euclid. It did receive an introductory paragraph, but that section primarily serves to justify the inclusion of the modern text.

[50] Clavius, *Euclidis Posteriores*, 2*r*-*v*. "Neque in eorum possum sententiam ire, qui putant ad eius intelligentiam esse necessariam eam partem Arithmetices, quae de radicibus numerorum, tam rationalibus quam irrationalibus, ut vocant sermonem instituit: Immo contra persuasum mihi prorsus habeo, cognitionem perfectam illius partis Arithmetices pendere ex hoc 10 librum tantum abest, ut existemem, tractationem illa radicum requiri, ut facilius hic liber intelligatur."

mathematical arts. Instead, he simply allowed *The Elements* to appear as foundational to a sixteenth-century mathematics curriculum. Indeed one of the justifications he gave for writing his commentary was that he perceived a need for a version of Euclid's text that was easier to read, especially for those just beginning their mathematical studies.[51]

Clavius's consideration of the role of Euclid's text in his curriculum is most evident in the longest of his opening comments, the introduction to the eleventh book. There he provided a summary of what the previous ten books had covered and indicated what books thirteen, fourteen, and fifteen would cover as the payoff for the study of books eleven and twelve.[52] Because he taught *The Elements* in segments at different times in a mathematics course, the summary may well have been intended to reorient the reader to Euclidean geometry after his course of study had taken him to other subjects for some time. In Clavius's 1581 curriculum, the eleventh book of *The Elements* was the starting point for second-year mathematics students, meaning that several months may have passed since students had studied any mathematics, let alone

[51] Clavius, *Euclidis Elementorum,* a6*r*. "Ita enim, nostra sententia, Euclides facilius a studiosis, iis praesertim, qui ceu tyrones, haec Mathematica studia nunc primum auspicantur, ac maiore voluptate, utilitateque cognoscetur."
[52] Clavius, *Euclidis Posteriores,* 117*r*. "Postquam Euclides in prioribus sex libris abunde de ea Geometria parte disseruit, quae circa plana versatur, sibique nomen Geometriae, tanquam proprium, usurpavit; In subsequentibus vero tribus diligenter ea de passionibus numerorum docuit, quae necessaria videbantur ad intelligendam doctrinam linearum commensurabilium, ac incommensurabilium, quas idcirco luculentissime in libro decimo deinde exposuit, ut constructio, atque natrua quinque corporum regularium, de quibus in postremis tribus libris, nimirum in 13, 14, & 15 subtilissime disseritur, perfectius posset cognosci: Nunc tandem aggreditur in hoc libro undecimo eam partem Geometriae, quae corpora, sive solida considerat, proprioque vocabulo Stereometria est appellata."

Euclid.[53] The opening summary could thus serve as a reminder of the previous year's study and as a promise of the new year's syllabus.

The Foundations of Geometry: Definitions in Book I

While it is to be expected that scholars with different agendas would create unique presentations of *The Elements* by tailoring the structure of their commentaries and carefully selecting additions to the ancient text that were in keeping with their individual goals, it is somewhat surprising that the contents of the core texts are far from uniform. After all, mathematicians in the sixteenth century often professed that *The Elements* had not been substantially changed since it was written.[54] Nevertheless, while the differences found in the main text of the three versions studied here are small, they are able to bring Euclid's mathematics into the service of the commentators' various projects.

Already in the enunciation of the first definition, the authors diverge slightly from one another. Clavius's formulation, which appears in most Latin texts, is the simplest, "A point is that for which there is no part."[55] Billingsley's is not much

[53] Christopher Clavius, "Ordo servandus in addiscendis disciplinis mathematicis (1581)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 110-115. While Clavius did not write his curriculum until 1581, when his first edition of *The Elements* was published in 1574, he had been teaching mathematics at the Collegio Romano for over a decade. It is possible that he was already using a curriculum similar to that which he would later provide to the Jesuits' curriculum committee. In the shortest curriculum in which he outlined how much time should be spent on each section, he suggested that books 5 and 6 of Euclid be taught in the seven weeks between Easter and Pentecost of the first year. Perspective and horology were taught from Pentecost until the end of the first year. No further study of Euclid was required until books 11 and 12 were introduced at the start of the second year. Those two books would have been taught from then until Christmas (p. 114).

[54] See epigraph and Chapter 1.

[55] Clavius, *Euclidis Elementorum,* 1*r*. "Punctum est, cuius pars nulla est."

different, but he does include an alternative name. He says, "A signe or point is that, which hath no part."[56] Commandino provides two formulations: "A point is that of which there is no part, or that which has no magnitude."[57] All three definitions clearly express the same idea, but, given the commentators' assertions that mathematics' certainty allowed it to survive unchanged across the centuries, should it not be the case that all authors express complete agreement on the definition that begins the study of mathematics?

Within the first definition, Billingsley's and Commandino's adjustments to the standard formulation found in Clavius's text respectively emphasize the physical and the abstract components of mathematics, thereby aligning mathematics with their distinct goals. Billingsley's use of the alternative name "signe," from the Latin root "signum," conveys the materiality of a point, allowing even the foundation of pure mathematics to be an object that could be manipulated by enterprising artisans.[58] Because "sign" and its Latin root, "signum," are used primarily to denote symbols, including this word choice suggests that a dot drawn to represent a point can, for all practical purposes, be accepted as an actual point. Furthermore, the use of the word "sign" to mean "point" was not common in English. According to the *Oxford English Dictionary*, the geometrical use of the word "point" appeared in English as early as 1398, and in 1551, Robert Recorde used "poynt or prycke" to convey the geometrical

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[56] Billingsley, *Elements of Geometrie,* 1.

[57] Commandino, *Euclidis Elementorum,* 1. "Punctum est, cuius nulla est pars, vel quod magnitudinem nullam habet."

[58] Billingsley was not the first to use this term in *The Elements*. Candalla, whom Billingsley cited frequently, used "signum" instead of "punctum" in his definition. See Candalla, *Euclidis Megarensis mathematici* (1566)*,* 1.

entity in his *Pathway to Knowledge*.[59] While both Recorde's use of the word "prycke" and Billingsley's use of the word "signe" allow a point to have a physical nature, Billingsley's term makes that nature more explicit. Recorde's word suggests a means to produce a point (pricking a pen to the page), but it does not explicitly designate the resulting pinprick as a visible, manipulable point. Billingsley's use of the word "signe" effectively equates the concrete symbol, a dot (which may be much larger than a pin-prick), with the abstract concept of a point.

In contrast, Commandino's definition introduced a point as the negation of magnitude, thereby tying the definition into an abstract discussion of the nature of the foundation of geometry, the study of magnitudes. In the quadrivium, the most common division of mathematics in the early modern period, geometry was defined as the study of continuous quantity, i.e. magnitudes, as opposed to discrete quantity, i.e. numbers.[60] As a continuous quantity, magnitude has dimension and, so, can be measured. Defined as an entity without magnitude, then, a point cannot be measured. Since all material is measurable, Commandino denied the possibility of representing a point in any material way. For him, it was simply an idea from which the definition of magnitude as a measurable entity could become clear.

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[59] *Oxford English Dictionary,* 3rd ed. (online version, updated 2006), s.v "point." http://www.oed.com/view/Entry/146609?rskey=4X3mbE&result=1&isAdvanced=false#eid; Robert Recorde, *Pathway to Knowledge*, (London: Reynold Wolfe, 1551), A*r.*

[60] Note that numbers as defined in the quadrivium included what we call rational numbers, not all real numbers. Furthermore, the rational numbers are understood as they can be generated from whole numbers. Thus, one, or unity, serves as the first number from which all other numbers are generated, which excludes 0. Irrational numbers are discussed in the tenth book of Euclid, but they are developed geometrically as the measurements of magnitudes. Today, real numbers are often represented on a continuous line, and rational numbers are discrete points along such a line.

In their commentaries on the definition, the authors cemented their visions of a "point." Billingsley's commentary begins by distinguishing a point from "quantitie (whereof Geometry entreateth)."[61] Unlike quantity, whose nature – according to Billingsley – was to be divided, a point was indivisible. But, while a point was not properly the subject of geometry, it was still a physical entity; specifically, it was "the least thing that by minde and understanding can be imagined." To help the reader imagine a point, Billingsley provided an example in the margin of his text. "Point A in the margent" was presented as a point, not just a representation thereof.[62] Furthermore, in the second part of his commentary, Billingsley explicitly claimed that a point had material. This section began with an alternative definition, which Billingsley attributed to Pythagoras: a point is "unitie which hath position."[63] Just as unity is the source of all number, a point is "the beginnyng of magnitude," and like magnitudes, it can be assigned to a position, and, therefore, must be material.[64]

Commandino's discussion of a point denies the geometric concept the possibility of physical reality. He began his commentary with the claim that, as the foundation of Euclid's study of geometry, a point itself must be the negation of the objects of that study, magnitudes.[65] Commandino credited this argument to Proclus's

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[61] Billingsley, *Elements of Geometrie,* 1*r*.
[62] Ibid., 1*r*. The word "thing" needs to be understood as a physical entity in this context because the word "least" conveys an ability to measure things and compare their sizes.
[63] Ibid., 1*r*. Billingsley's commentary on the Pythagorean version of the definition could be drawn from Proclus's commentary on the definition of a point. While he does not cite Proclus in this definition, he does cite the fifth-century author in other places in Book 1, suggesting that he had access to some version of Proclus's commentary.
[64] Ibid, 1*r*.
[65] Commandino, *Euclidis Elementorum,* 1*r*. "Euclides per negationem partium signicavit nobis punctum, quod est principium totius proposite contemplationis. Cum enim principia aliam rationem habent ab iis, quorum sunt principia, & eorum negationes illorum quodammodo naturam ostendant; non

commentary on the definition, in which the fifth-century Neoplatonist cited Parmenides's belief that only through denying the essences of the subject of study can the foundation of that subject illuminate the unique properties of those essences.[66] Thus, since magnitudes are physical entities, a point cannot have physical materiality. While Commandino continued his commentary with a discussion of the Pythagorean definition of a point, he left out the argument, which Proclus attributed to the Pythagoreans, that the positionality of a point required it to be material.[67] Instead, in his presentation of unity and a point as analogous sources for discrete and continuous quantity, respectively, both foundational concepts were taken to be immaterial, philosophical constructs.

Clavius's commentary falls between the views found in his two contemporaries' texts. Like Billingsley, Clavius provided a discussion of physical quantity and an image to enable his reader to imagine a point, but like Commandino, he insisted that a point was immaterial. Thus, his commentary offered a pedagogical discussion that was designed to lead students to the abstract truths of mathematics from an exploration of more easily understood physical entities. He began his discussion with definitions of the three components by which continuous quantity could be measured – length, breadth and depth – and built to their negation to define a

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immerito negantes sermones principiis ipsis convenire comperti sunt: quod etiam asserit Proclus auctoritate Parmenidis."

[66] Proclus, *Commentary on the First Book,* 77.

[67] Ibid.,78. "By contrast the point is projected in imagination and comes to be, as it were, in a place and embodied in intelligible matter. Hence the unit is without position, since it is immaterial and outside all extension and place; but the point has position because it occurs in the bosom of imaginations and is therefore enmattered."

point. Citing Ptolemy's *Analemmata* and Simplicius's work on dimension, Clavius explained that the three spatial dimensions are the only possible dimensions for any magnitude.[68] After describing the dimensions, he instructed his students to imagine – since it cannot be observed – a quantity that lacked any of the three dimensions; that entity is a point. Like Billingsley, Clavius included an illustration of a point in the margin of his text to help his readers exercise their imaginations. However, its placement near the sentence in which Clavius explained that a point cannot be found in anything material made it clear that the drawing was just a useful representation of a concept that cannot properly be illustrated.[69] In order to provide his students with another tool with which to understand a point, Clavius ended his commentary with an analogy between a point and unity, but he never cited the Pythagoreans or gave their definition. He simply said that a point is to magnitude what unity is to number or an instant is to time without dwelling on the connection between magnitude and the abstract concept of number or the more concrete notion of time.[70] Clavius likely left out the formal statement of the Pythagoreans' definition of a point because, in using unity to define a point, it violated his requirement that nothing in mathematics rely on

[68] Clavius, *Euclidis Elementorum,* 1*v*. "Neque aliam dimensionem habere potest res ulla quanta, ut recte demonstravit Ptolemaeus in libello de Analemmate, opera Federici Commandini Urbinatis nuper in pristinam dignitatem restituto, necnon, ut ait Simplicius, in libello de Dimensione, qui ciudem, quid sciam, hactenus nondum est excusus."

[69] Ibid., 1*v*. "Huius exemplum in rebus materialibus reperiri nullum potest, nisi velis extremitatem alicuius acus acutissimae, similtudinem puncti exprimere; quod quidem omni ex parte verum non est, quoniam ea extremitas dividi potest, & secari infinite, punctum vero individuum prorsus debet existimari."

[70] Ibid., 1*v*. "Denique in magnitudine id concipi debet esse punctum, quod in numero unitas, quodque in tempore instans."

a concept not previously defined or demonstrated. Unity is not defined until the fifth book.

The Foundations of Geometry: Postulates and Axioms

In the first book of *The Elements* postulates and axioms follow the definitions, thereby rounding out the three kinds of principles that must be accepted in order to prove the propositions that follow. While definitions supply a vocabulary of concepts that do not rely on reason, postulates and axioms are principles that require some consideration, but no demonstration, before assenting to their truth. Aristotle described axioms as common notions that are universally acknowledged as true and postulates as assumptions that may go against the belief of the student. Both were classed as necessary first principles of any discipline.[71] As with the definitions, the authors did not express complete agreement on the content of these principles. In fact, Clavius added one postulate and seven axioms to the content found in his contemporaries' texts. The additions all follow from the enunciations common to all three texts and, so, seem to be a pedagogical choice to make his text more accessible to readers who were not eager to think through the implications of the other enunciations.[72] More telling are the differences in classification. Billingsley's and Commandino's postulates and axioms differed only in the classification of one enunciation, and Clavius reclassified two enunciations found among the postulates in his

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[71] See the extended quotation of Aristotle in Thomas Heath, *The Thirteen Books,* vol. 1, pp. 117-119.
[72] The postulate Clavius adds says that for any magnitude a greater or a lesser magnitude can be found. All but one of the additional axioms included by Clavius deal with equality of sets of entities. They are very much in keeping with the nature of the nine axioms shared by all three authors. See Table 1.

contemporaries' works as axioms. For a complete list of the postulates and axioms found in the texts, see Table 1.

The differences in the commentators' organization of the enunciations into the two categories arose from their interpretations of the two possible modes of distinction they all cited: that postulates have content specific to geometry, while axioms are general to all knowledge and that postulates call for some kind of construction, while axioms require only reason. These distinctions are clearly drawn from Proclus's commentary, which Commandino and Clavius both cite. In Proclus's text they are offered as two unique classification systems, each of which fails to fully account for the ordering of the enunciations in Euclid's text.[73] Thus, all three authors sought to clarify the distinction between postulates and axioms either by tweaking the classification schemes suggested by Proclus, or by changing the classification of enunciations.

For Commandino, whose text follows the order that Proclus presented as Euclid's, the preservation of the ancient order was too important to reclassify any of the enunciations. Therefore, combining the two suggestions he drew from Proclus, he created a classification scheme based on the ease with which a claim could be accepted. His scheme was a variation on the first distinction from Proclus named above. An axiom depended only on general knowledge, while a postulate required

[73] Proclus, *Commentary on the First Book,* 142-143. Proclus also included a third possible distinction: postulates can be demonstrated, but axioms cannot. This he attributed to Aristotle, but it was only one part of the latter's discussion of the various kinds of first principles. None of the three commentators discussed here gave that distinction any weight. Indeed, Clavius even demonstrated the axiom that states all right angles must be equal.

Table 1: The Postulates and Axioms of Book One

Commandino's Postulates	**Billingsley's Postulates**	**Clavius's Postulates**
1. It is required to draw from any point to any point a straight line.	1. From any point to any point, to draw a right line.	1. It is postulated, that it is allowed to draw a straight line from any point whatever to any point.
2.To extend a bounded straight line in a continuous and straight manner.	2. To produce a right line finite, straight forth continually.	2. And to extend straight forth a bounded straight line in a continuous manner.
3. To describe a circle on any center and interval.	3. Upon any centre and at any distance, to describe a circle.	3. Likewise to describe a circle on any center and interval.
4. All right angles are equal to one another.	4. All right angles are equall the one to the other.	4. Likewise, for any given magnitude to be able to obtain another magnitude either greater or smaller.
5. And if a straight line falling on two straight lines, makes the interior angles on the same side less than two right angles, then those straight lines extended to infinity will meet on that side in which the two angles are less than two right angles.	5. When a right line falling upon two right lines, doth make on one and the selfe same syde, the two inwarde angles less then two right angles, then shall the two right lines beying produced in length concurre on that part, in which are the two angles lesse then two right angles.	
	6. That two right lines include not a superficies.	

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Table 1: The Postulates and Axioms of Book One (continued)

Commandino's Axioms	**Billingsley's Axioms**	**Clavius's Axioms**
1. Those things which are equal to the same thing are equal to each other	1. Thinges equall to one and the selfe same thyng: are equal also the one to the other.	1. Those things which are equal to the same thing are equal to each other
2. And if equal things are added to equal things, the wholes are equal.	2. And if ye adde equall thinges to equall things: the whole shalbe equall.	2. And if equal things are added to equal things, the wholes are equal.
3. And if equal things are taken from equal things, the remainders are equal.	3. And if from equall thinges, ye take away equall thinges: the things remayning shall be equall.	3. And if equal things are taken from equal things, the parts that remain are equal.
4. And if equal things are added to unequal things, the wholes are unequal	4. And if from unequall thinges ye take away equall thinges: the thynges which remayne shall be unequall.	4. And if equal things are added to unequal things, the wholes are unequal.
5. And if equal things are taken from unequal things, the remainders are unequal.	5. And if to unequall thinges ye adde equall thinges: the whole shall be unequal.	5. And if equal things are taken from unequal things, the remainders are unequal.
6. And things that are double to the same thing are equal to each other.	6. Thinges which are double to one and the selfe same thing: are equall the one to the other.	6. And things that are double to the same thing, are equal to each other.
7. And things that are half of the same thing, are equal to each other.	7. Thinges which are the halfe of one and the selfe same thing are equal the one to the other.	7. And things that are half of the same thing, are equal to each other.
8. And those things which coincide with each other are equal.	8. Thinges which agree together; are equall the one to the other.	8. And those things which coincide with each other are equal. .
9. The whole is greater than its part.	9.Every whole is greater then his part.	9. And the whole is greater than its part.
10. Two straight lines do not include a space.		10. Likewise, all right angles are equal to each other.

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Table 1: The Postulates and Axioms of Book One (continued)

		Clavius's Axioms continued: 11. And if a straight line falling on two straight lines, makes the internal angles on the same side less than two right angles, then the due lines extended to infinity will mutually fall on each other on the side where the angles are less than two right angles. 12. Two straight lines do not include a space. 13. Two straight lines do not have one and the same segment in common. 14. If unequal things are added to equal things, the excess between the totals will be equal to the excess between the things added to the equals. 15. In equal things are added to unequal things, the excess between the totals will be equal to the excess between those things which diverged in the beginning. 16. If unequal things are taken from equal things, the excess between the residuals will be equal to the excess between the things taken from the equal things. 17. If equal things are taken from unequal things, the excess between the residuals, will be equal to the excess between the wholes. 18. All of a whole is equal to the all of its parts taken at the same time. 19. If a whole is double another whole and things double to one another are taken from those wholes, the remainder of the larger pair is double the remainder of the smaller pair.

some peculiarly geometric concept so specialized that it cannot simultaneously exist as a common sense definition. It was the other distinction from Proclus – the requirement of a construction – that removed a geometric concept from the realm of common sense. Therefore, in Commandino's classification, an axiom ought to be immediately obvious to anyone while a postulate may be foreign to a novice mathematician or an outsider, and, incorporating the other distinction, could only become evident if the geometric entities described were constructed.[74] The only enunciation Proclus had identified as incorrectly categorized based on the distinction between general and particular knowledge was Commandino's tenth axiom, which states that two straight lines cannot enclose an area. According to Proclus, this axiom clearly treated a uniquely geometrical subject.[75] However, since the only term in Commandino's formulation of that axiom with a technical definition is "rectae lineae," or straight lines, which can be understood without a technical definition, he believed that it could be understood by anyone even without a construction.[76] Thus, Commandino saved the ancient ordering, since his separation of general and particular knowledge allowed a geometrical concept that did not require construction to fit in the former category.

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[74] Commandino, *Euclidis Elementorum,* ***v*. "Cum enim is, qui audit propositionem aliquam, statim sine doctore ut veram admittit, elue certessimam fidem adhibet, hoc Axioma appellatur … Cum autem rursus & ignotum sit addiscenti, quod dicitur, & tamen eo assentiente assumatur, tunc id postulatum appellamus."

[75] Proclus, *Commentary on the First Book* 143.

[76] Commandino, *Euclidis Elementorum,* 6*v*. "Duae rectae lineae spacium non comprehendunt." The definition of a straight line is "a line that lies equally between its points." While the definition is precise, the idea of a straight line as a line that does not curve certainly falls within the realm of common knowledge. The lack of curvature is all that is necessary to recognize that two lines cannot enclose a space. Commandino does use the word "spatium" for area, which is not a specialized term.

Billingsley equated the two distinctions from Proclus, i.e. that axioms are general to all knowledge while postulates are specific to geometry and that axioms require only reason while postulates require a construction. Therefore, he reclassified the problematic axiom as a postulate (his sixth). His classification treats geometry as a physical study of constructible entities. A concept is particular to geometry if it *can* be constructed, i.e. the postulates are those enunciations that can be recognized in an image. This is a much looser requirement than Commandino's since some concepts that can be constructed, e.g. straight lines, need not be constructed to be understood. Thus, in Billingsley's classification, the principle that two straight lines cannot enclose a surface is a postulate because a straight line, as a concept that can be drawn, is particular to geometry. He offered a diagram (Figure 4) to help his reader assent to the claim. Since the diagram illustrates an impossibility rather than a concrete entity, Billingsley suggested that the reader imagine moving the lines to attempt other configurations, cementing the image of geometry as a concrete study of physical, manipulable bodies.

Unlike his two contemporaries, who found distinct ways to combine Proclus's distinctions, Clavius chose one of the ancient distinctions as his guiding principle: postulates required something to be constructed, while axioms required only the use of reason. In fact, he explicitly rejected the distinction that postulates were specific to geometry while axioms were general truths on the grounds that the postulates which address the extension of magnitudes could be recast as general to all kinds of quantities, and the axioms which address the physical configurations of magnitudes

6 *That two right lines include not a ſuperficies.*

If the lines *A B* and *A C*, being right lines, ſhould incloſe a ſuperficies, they muſte of neceſſitie bee ioyned together at both the endes, and the ſuperficies muſt be betwene the. Ioyne them on the one ſide together in the pointe A, and imagine the point *B* to be drawen to the point C, ſo ſhall the line *A B*, fall on the line *A C*, and couer it, and ſo be all one with it, and neuer incloſe a ſpace or ſuperficies.

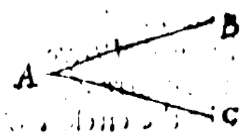

Figure 4: Billingsley's Sixth Postulate and its Construction.

Note that the drawing is intended to convey the impossibility of enclosing a space. This diagram is a "proof" by contradiction.

only applied to geometry.[77] By abandoning the stipulation that postulates be more specialized than axioms, Clavius was able to offer a precise and easily identifiable distinction between the two kinds of principles. Postulates require something to be done and are, therefore, presented as a permitted task. For example, the first postulate says, "It is postulated that it is allowed to draw a straight line from any point whatever to any point."[78] In contrast, axioms rely only on reason and are written as true claims. For example, the first axiom says "Those things which are equal to the same thing are equal to each other."[79] Proclus had noted two enunciations that failed to meet this distinction in Euclid's ordering. Those were the fourth and fifth postulates. The fourth postulate says that all right angles are equal.[80] The fifth postulate is the parallel postulate which says that if two straight lines are cut by a third such that the interior angles made by each of the first two lines with the third on the same side sum to less than two right angles, the first two lines will eventually intersect on that side of the third line.[81] Since neither can be reduced to a task, Clavius reclassified these entities

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[77] Clavius, *Euclidis Elementorum,* 20*v*. "Constat quoque Postulatorum alia propria esse Geometriae, qualia sunt illa tria, quae Euclides nobis proposuit; quaedam vero communia & Geometriae, & Arithmeticae, cuiusmodi est hoc, Quantitatem posse infinite augeri. Tam enim numerus, quam magnitudo per additionem augeri potest, ita ut nunquam huius incrementi finis reperiatur. Idem dices de Axiomatis, sive pronunciatis." The axioms in question are the eighth, tenth, eleventh, twelfth, and thirteenth. See Table 1 for the enunciations.

[78] Ibid., 14*r*. "Postuletur, ut a quovis puncto in quodvis punctum, rectam lineam ducere concedatur."

[79] Ibid., 15*r* "Quae eidem aequalia, & inter se sunt aequalia."

[80] Clavius had already included a demonstration for this axiom in his commentary on the definition of right angles. He made no mention of that demonstration in his discussion of the axiom. (Clavius, *Euclidis Elementorum,* 17*r*.)

[81] In the sixteenth century, mathematicians recognized the parallel postulate as somewhat problematic because it is much less obvious than the other postulates and axioms. Indeed, its converse (the claim that if the internal angles do sum to two right angles, the lines are always separated by the same space) is later proven as one of the propositions. Commandino and Clavius both noted the doubts about the status of the parallel postulate as obvious enough to be classed among postulates or axioms (Commandino, *Euclidis Elementorum,* 6*v*; Clavius, *Euclidis Elementorum,* 17*v*-18*r*.) Many mathematicians, both ancient and modern, made efforts to prove the parallel postulate. Clavius

as axioms without any discussion of that choice. Furthermore, in Clavius's eyes, the enunciation on which Commandino and Billingsley differed, clearly belonged with the axioms. It is true that two straight lines cannot enclose a space, but it cannot be presented as a construction to be done, so it does not matter whether it is seen as particular to geometry.

Besides eliminating the need to define what it meant for a concept to be particular to geometry, Clavius's distinction served to combine the Platonic reason-based approach to geometry emphasized by Commandino and the physical approach emphasized by Billingsley. In Commandino's distinction, physical constructions served as a tool to extend the reach of reason. In Billingsley's, reason operated as a tool to make sense of physical constructions. By separating the two approaches to geometry from one another, Clavius was able to give them equal weight and to provide geometry a role in the development of both abstract and concrete knowledge, allowing it to be a versatile discipline. Since, as Proclus and the three commentators studied here acknowledged, the same distinction was used to differentiate between the two types of propositions – problems, which require something to be constructed or done to a figure and theorems, which require the demonstration of a general property of a geometrical concept – the balance Clavius struck carried over into the rest of the

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included a proof from Proclus's work after Euclid's demonstration of the converse. (Clavius, *Euclidis Elementorum,* 49*r* – 50*r*). Clavius made no comment as to the validity of the proof he presented. His inclusion of the proof suggests that he thought it was perfectly sound. Since he had already included a proof for another axiom, he clearly did not see a problem with including axioms that could be proven. Commandino did not include a proof for the postulate. His choice to leave out any proof seems to be a choice to preserve the ancient Euclidean text as best as possible. He also may have been concerned that Proclus's proof was invalid and did not want to include something that could damage his claims about the unquestionable truths found in mathematical demonstrations. Billingsley makes no comment on any of the doubts surrounding the parallel postulate. Nor does he include any demonstrations for it.

text and enabled him to show early on that each of Euclid's enunciations was necessary piece of the foundation for the building of the mathematical disciplines.[82]

Propositions

Propositions differ from the first three kinds of Euclidean principles in that they are not immediately obvious and require proof in order to be accepted. Problems, propositions which instruct the reader to do something, require first a construction to fulfill the given task and then a demonstration that that construction did indeed solve the problem. Theorems, propositions which state a relationship between geometric entities, require a demonstration that the claim is necessarily true. It is through these demonstrations, which constitute the majority of the Euclidean text, that geometry builds its knowledge from the fundamental principles found in the definitions, postulates, and axioms to non-obvious claims like the Pythagorean Theorem. Their

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[82] See Clavius, *Euclidis Elementorum,* 20*r-v*. "Colligi potest ex dictis cum Proclo, & Gemino hoc discrimen inter postulata, & Axiomata, quod cum utraque sint per se nota, & indemonstrabilia, illa naturam sapiunt Problematum, propterea quod aliquid fieri exposcant; haec vero, Theoremta imitantur, cum nihil fieri petant, sed solum sententiam aliquam notissimam proponant."; Commandino, *Euclidis Elementorum*, 6*r*. "At differunt axiomata a postulatis eodem prorsus modo, quo theoremata a problematibus."; Billingsley does not explicitly acknowledge the similarity, but it is obvious from a quick comparison of his descriptions of the difference between postulates and axioms and of problems and theorems. Billingsley, *Elements of Geometrie,* On the difference between postulates (peticions) and axioms (common sentences): "Peticions also are very manifest, but not so fully as are the common sentences, and therfore are required or desired to be graunted. Peticions also are more peculiar to the arte wherof they are: as those before put are proper to Geometry: but common sentences are generall to all things wherunto they can be applied. Agayne, peticions consist in actions or doing of somewhat most easy to be done: but common sentences consist in consideration of mynde, but yet of such thinges which are most easy to be understanded, as is that before set." (p. 6*v*). On problems and theorems: "A Probleme, is a proposition which requireth some action, or doing: as the making of some figure, or to devide a figure or line, to apply figure to figure, to adde figures together, or to subtrah one from another, to describe, to inscribe, to circumscribe one figure within or without another, and such like." "A Theoreme is a proposition, which requireth the searching out and demonstration of some propertie or passion of some figure: Wherin is onely speculation and contemplation of minde, without doing or working of any thing." (p. 7*v*).

certainty was the grounds on which mathematical knowledge could be accepted.[83] Thus, in order to understand how each author defined mathematical knowledge, it is necessary to understand how they approached propositions and explained what was required for a complete demonstration.

All three authors included general descriptions of the parts of a proposition early in the first book. The differences in the content and presentation of these descriptions illustrate each author's vision of mathematics. Even though Commandino and Billingsley both drew their six-part anatomy of a proposition from a common source, Proclus's commentary on the first book, their uses of the ancient text reveal Billingsley to be interested solely in making the content of *The Elements* accessible, and potentially useful and show that Commandino sought to present geometry as a coherent system of knowledge.[84] In both of their texts a complete proposition is said to consist of the proposition (the concept that is to be proved), the exposition (any given parameters), the determination (the declaration of what needs to be done), the construction (the diagramming of whatever is necessary to do the problem or prove the theorem), the demonstration (the reasoning and proof for the proposition), and the conclusion (just the proposition, again). Like Proclus, both Commandino and Billingsley noted that of these six parts only the proposition, demonstration, and conclusion were always necessary.[85] However, while Billingsley stopped at that

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[83] For a discussion of sixteenth-century arguments for mathematical certainty, see Chapter 1.
[84] Proclus, *Commentary on the First Book,* 157-167.
[85] Billingsley, *Elements of Geometrie,* 8*v,* "But all those partes arte not of nessitie required in every Probleme and Theorem. But the *Proposition, demonstration, and conclusion,* are necessary partes, & can never be absent: the other partes may sometymes be away."; Commandino, *Euclidis Elementorum,* 7*r,* "Maxime autem necessariae, & quae in omnibus insunt Propositio, Demonstratio, & Conclusio:

point, Commandino continued to follow Proclus to include an explanation for why the exposition, determination, and construction could be left out. Thus, it is clear that Billingsley, who only presented the components a reader could expect to find in a proposition, used Proclus's explanation to provide his reader with a practical guide to the text, not to describe the nature of mathematical knowledge.[86] In his eyes, the content of the propositions mattered more than their philosophical status.

In contrast, Commandino, in order to present Euclidean geometry as a complete system of knowledge, needed to assure his reader that the each proposition was in and of itself complete. Thus, he could not be cavalier about discrepancies between his description of a complete proposition and the structure of the propositions. Following Proclus, he observed that the exposition and the determination were simply restatements of pieces of the proposition. Furthermore, in some propositions nothing is given, so an exposition becomes unnecessary. And finally, while the construction is, by definition, always required in a problem, some theorems, especially those in the books on number theory, do not require a construction because the exposition in the proposition is sufficient basis for a reasoned demonstration of the claim.[87]

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oportet enim ante cognoscere quaesitum, perque media ostendere." ; Proclus, *Commentary on the First Book,* 159.

[86] Billingsley's willingness to ignore Proclus's descriptions of the nature of mathematical knowledge was somewhat countered by John Dee's "Mathematicall Preface," which is devoted to a description of the nature of mathematics in all of the branches identified by Dee. However, Dee did not discuss the structure of mathematical demonstrations found in Euclid.

[87] Commandino, *Euclidis Elementorum,* 7*v*; Proclus, *Commentary on the First Book,* 159-161.

For Clavius the description of a proposition served to identify the ways in which Euclidean demonstrations created knowledge: physically through constructions, and rationally through logical demonstrations. He did not include Proclus's anatomy of a proposition, and instead only named the construction and the demonstration as the two essential pieces of most propositions, although he acknowledged that some theorems could be demonstrated without a construction.[88] By simplifying the description of a proposition, he drew the readers' attention to the ways in which geometry propositions developed new knowledge. Even in Proclus's six-part division, the constructions and demonstrations are the only parts that go beyond the enunciation of the proposition. Constructions provide knowledge through the creation of physical diagrams; demonstrations use reason to uncover universal truths about geometrical entities (with or without a diagram). Thus, in Clavius's explanation of a proposition, mathematics is seen as a versatile combination of concrete physical study with rational demonstrations that lead to universal truths.

The placement of each author's discussion of the parts of a proposition further suggests his approach to mathematics. Commandino was the only commentator who positioned his discussion of propositions as an abstract introduction to the structure of

[88] Clavius, *Euclidis Elementorum,* 21*v,* "Quod idem in aliis problematis perspici potest. Haec etiam duo reperiuntur fere in omni Theoremate. Saepenumero enim ut demonstretur id, quod proponitur, construendum est, ac efficiendum prius aliquid, ceu manifestum erit in sequentibus Pauca vero admodum sunt theormata, quae nullam requirant demonstrationem." Because Clavius included his brief discussion of propositions immediately following the enunciation of the first proposition, which is a problem, he stated that constructions and demonstrations are necessary to all problems. He did mention that most theorems include the same two parts, but not all theorems require a construction. Clavius's placement of his description of propositions between the enunciation of the first problem and its proof also provided a pedagogical reason for limiting the discussion to two parts of propositions. While he prepared the reader for what to expect in the subsequent proof, he did not take a long tangent on the nature of propositions.

mathematical knowledge as it was developed in *The Elements.* He placed it in between the last axiom and the first proposition, which established it as a treatment of the nature of Euclidean mathematics, rather than a description of a kind of Euclidean enunciation. In fact, in addition to his description of propositions, Commandino's discussion provided an extensive examination of the kinds of information that can be given, and various extensions that can be added to a proposition, namely, demonstrations for multiple cases, lemmas, and corollaries. While it served to prepare the reader for the material to come, it also created a clear picture of geometry as the product of a rational development of simple pieces of given knowledge. From given positions (i.e. points), magnitudes (e.g. lines) and kinds (e.g. right, acute, or obtuse angles) propositions revealed mathematical truths, and their demonstrations opened the door to new claims, lemmas and corollaries, which could transition into new propositions.[89] Thus, he showed mathematics to be a coherent body of knowledge, not just a collection of unrelated facts, earning it a place among the sciences of philosophy as a means to make sense of the world.

In contrast, Billingsley placed his description after the proof for the first proposition had been presented, allowing him to use the first proposition as a model of all propositions. While his discussion named and defined the various parts of the proposition and made mention of the possibility of multiple cases, it offered no explanation of the role each part of a proposition played in building mathematics into a coherent system of knowledge from simple assumptions. Instead, after his

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[89] Commandino, *Euclidis Elementorum,* 7*r*-*v*.

description of a proposition, he simply pointed out the locus of each of the six parts within the text of the preceding proof. For example, he identified the exposition as the opening words of the proof: "Suppose that the right line geven be AB." As Billingsley observed, this sentence "declareth onely the thing geven," which was the sole task of the exposition.[90] Thus, his discussion was simply an instruction tool showing how to read a proof. His interests lay in making the information contained within mathematics accessible to his readers, regardless of whether they saw the discipline as a complete system of knowledge or a collection of independent facts.

Like Billingsley, Clavius used the first proposition as a concrete example to anchor his discussion of the nature of propositions. However, he placed his description between the enunciation and the proof, thereby allowing him to use the first proposition as a tool to guide his reader through a discussion on the development of mathematical knowledge and its validity. In fact, Clavius ignored the specifics of the proof, and only used the example to clarify the two necessary components of a proposition: the construction, which relied on the postulates, and the demonstration, which relied on the axioms and definitions. He explained that in the case of the first proposition, which requires that an equilateral triangle be drawn on a given line, the construction of the triangle fulfilled the request of the proposition, and the

[90] Billingsley, *Elements of Geometrie,* 8*r*-8*v*. The breakdown of the first proposition also appears in Commandino's commentary. Like the discussion of the parts of a proposition, it is very similar in the two texts, and appears to have been drawn from Proclus's work (Proclus, *Commentary on the First Book,* 162-164). While Billingsley indicated the exact words from the proof for all six parts of the proposition, Commandino did so only for the exposition, determination, and conclusion. He indicated the exact words that begin the construction, but he only described the transition from construction to demonstration and from demonstration to conclusion. Billingsley went on to indicate the parts of each proposition with marginal notes in the rest of his text.

demonstration showed that the construction was successful, both in that the appropriate triangle had been produced and in that the method used would always produce such a figure. According to Clavius, all future problems and most theorems followed this pattern of demonstration. (Some theorems did not require a construction). Thus, for Clavius, the first proposition was an example of the process through which mathematics combined physical constructions and logic to build a coherent system of knowledge from first principles to complicated propositions. And since he emphasized that Euclidean demonstrations continually built on certain, previously accepted truths, his description showed the validity of Euclidean demonstrations as sources of new knowledge.

While Clavius was the only one of the three authors to address the method used in Euclidean demonstrations in his description of propositions, all three used their commentary on the first proposition to illustrate how Euclidean proofs were to be completed. For Commandino, that meant emphasizing the exclusive use of previously accepted claims in building logical arguments from first principles. To do so, Commandino used the first proposition as an exemplar of his earlier description of propositions and the way in which they generated knowledge. His commentary was structured as a description of the six parts of the first proposition. However, Commandino's discussion focused on the construction and the demonstration by offering an explanation of the reasoning that made the proof possible and why that reasoning should be accepted as a proof of the claim made.[91] In those sections he

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[91] Commandino, *Euclidis Elementorum,* 8*r.*

reminded the readers of the specific postulates that made the construction possible - that a circle can be drawn with any center and any radius, and that a straight line can be drawn from any given point to any other point. Then he briefly alluded to the definition – that of a circle – and the axiom – things that are equal to the same thing are equal to each other - that made it possible to demonstrate the equality of the three sides of the triangle. Instead of just giving the numbers of the relevant postulates and axioms, as Clavius and Billingsley did within their proofs, he provided their content, allowing the reader to recognize that the knowledge claims required by the proof were indeed first principles to which he had already assented. Thus, Commandino's use of the first proposition exemplified Euclid's certain method, not just a complete proposition.

While Commandino sought to convince his reader of the certainty of the knowledge created with the method used in Euclidean proofs, Billingsley sought only to show his reader what kinds of arguments they could expect in *The Elements*. To do so, in his commentary to the first proposition, Billingsley included a description of the three kinds of demonstrations found in *The Elements:* composition, resolution, and a demonstration leading to impossibility.[92] In his view, composition was the most common kind of demonstration found in *The Elements*. It started with first principles

[92] Resolution and composition were not unique to mathematics. They are both found in Aristotle's *Posterior Analytics.* In the mid-sixteenth century, the Paduan philosopher Jacopo Zabarella developed their use in natural philosophy as part of his contributions to contemporary debates over the means through which knowledge could be created. For a discussion of the significance of Aristotelian rational philosophy in the sixteenth century see Luca Bianchi, "Continuity and Change in the Aristotelian Tradition" in *The Cambridge Companion to Renaissance Philosophy* ed. James Hankins (Cambridge University Press, 2007), 49-71.

and built to the conclusion. Resolution did the opposite, beginning with the conclusion and working backwards to first principles. Finally, he described a demonstration leading to impossibility as an argument in which an impossible conclusion was reached when the principle to be proven or any of its premises were assumed to be false.[93] Like Commandino, Billingsley relied on examples to illustrate the modes of Euclidean demonstration – he showed how the first proposition could be done by either composition or resolution, and he told the reader that the fourth proposition of the first book would provide an example of a demonstration leading to impossibility.[94] However, unlike Commandino, he did not place those examples into the service of an argument for the certainty of Euclidean geometry as a whole. While the examples asserted the validity of individual proofs, the three methods were not united by some underlying principle (such as reliance of first principles), and, therefore, Billingsley's descriptions and examples served only to prepare the reader for the kinds of demonstrations used in the rest of the text.

Since Clavius had already established the sole dependence of demonstrations on first principles and prepared his reader for the kinds of demonstrations to expect

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[93] Billingsley, *Elements of Geometrie,* 9*r.*

[94] Ibid., 9*r* – 9*v*. In order to complete the demonstration by composition Billingsley argues from the definition of a circle that the given line and one new side are equal because they are both radii of the same circle. The same argument applies to the given line and the second new side. The two new sides are then equal to each other because they are both equal to the given line. Therefore all three lines are equal to each other which means that the triangle they form is equilateral, which was the conclusion that needed to be reached. In order to complete the proof by resolution, Billingsley reversed the order of the arguments. Since an equilateral triangle has three equal sides, the three lines must be equal. Since all three lines are equal, each of the new sides is equal to the given line. This he proved by making the parallel arguments about each new side and the given line being radii of the same circle. The definition of a circle requires that the radii of the same circle are equal. That definition is the undeniable first principle reached through resolution.

within his description of proposition, his discussion of the method of Euclidean demonstrations was quite different from either of his contemporaries'. In it he sought to justify the place of mathematics within philosophy by illustrating that mathematical demonstrations could be made to fit the scholastic standard of proof of syllogisms found in Thomas Aquinas' work, which Jesuit schools used as their basis for much of their education.[95] He used his commentary to work through the demonstration of the first proposition using syllogisms through a process called resolution. That process works by breaking a universal claim into a more basic set of claims, including a particular claim about the proposition in question until a definition that cannot be resolved is reached.[96] In the first proposition, the first syllogism begins with the universal claim that all triangles that have three equal sides are equilateral. Its particular claim says that the three sides of the constructed triangle are all equal. Therefore, the triangle is equilateral. The next syllogism confirms the particular claim of the previous syllogism, and the process continues until the particular claim cannot be broken down further as it is a first principle – in this case, the postulate that allows

[95] See George Ganss, *St. Ignatius's Idea of a Jesuit University: A Study in the History of Catholic Education* (Milwaukee: The Marquette University Press, 1956), 54-56. Ganss notes that Thomas Aquinas's *Summa Theologicae* was the foundation of Jesuit "scientific theology." On page 120 Ganss notes that Ignatius "preserved what was of perennial value in the preceding education, especially the philosophy and theology which found its best expression in St. Thomas Aquinas." Marcus Hellyer notes that in natural philosophy the Jesuits were not hegemonically Thomistic. (Marcus Hellyer, *Catholic Physics: Jesuit Natural Philosophy in Early Modern Germany* (Notre Dame: University of Notre Dame Press, 2005), 78.) However, since Aquinas's methods defined the Jesuit approach to theology, it is reasonable to assume that the standard of syllogisms extended into the lower discipline of natural philosophy.

[96] Clavius, *Euclidis Elementorum,* 22*r.* The use of syllogisms outlined by Clavius is identical to Billingsley's proof by resolution.

the drawing of a line from any point to any other.[97] Having illustrated that the first proposition could be demonstrated using syllogisms, Clavius immediately inferred that all mathematical proofs could be resolved in the same way. However, he explained that most mathematicians neglected to do so because it was easier and more concise to build to the final conclusion from first principles as he had done in the original proof for the first proposition.[98] Thus, Clavius's syllogistic resolution of the demonstration for the first proposition can be seen as an effort to illustrate that mathematical demonstrations could be made to fit a contemporary standard of proof, namely, syllogisms, even while he embraced the more practical mathematical convention of building proofs from first principles to various propositions.

After the first proposition, none of the authors devoted further discussion to the nature of propositions or the methods of demonstration. However, their various approaches to mathematics and *The Elements* remain evident throughout their work.

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[97] Ibid., 22*r*. The whole syllogistic proof is as follows: Syllogism 1: All triangles that have three equal sides are equilateral. The three sides of the triangle constructed for the first proposition are all equal. Therefore, the triangle is equilateral. Syllogism 2: If two magnitudes are each equal to a third magnitude, then they are equal to each other. The two new sides are each equal to the given line. Therefore, all three sides are equal. Syllogism 3a: Straight lines drawn from the center of a circle to its circumference are equal. The given line, AB, and one new side of the triangle, AC, are both drawn from the center of a circle to its circumference. Therefore, AB and AC are equal. Syllogism 3b: Straight lines drawn from the center of a circle to its circumference are equal. The given line, AB, and one new side of the triangle, BC, are both drawn from the center of a circle to its circumference. Therefore, AB and BC are equal. The particular claims in the final two syllogisms are just versions of the postulate that allows the drawing of a straight line from any point to any other. They cannot be further reduced, so the demonstration is complete.

[98] Ibid., 22*r*-22*v*. "Non aliter resolui poterunt omnes aliae propositiones non solum Euclidis, verum etiam caeterourm Mathematicorum. Negligunt tamen Mathematici resolutionem istam in suis demonstrationibus, eo quod brevius, ac facilius sine ea demonstrent id, quod proponitur, ut perspicuum esse potest ex superiori demonstratione." Clavius had originally done the demonstration using the method Billingsley described as composition. It is not clear why Billingsley separated composition and resolution since it seems where one would work, so would the other. Clavius treated the two techniques as fundamentally the same, with composition being easier to understand.

In the rest of this chapter I will analyze the three authors' approaches to the Pythagorean Theorem, the forty-seventh proposition of the first book, to provide a complete example of how each author uses the Euclidean text.

Book I, Proposition 47: The Pythagorean Theorem

The "most celebrated invention of Pythagoras,"[99] the Pythagorean Theorem, was (and remains) the most famous proposition in *The Elements*. As is well-known, it relates the lengths, *a* and *b,* of the legs of a right triangle to the length, *c,* of the triangle's hypotenuse and can be interpreted by the now well-known formula $a^2+b^2 = c^2$. In Billingsley's words, the Pythagorean Theorem reads, "In rectangle triangles, the square whiche is made of the side that subtendth the right angle, is equal to the squares which are made of the sides contaying the right angle."[100]

All three authors provide a similar proof and follow the same logical arguments. Billingsley and Commandino include identical proofs that have their formal six-part structure for propositions. Immediately following the statement of the proposition, they include the exposition, identifying the right triangle as ABC with the right angle BAC. (See Figure 5 for their diagrams). They then give the determination, stating that the square of side BC is equal to the sum of the squares on the sides AB

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[99] Ibid., 73*v*. "….ex celeberrimo hoc Pythagorae invento…"
[100] Billingsley, *Elements of Geometrie,* 57*v.* It is a word-by-word translation of the Latin found in both Commandino and Clavius's texts. Their enunciations read "In rectangulis triangulis, quadratum, quod a latere rectum angulum subtendente describtur, aequale est eis, quae a lateribus rectum angulum continentibus." Clavius, *Euclidis Elementorum,* 72*r*; Commandino, *Euclidis Elementorum,* 27*v.*

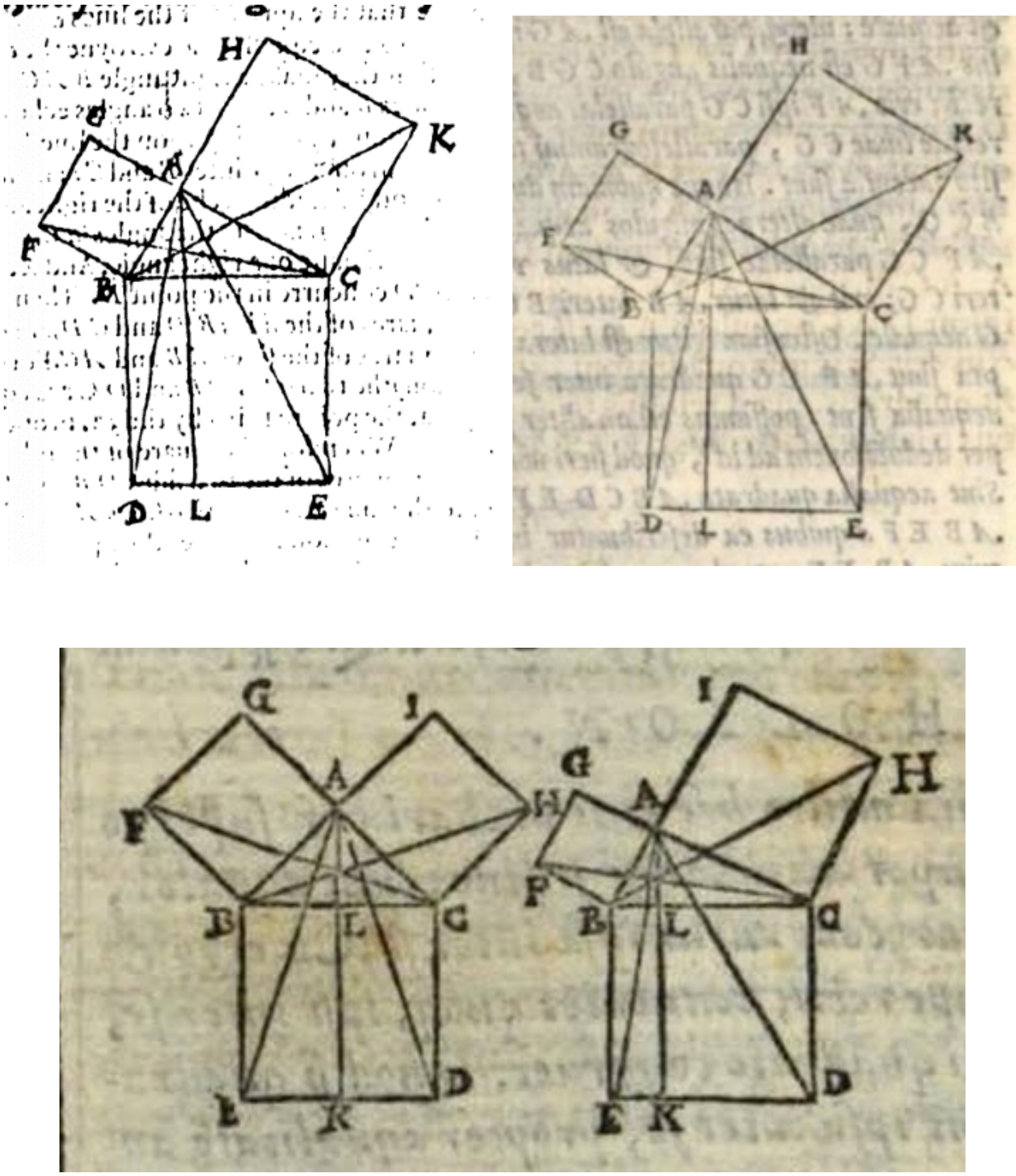

Figure 5: Diagrams for the Pythagorean Theorem.

Billingsley's (top left), Commandino's (top right), and Clavius's (bottom) diagrams are all quite similar. Clavius included two diagrams because he made explicit note of the fact that the theorem holds for both scalene and isosceles right triangles. His labels are also slightly different because he chose to label the point at which the line extended from point A intersected line BC in addition to the point at which it intersected line DE.

and AC.[101] The construction then begins by drawing squares on all three sides of the triangle. In the diagram, these are the squares ABFG, ACKH, and BCDE. It then includes the drawing of lines AL, AD, and FC before moving on to the demonstration. The lines AE and BK are put off until later in the proof because they are unnecessary in the first part of the demonstration. The demonstration shows that the square ABFG is equal to the rectangle BL (named for the diagonal corners). Arguments based on previous theorems demonstrate that triangles FBC and DBA are equal to one another and that DBA and FBC are equal to half of BL and half of the square ABFG, respectively. Since FBC and DBA are equal, FBC is also equal to half of BL. Thus, ABFG and BL are both double FBC. It then follows that square ABFG equals rectangle BL. For the details of all of the arguments see Appendix C in which I present a translation of Clavius's proof for the Pythagorean Theorem. At this point, the two authors instruct the reader to draw lines AE and BK to observe that the same pattern of arguments used in the demonstration can be show that square ACKH equals rectangle CL. They do not repeat the arguments. Since BL and CL make up the square BCDE, it follows that BCDE equals the sum of ABFG and ACKH. This final claim, which both authors restate as their conclusion, is exactly the proposition.

Clavius's proof relies on the same arguments as those found in Billingsley's and Commandino's texts, but he reorders the elements of the proof in order to improve

[101] Billingsley, *Elements of Geometrie,* 57*v.* "Suppose that ABC be a rectangle triangle havying the angle BAC a right angle. Then I say the square which is made of the line BC is equall to the squares which are made of the lines AB and AC." Commandino, *Euclidis Elementorum,* 27*v.* "Sit triangulum rectangulum ABC, rectum habens BAC angulum. Dico quadratum descriptum a recta BC aequale esse quadratis, quae ab ipsis BA AC describuntur."

its clarity.[102] Like his contemporaries, Clavius opened with the instructions to draw squares on the three sides of right triangle ABC. However, instead of leaving two lines out of his initial construction, he immediately instructed his reader to draw all of the necessary lines to create all of the required triangles and rectangles. Only once the entire construction is complete did Clavius begin the demonstration, and, unlike Billingsley and Commandino, he presented it in full.

Although Clavius's changes to the proof were not substantive, they did allow him to provide greater guidance to his reader. In preserving the distinction between the construction and the demonstration, he helped eliminate possible confusion owing to lines that appeared in the diagram but were not mentioned in the original construction. His approach also allowed him to present the arguments for both halves of the demonstration side-by-side in only a few more words than Billingsley and Commandino had used for their presentations of just half of the proposition. By leaving nothing to the reader, Clavius ensured that neither struggling students nor underqualified teachers would err in the development of the second half of the demonstration.

Once the text of the proof was completed, all three authors included some commentary on the proposition. Each began with the same basic story about Pythagoras's discovery of the theorem: he was so overjoyed by the discovery of this

[102] Clavius's ordering of the elements of the demonstration seems to have been unique. The order found in Commandino's and Billingsley's texts also appears in Campanus's, Candalla's, and Peletier's. They also all leave the second half of the proof to the reader. For Campanus, see H.L.L Busard, *Campanus of Novara and Euclid's Elements, Vol 1*. (Germany, Franz Steiner Verlag, 2005), 92-93. For Candalla, see Candalla, *Euclidis Megarensis* (1566), 14*r-v*. For Peletier see, Peletier, *In Euclidis Elementa,* 46.

theorem that he immediately offered a sacrifice to the gods. They differed as to the sacrifice's extent. For Commandino and Billingsley it was a single ox, but according to Clavius, there were those who increased that number a hundredfold.[103] Regardless, the anecdotes establish that the ancients, whose authority was a significant part of arguments for the status of mathematics, valued the theorem without indicating why.[104] On that point, the three commentaries diverge. Commandino focused on the theorem's generalizability, giving it value as the foundation for a more universal claim. Billingsley emphasized problems that could be solved using it, thereby privileging its practical implications. Clavius, whose commentary was the most extensive and included everything found in his two contemporaries' works, did both.

For Clavius and Commandino, the Pythagorean Theorem reflected more general claims about the relationships between figures drawn on the sides of triangles. Clavius offered two generalizations as the bookends of his commentary, while Commandino restricted his to a reference to a more general proposition that appears in

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[103] Commandino, *Euclidis Elementorum,* 28*r,* "Hoc theorema ad pythagoram referunt, dicuntque eum cum illud invenisset, bovem immolasse." Commandino gave no citation for the story of the sacrifice.; Clavius, *Euclidis Elementorum,* 72*v,* "Inventio huius theorematis ad Pythagoram refertur, qui, ut scribit Vitruvius lib. 9 hostias Musis immolavit, quod se in tam praeclaro invento adiuverint. Sunt qui putent, eum immolasse centum boves; sit amen Proclo credendum est, unum tatummodo obtulit." Billingsley, *Elements of Geometrie,* 58*r*; "This most notable and excellent theorem was first invented by the philosopher Pithagoras, who for the exceeding joy conceived of the invention therof, offered in fair sacrifice an Oxe, as recorde Hierone, Proclus, Lycius, and Vitruvius." Although Billingsley did not cite Peletier at this point in his commentary, his version of the story seems to be based on the French mathematician's work. Peletier's text reads "Haec est illa tam celebris Demonstratio a Pythagora Philosopho pervestigata: ob quam per gaudio bovem Daemonibus immolavit, si Heroni, Proclo, Lycio, et Vitruvio credinus." (Peletier, *In Euclidis Elementa,* 47.)

[104] For some discussion of the significance of the authority of ancient scholars in establishing the value of mathematics, see chapter 1 in which I presented the antiquity of mathematics as one argument humanist scholars made for the nobility of their discipline.

Euclid's sixth book.[105] That theorem, also the first bookend in Clavius's commentary, claims that any rectilinear figure built on the hypotenuse of a right triangle will be equal to the sum of similar figures built on the legs. While Commandino did no more than point the reader to this more general claim, Clavius used this opportunity to discuss how mathematicians might develop their theorems. For him, the Pythagorean Theorem was not only a particular case of more general claims; it was also a stepping stone to their discovery.

Clavius speculated that Pythagoras arrived at his theorem through the analysis of sets of numbers known today as "Pythagorean triples." Upon noticing that the numbers 3, 4, 5 share the relationship between squares described in the theorem, and observing that when formed into a triangle, the triangle has a right angle, Pythagoras investigated other such sets of numbers and eventually all right triangles to develop his famous theorem. Clavius argued that the more general proposition in the sixth book came from pushing this inquiry a step further. If the relationship holds true for squares, why not investigate other figures? Through this discussion Clavius created an image of mathematics as a field of universal truths that can be found by observing and analyzing the relationships between numbers or magnitudes.

Clavius's second bookend is a further generalization of the theorem, namely, Pappus's area theorem, that describes the conditions necessary for the sum of the areas

[105] Commandino, *Euclidis Elementorum,* 28*r.* "Quod autem ab Euclide in sexto libro conscribitur multo unviersalius est. ostendit enim in rectangulis triangulis figuram, quae sit a latere rectum angulum subtendente aequalem esse figuris, quae a lateribus rectum angulum continentibus, priori illi similes, & similiter positae, describuntur." The referenced proposition is VI.31 This quotation is the second and third of the three sentences that comprise Commandino's commentary on the Pythagorean Theorem.

of arbitrary parallelograms drawn on two sides of any triangle to be equal to the area of a parallelogram drawn on the remaining side of the triangle.[106] Because Pappus's area theorem describes a particular set of conditions, it could have been written in the form of a problem requesting the construction of the third parallelogram with a diagram-based description of the necessary steps to construct the third parallelogram. However, Clavius noted that he chose to write it as a theorem for increased clarity and because he judged it to be more general in that form.[107] Using the semantic form of a theorem ensured that his reader could easily recognize that the generalization was indeed a universal truth with wider applications than the Pythagorean Theorem.

For Clavius and Billingsley, the theorem's utility was also paramount. Between his universalizing bookends, Clavius introduced eight claims with the

[106] Commandino also included this generalization in his text, but he placed it in his commentary on the generalization of the Pythagorean Theorem found in Book Six (Commandino, *Euclidis Elementorum,* 86*r*). In his discussion of the Pythagorean Theorem, Commandino made no mention of this generalization, suggesting that he thought the more general claim found in the sixth book was more worthy of expansion and, therefore, more significant than the Pythagorean Theorem. It is possible that Commandino's text was Clavius's source for this particular generalization. In his commentary on the proposition in the sixth book, Clavius noted that he included Pappus's area theorem in his discussion of the Pythagorean Theorem, which suggests that he was aware of his contemporary author's placement of the theorem in the sixth book (Clavius, *Euclidis Elementorum,* 222*r*).

[107] Clavius, *Euclidis Elementorum* 75*v.* "Theoremate vero hoc Pythagoreo multo universalius est illud, quod a Pappo demonstratur in omni triangulo, sive illud rectangulum sit, sive non, & de quibuscunque parallelogrammis super latera trianguli constructis, tam rectangulis, quam non rectangulis, etiamsi non sint inter aequiangula. Quod nos in formam theorematis redigentes, clarius hoc modo proposuimus, & meo iudico generalius." In Clavius's text the theorem reads "In omni triangulo, parallelogramma qauecunque super duobus lateribus descripta, aequalia sunt parallelogrammo super reliquo latere constituto, cuius alterum latus aequale sit, & parallelum rectae ductae ab angulo, quae duo illa latera comprehendunt, ad punctum, in quo convenient latera parallelogrammorum lateribus trianguli opposita, si ad partes anguli dicti producantur." In contrast, in Commandino's text Pappus's area theorem appears as the solution to an unstated problem. It describes the construction of the desired parallelograms on a particular triangle. "Si sit triangulum ABC, & ab ipsis AB BC describantur quaevis parallelogramma ABED BCFG, & lineae DE FG prducantur ad H, iungatur que HB fient parallelogramma ABED BCFG aequalia parallelogrammo contento AC HB, in angulo, qui utrisque BAC DHB sit aequalis." (Commandino, *Euclidis Elementorum,* 86*r*).

assertion that they exhibit that utility.[108] Although he gave no concrete examples of how or when any of the claims might actually be used, by presenting them as consequences of the Pythagorean Theorem, he imbued them with utility simply by virtue of being applications. Billingsley must have held a similar notion of potential utility since his commentary consisted of four such applications.[109] In both Clavius's and Billingsley's works, most of the additions, five of Clavius's and three of Billingsley's, were presented as problems, which require the completion of a particular task, and so are useful. Furthermore, five of Clavius's claims and three of Billingsley's address relationships between squares, extending Euclid's discussion of triangles to other planar figures, a clear reminder that the propositions found in Euclid

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[108] Clavius, *Euclidis Elementorum,* 73*v* – 75*r*. "Colliguntur ex celeberrimo hoc Pythagorae invento plurima scitu non iniucunda tam theoremata, quam problemata, e quibus visum est ea duntaxat in medium proferre, quae utilitatem magnam rebus Geometricis allatura creduuntur, initium hinc sumentes." The eight claims are as follows. 1. Si in quadrato quovis diameter ducatur, quadrtum a diametro descriptum duplum erit praedicti quadrati. 2. Quadratum diametri figurae altera parte longioris aequale est duobus quadratis laterum inaequalium. 3. Si fuerint duo triangula rectangula, quorum latera rectis angulis opposita sint aequalia, erunt duo quadrata reliquorum duorum laterm unius trianguli aequalia duobus quadratis reliquorum duorum laterum alterius. 4. Duobus quadratis inequalibus propositis, invenire alia duo quadrata, quae & aequalis sint inter se, & simul sumpta aequalia duobus inaequalibus propositis simul sumptis. 5. Propositis duabus lineis inaequalibus, invenire id, quo plus potest maior, quam minor. 6. Propositis quotcunque quadratis, sive aequalibus, sive inaequalibus, invenire quadratum omnibus illis aequale. 7. Propositis duobus quadratis quibuscunque, alteri illorum adiungere figuram, quae reliquo quadrato sit aequalis, ita ut tota figura composita sit etiam quadrata. 8. Cognitis duoobus lateribus quibuscunque trianguli, in cognitionnem reliqui lateris pervenire.

[109] Billingsley, *Elements of Geometrie* 58*v*-59*r*. The additions presented are as follows: 1. To reduce two unequall squares to two equall squares. 2. If two right angled triangles have equall bases, the squares of the two sides of the one are equal to the squares of the two sides of the other. 3. Two unequall lines being geven, to know how much the square of the one is greater than the square of the other. 4. The diameter of a square being geven, to geve the square thereof. This last addition is accompanied by a corollary: "Hereby it is manifest, that the square of the diameter is double to that square whose diameter it is." Only the last of Billingsley's additions is not found in Clavius's text. However, Clavius did include the corollary as one of his additions, but, since that corollary was not written in the form a problem, it shows his own biases towards illustrating the nobility of mathematics with universal claims rather than the demonstrating the utility of mathematics with task-specific problems.

could be applied beyond their immediate topics. In both texts, the additions lend a handbook quality to the commentaries. While neither author presumed to guess when these particular tasks might be required, both supplied them, so that the reader who found a need to double a given square or find a square with an area equal to the sum of areas of any number of given squares could return to the commentary and find the relevant addition.

Billingsley clearly made a conscious decision to include only concrete and potentially applicable claims in his commentary on the theorem. The one claim that is not written as a problem could not take that form, but was clearly related to the two problems surrounding it. It states that if two right triangles have the same hypotenuse, the sums of the squares of the sides of each triangle are equal to each other. It follows immediately from the demonstration of the previous problem, which requires that two unequal squares be reduced to two equal squares, and could have been called a corollary to that problem.[110] It also was necessary to the completion of the problem laid out in the following addition, which asked the reader to find the difference between squares on unequal lines. Furthermore, Billingsley used Jacques Peletier's 1557 commentary as his source for additions to the Pythagorean Theorem, but the four claims he added do not include all of the French mathematician's commentary on the theorem.[111] Besides the additions Billingsley translated, Peletier offered his reader a

[110] Reducing two unequal squares to two equal squares means finding two equal squares which sum to the same total as the two unequal squares.
[111] Billingsley, *Elements of Geometrie,* 58*v*-59*r*. He did explicitly credit Peletier's work for all four of his additions. Each one is introduced with a heading that says "An addition of Peletier" or "Another addition of Peletier."

lengthy argument for the validity of the Pythagorean Theorem for both isosceles and scalene right triangles. In this discussion he included alternate proofs of the theorem for isosceles and scalene right triangles and demonstrations of relationships between angles and sides within right triangles.[112] This section examines properties of right triangles to justify that the theorem holds true for all right triangles. It does not extend the Pythagorean Theorem to possible applications. Billingsley's likely left it out because it could not provide concrete benefits beyond what the theorem itself already covered.

While Commandino and Billingsley both took a narrow approach to their commentaries on Pythagoras's theorem, Clavius not only included everything his contemporaries discussed, but he attempted to combine the nobility of mathematics as a source for universal truths found in Commandino's generalizations of the Pythagorean Theorem with the utility found in Billingsley's consequences. In one attempt at combining the two facets of mathematics, he included a practical description of two methods for generating Pythagorean triples. One starts from any odd number and the other starts from any even number, thereby providing readers with a quick way to create a right triangle with any given length for the shortest side. Yet, he also suggested that such practical tricks for finding Pythagorean triples may have

[112] Peletier, *In Euclidis Elementa,* 47-48. The alternate proof for isosceles triangle relies on dividing the square on the hypotenuse into four equal triangles each of which is shown to be half of the square on one of the legs. The alternate proof for scalene triangles also holds for isosceles triangles. It relies on using the height of the original triangle to divide it into two smaller right triangles. The two smaller triangles and the original triangle are all similar to one another. The ratio between each of the smaller triangles and the large triangle is the ratio between one of the legs and the hypotenuse. Using the principles of similarity and the ratios between the sides of the three triangles, Peletier deduces the Pythagorean Theorem.

helped Pythagoras develop his universal theorem in the first place. When introducing the intermediate eight claims Clavius describes them as "not unpleasant to know" before claiming that they illustrate the utility of geometry.[113] By including elements of the two facets of mathematics throughout his commentary, Clavius allowed his students to see that mathematics was valuable both for its ability to uncover universal truths and for its potential for mundane applications.

Conclusion

The three commentaries on *The Elements* studied here were part of a widespread sixteenth-century renaissance in mathematics that included the publication of myriad versions of Euclid's foundational text. And while mathematicians across Europe agreed that the content of *The Elements* had retained its value as true knowledge over the two millennia since Euclid had written, each commentator designed his own version of the text to advance a particular vision of mathematics and its value. For Commandino and Billingsley, those projects were well-defined. The former wrote his commentary as part of his project to restore ancient mathematics. His humanist goals were well-suited to his position as a tutor in the Urbino court and allowed him to present mathematics as a branch of philosophy, intermediate between natural and divine philosophy, worthy of study for its ability to uncover universal truths. Billingsley's project could not have been more different. His commentary on

[113] Clavius, *Euclidis Elementorum,* 73*v*. "Colliguntur ex celeberrimo hoc Pythagorae invento plurima scitu non iniucunda tam theoremata, quam problemata, e quibus visum est ea duntaxat in medium proferre, quae utilitatem magnam rebus Geometricis allatura creduuntur, initium hinc sumentes."

Euclid was intended as a resource for enterprising English artisans, and as such, it contained a variety of additions, ancient, medieval, and modern, in order to provide readers with as much potentially useful information as possible. For him, mathematics was worthy of study as a tool to guide manipulations of the physical world.

In contrast, Clavius was writing a textbook for a growing school system which did not yet have a clearly defined place for mathematics. Thus, as part of his efforts to secure his disciplines' status within the schools, he included elements similar to those found in both Commandino's and Billingsley's texts, illustrating the variety of benefits his discipline could offer to his Order. His hope was that his students would be able to build on the ancient, medieval, and modern mathematics he presented to further their own studies in mathematics and beyond in order to benefit the Society in Europe and abroad. The end result was an extensive two-volume commentary in which Clavius revised Euclidean proofs to improve their clarity and included additions from ancient, medieval, and modern sources, providing his students with a rich picture of mathematics as both a noble and useful study.

Although the commentators made their arguments for the value of mathematics most explicitly in their prefaces to their works, which I discussed in chapters one and two, their beliefs about mathematics clearly define their approach to the presentation of the Euclidean content itself. This chapter has focused on the first book of *The Elements,* but the three commentators studied here continued in much the same ways throughout their texts. Billingsley did everything he could to provide potentially useful information to his reader, frequently adding propositions, especially problems, collected from other commentaries. Sometimes he included alternate proofs that he believed were easier to understand than Euclid's demonstrations. In contrast, Commandino limited his commentary to clarifications of the proof and propositions, especially theorems, developed by ancient authors. He frequently emphasized the universal truth of mathematical claims. Clavius continued to include almost everything found in his two contemporaries' works along with his own sometimes pedantic explanations of the text, showing mathematics to be both a means to access universal truths, a branch of philosophy, and a foundation to the practice of a variety of useful arts. The differences in the texts illustrate that the arguments the authors made to defend their discipline were not simply rhetoric. Clavius, Commandino, and Billingsley were each so convinced of the visions of mathematics that they painted in their prefaces that they tailored the mathematical content of Euclid's text to embed those visions in the very foundations of mathematical study.

Chapter Four

Euclid's Elements of Arithmetic?

"Since, truly, for many, this book is obstructed by difficulties on account of the lines which it discusses, I have therefore fought against obscurity with all of my industry, so that from these things which have been demonstrated by Euclid up to this point, it is rendered level and easy, such that it can be learned without much labor by anyone who rightly understands the demonstrations of the preceding books. And I cannot go according to the opinion found in conversation of those who think that that the part of arithmetic which treats the roots of numbers, both rational and irrational as they are called, is necessary to the understanding of this book." [1]

Clavius, on Book Ten of *The Elements,* 1574

In their prefaces to *The Elements*, Commandino, Clavius, and John Dee (who wrote the preface to Billingsley's commentary) all provided discussions of the divisions of mathematics. Commandino adhered to the quadrivium; Clavius offered a division of mathematics based on intelligible and sensible branches, and Dee created his own extensive division.[2] All three authors maintained the separation between geometry and arithmetic as the two branches of pure mathematics.[3] Geometry was

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[1] Christopher Clavius, *Euclidis Posteriores libri sex ad XV. Accessit XVI de solidorum rgularium comparatione*, (Rome: Vincentium Accoltum, 1574), 2*r*-*v*. "Quoniam vero hic liber multis obstructus est difficultatibus, ob linearum, de quibus disserit, obscuritatem omnes nervos industriae meae in eo contendam, ut ex his, quae hactenus ab Euclide sunt demonstrata, ita planus reddatur, ac facilis, ut sine multo labore a quovis, qui praecedentium tamen librorum demonstrationes recte intellexerit, possit percipi. Neque in eorum possum sententiam ire, qui putant ad eius intelligentiam esse necessariam eam partem Arithmetices, quae de radicibus numerorum, tam rationalibus quam irrationalibus, ut vocant, sermonem instituit."

[2] See Chapter 2 for a discussion of Clavius's and Dee's divisions of mathematics. See chapter 1 for a discussion of the quadrivium.

[3] Clavius used the word "pure" to describe geometry and arithmetic in opposition to the "mixed" branches of astronomy and music in the quadrivium. (Christopher Clavius, *Euclidis Elementorum Libri XV Accessit XVI de solidorum Regularium comparatione* (Rome: Vincentium Accoltum, 1574), a7*v*, "Ad has autem quatuor scientias Mathematicas, quarum Arithmetica & Geometria purae, Muscia vero,

taken to be the study of magnitude, i.e. continuous quantity, and arithmetic was seen as the study of number, discrete quantity.[4] In the sixteenth century number was usually understood to cover what are today called natural numbers, excluding one, which had special status as unity. Today's rational numbers were understood as ratios between numbers, rather than being treated as numbers themselves.[5] They are discrete quantities. Magnitudes were understood as continuous quantities, i.e. lines, and they could have two or three dimensions, i.e. surfaces or solid bodies. Within these divisions, there seemed to be little if any reason for geometry and arithmetic to overlap. Indeed, the first six books of *The Elements* enforce the separation of geometry and arithmetic because they treat a wide variety of shapes and figures in one and two dimensions without the use of numbers. Even the study of proportion in the fifth and sixth books is presented through the relationships between lengths of lines and areas of figures. However, the next three books of *The Elements* address the study of number, and the propositions found in those books are frequently used in the

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atque Astronomia mixtae dicunt." Dee and described geometry and arithmetic as the "principall" branches of mathematics. (John Dee, "Mathematicall Preface" in Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), *j*r.* "Of *Mathematicall* thinges, are two principall kindes: namely, *Number,* and *Magnitude.*" On the Groundplat arithmetic and geometry are labeled "Principall." Commandino, in his description of the intelligible and sensible branches of mathematics called arithmetic and geometry the two "leading" branches of mathematics. Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), *4*v*, "Atque huius sane generis duas principes, longeque praestantes ponimus species, Arithmeticam 7 Geometriam.") "Pure mathematics" has proved to be a long-lasting term, although today it applies to a great deal more than geometry and arithmetic.

[4] I will use "number" to indicate a subject of study and "numbers" to indicate the collection of entities that make up that subject.

[5] See Stillman Drake, "Euclid Book V from Eudoxus to Dedekind," in *Essays on Galileo and the History and Philosophy of Science,* vol. 3. Selected by N.M. Swerdlow and T.H. Lawrence, (Toronto: University of Toronto Press, 1999), 61. Drake describes the definition of number at the time of Eudoxus of Cnidos in the fourth century BCE since Book Five of *The Elements* is believed to be mostly Eudoxus' work. That definition of number seems to have been transmitted to the sixteenth century along with the ancient texts.

demonstrations for the later propositions, especially in the study of commensurability in Book Ten. Thus, by the end of *The Elements*, it is not clear that the divide between arithmetic and geometry was necessarily seen as rigid in the sixteenth century.

The problem of the relationship between arithmetic and geometry is at least as old as *The Elements,* and is best seen in Book Five, in which Euclid defined proportion for magnitudes. Stillman Drake noted that some recent commentators on *The Elements,* including Thomas Heath, assumed that Euclid saw number, which could not be indefinitely divided, as a special case of magnitude, which could be indefinitely divided. However, Drake rightly pointed out that such an assumption, while an easy leap to make for a twentieth-century mathematics student who is familiar with the nineteenth-century definitions of the set of real numbers as a continuum, is not an accurate reflection of the Euclidean text, in which the divide between number and magnitude was seen as rigid.[6] Thus, Euclid's fifth and seventh books develop two distinct theories for proportion, one for magnitude and one for number. However, between the fourth century BCE and the sixteenth century, a crucial definition from the fifth book (the definition that identifies what it means for two magnitudes to have a ratio to one another) was lost, effectively destroying a geometric notion of proportion until Bartolomeo Zamberti translated Theon's version of *The Elements* along with the necessary definition in 1505.[7] Thus, in the sixteenth century, scholars working on Euclid's text were confronted with a numerical theory of proportion that had been

[6] Ibid., 62-63.
[7] Ibid., 64-68

used by medieval authors to make sense of Book Five, and a much less developed geometric theory of proportion that could be attributed solely to ancient authors.

Because medieval scholarship had necessarily blurred the line between arithmetic and geometry in order to make sense of Euclid's fifth book, as each sixteenth-century commentator worked his way through *The Elements*, he had to decide what role arithmetic played in the study of geometry. Was arithmetic necessary to geometry as a foundational discipline? Was it an equal and analogous study? Was it a study that itself depended on geometry? Was it simply a useful tool that could make the study of geometry easier? Answering these questions required the each mathematician to define for himself a particular vision of his discipline based on his assumptions about the relationships between number and magnitude. Thus, the role each author assigned to arithmetic within *The Elements* was an answer to the philosophical question of the nature of the subject of mathematical study.

As was the case with many questions in philosophy of mathematics, the question of the relationship between arithmetic and geometry came to bear on the question of the status of mathematics.[8] As I discussed in Chapter 1, arguments for the nobility of mathematics included arguments for the nobility of the subject being studied. For many humanist mathematicians, including Clavius and Commandino, those arguments centered on the nobility of magnitudes. The perfect entities that

[8] For a discussion of the importance of the philosophy of mathematics to the development of mathematical practice in the seventeenth century see Paolo Mancosu, *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century,* (Oxford: Oxford University Press, 1996). In Chapter 1, Mancosu traces the significance of the "Quaestio de Certitudine Mathematicarum" which was at the heart of the sixteenth-century debate over the status of mathematics in the development of seventeenth-century mathematics.

authors claimed gave mathematics its nobility of subject matter are circles and spheres, geometric entities. As the study of heavenly bodies, which were believed to be spheres, and their motions, which were believed to be circular orbits, astronomy examined precisely those entities. Even arguments for the certainty of mathematics usually focused on the cumulative nature of geometric proofs found in *The Elements*. Other arguments imbued number with spiritual qualities. John Dee, citing Boethius, argued that number was a link between the mind of the Creator and all creatures of nature and that it could be found in the human soul.[9] However, Clavius explicitly rejected such a belief as contrary to the Christian faith.[10] He instead argued for the value of arithmetic as a study that was necessary to civilization. In his preface to the *Epitome arithmeticae practicae*, Clavius paraphrased Plato making a claim that "those who remove arithmetic from their way of life, to such an extent remove good sense and all of civilization from the world." After all, no business can be conducted without arithmetic.[11]

The commentators' arguments for the nobility of their discipline shaped the ways in which each mathematician treated the relationship between geometry and arithmetic. For Commandino, who wished to establish mathematics as a branch of

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[9] Dee, "Mathematicall Preface," *j*v*. Dee discusses "Number numberyng" as number in the mind of the Creator and "Number numbered" as numbers in all creatures.

[10] Clavius, *Euclidis Elementorum,* a7*r*, "Pythagorei enim, atque Platonici existimantes, animas rationales certo quodam, ac determinato numero contineri, easque de corpore in corpus migrare, (quod tamen christiana fides falsum esse perspicue docet) testantur, eas nomen doctrinae, sive disciplinae obtinere, quod maxime ex ipsis nanciscamur recordationem, reminiscentiamque scientia, qua anima nostra (ut eorum est error) antequam corpus informaret, erat praeditae."

[11] Christopher Clavius, *Epitome arithmeticae practicae* (Rome: Dominici Basae, 1583), 3. "Itaque audacius illud quidem, sed tamen vere dixit Plato, prudentiam atque adeo humanitatem omnem e mundo eos tollere, qui Arithmeticam e vita tollant; cum sine ea neque publicae, neqe privatae res constare possint."

philosophy intermediate between natural and divine philosophy, geometry, whose perfect entities he believed could connect the imperfect physical world to the perfect world of the divine, was the foundation of all mathematical study. He used numbers as measurements for magnitudes, treating arithmetic only as a descriptive tool within geometrical study. The ancient divide between number and magnitude remained intact through the first six book of his commentary, and books seven through ten, as they develop the notion of incommensurable magnitudes, show that geometry, as the study of continuous magnitude, could transcend arithmetical description. Billingsley established the opposite relationship between geometry and arithmetic. As was the case in Dee's arguments that number was central to all Creation, in Billingsley's relationship between the two branches of mathematics, number was the means through which all objects, including geometrical magnitudes, could be most effectively studied. In fact, Billingsley took the presence of books seven through ten as evidence that geometry could not be understood without first grasping the more basic study of number. In so doing, he emphasized the practical value of geometry for its ability to describe the physical world. Clavius struck a balance between his contemporaries. By using numerical analyses as analogs to geometric proofs, Clavius allowed arithmetic to serve as a pedagogical aid to geometry, the study he believed to be foundational to all of mathematics, but still treated the former field as its own source of mathematical knowledge.

In this chapter I will explore the relationships between arithmetic and geometry as they were developed in the three editions of *The Elements*. To do so, I will study the use of number in the first six books and the presentation of the mathematics in the seventh through ninth books, the number theory books. Finally, I will conclude by illustrating each authors' union of arithmetic and geometry in the study of commensurability within the proof that the diagonal of a square is incommensurable with the side of the same square, which is found at the end of the tenth book.[12] I hope to show that in his definition of the place of arithmetic in *The Elements* Clavius took a middle road between the extreme approaches of his two contemporaries by treating arithmetic as a useful aid to the understanding of geometry through the analogies between the two fields. In so doing, he paved the way for the seventeenth-century algebraization of geometry, which Paolo Mancosu has identified as one of two shifts in mathematics that allowed seventeenth-century scholars to break away from classical mathematics and contributed to the development of calculus.[13]

Number in Plane Geometry: Barlaam's Arithmetic Versions of Ten Propositions from Book Two

The first four books of *The Elements* treat plane geometry through the study of various figures examining the properties of triangles, quadrilaterals, and circles. Because they present a study of pure magnitudes, there is no obvious reason for

[12] In modern arithmetical terms, this is the proof that the square root of two is an irrational number.
[13] Mancosu, p. 34. The other shift was the development and use of infinitary techniques. Mancosu's discussion of the algebraization of geometry, found in his third chapter, focuses on Descartes' *Géométrie*. Descartes had studied at the Jesuit school of Le Flèche. Another Jesuit, Gregory Saint-Vincent (1584-1667), who was a student of Clavius's has been credited with the "clearest early account of the summation of a geometric series." His work is a known source for Gottfried Leibniz (1646-1716). See Margaret Baron, *The Origins of Infinitesimal Calculus*, (Oxford: Pergamon Press, 1969), 134.

arithmetic to have a presence in these books. Indeed, as can be seen in Table 2, the three authors compared here offer only a few numerical examples to illustrate geometrical claims, and those only appear in the first and second books. Some of these uses occur in extensions of the Euclidean proofs, allowing arithmetic to build on a geometric foundation. For example, Commandino's only use of numbers in these books was in an example accompanying instructions to the reader on how to find the area of an obtuse triangle. This numerical example can be found in his commentary on a proposition about the relationship between the squares on the sides of an obtuse triangle, a concept which has no immediate connection to the area of the triangle itself.[14] Clavius and Billingsley made similar extensions to a few propositions, and they also included a few numerical examples of the concepts described by Euclid. For example, Clavius provided such numerical instances for two axioms addressing the differences between two pairs of magnitudes to show the reader that the relationships described by the axioms were indeed accurate.[15]

The only sustained use of number in these books appears in Billingsley's presentation of numerical versions of the first ten propositions of the second book,

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[14] Commandino, *Euclidis Elementorum*, 34*r-v*. This example can be found in his commentary on the twelfth proposition of the second book which states, "In obtusiangulis triangulis, quod a latere obtusum angulum subtendente fit quadratum maius est quam quadrata, quae fiunt a lateribus obtusum anglum contientibus, rectangulo conteno bis uno laterum, quae sunt circa obtusum angulum, in quod scilicet protractum perpendicularis cadit, et linea assumpta exterius a perpendiculari ad angulum obtusum." The proposition makes it possible to find the height of an obtuse triangle. From there Commandino describes what the modern reader would recognize as the formula for the area of a triangle, half of the product of the base and the height. Clavius gives a similar example in his commentary to the following proposition, where he also provides examples for finding the areas of right and acute triangles.

[15] Clavius, *Euclidis Elementorum*, 19*v*-20*r.* The axioms are the seventeenth (If from unequal magnitudes, equal magnitudes are taken away, the difference between the residues will be equal to the difference between the totals) and nineteenth (If one whole is double another, and parts are taken from each such that the part taken from the first is double the part taken from the second, then the remainders are such that the one is double the other).

Table 2: The Uses of Number in the First Four Books of *The Elements.*

N.B. Clavius's early use of numbers in the axioms of the first book is indicative of his pedagogical goals. All three axioms for which he included numerical examples were about the equality of magnitudes, and could be readily seen by performing the described operations on numbers. The numbers thus serve as shorthand for magnitudes.

	Billingsley	**Clavius**	**Commandino**
Book One	**Proposition 4**: Numerical lengths given to lines to clarify proposition and additional properties of triangles.	**Axiom 5**: Numerical example to clarify text. **Axiom 17**: Numerical example to clarify text. **Axiom 19**: Numerical example to clarify text. **Proposition 4**: Numerical lengths given to lines to clarify proposition. **Proposition 47**: Discussion of Pythagorean triples as historical source of Pythagorean Theorem. Numerical example given to clarify additional claim in scholion.	NONE
Book Two	**Definition 1**: Extension of definition from geometric into arithmetic form **Propositions 1 – 10**: Numerical examples offered as proofs; Barlaams' arithmetical versions of propositions, including proofs **Proposition 11**: Numerical example showing use of irrational numbers	**Definition 1**: Extension of definition from geometric into arithmetic form **Proposition 13**: Numerical lengths assigned to magnitudes to clarify additional claim in scholion.	**Proposition 12**: Numerical lengths assigned to magnitudes to clarify additional claim in scholion.
Book Three	NONE	NONE	NONE
Book Four	NONE	NONE	NONE

which, while they lack an obvious need for arithmetic, have clear analogies in number, making it possible to easily introduce arithmetic into the second book. However, of the three authors considered here, only Billingsley did so, showing his view of arithmetic as the foundational branch of mathematics. For him, even Euclid's early geometric claims were easier to understand and use numerically. He accompanied each of the first ten propositions in the second book with a numerical example and an arithmetical version of the proposition, complete with demonstration, that he credited to Barlaam, a fourteenth-century Greek monk, whose treatise giving arithmetical versions of the first ten propositions of Book Two was published in 1564 by the Strasbourg-based printer and author Konrad Dasypodius.[16] The propositions in question address the equality of rectangles and squares built on lines with given relationships. If one treats the rectangles and squares as plane numbers (i.e., numbers in which the units are arranged in a two-dimensional array), the properties of multiplication can be used to show that the same principles hold in arithmetic.[17]

For example, in Billingsley's text, the first proposition in the second book reads "If there be two right lines, and if the one of them be devided into partes howe many soever: the rectangle figure comprehended under the two right [straight] lines, is

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[16] Barlaam, Barlaam Monachi Arithmetica Demonstratio eorum, quae in secundo libro Elementorum sunt in lineis & figuris planis demonstrata, in *Euclidis quindecim elementorum geometriae secundum: ex Theonis commentariis Graece & Latine,* (Strasbourg: Konrad Dayspodius, 1564), 71-116.

[17] A plane number is a number in which the units are arranged in a two-dimensional array. According to the definitions found in the seventh book of *The Elements* a number is a collection of units and a unit is the source of all numbers. It is similar to the integer number 1, but in the sixteenth century a unit was not considered to be a number. For example 12 is a plane number as 3 x 4 or 2 x 6. Plane numbers do not have to be rectangular. They can be any shape. For example 6 is easily represented as a triangular plane number with one unit in the first row, two in the second, and three in the third. However, when studying the properties of multiplication plane numbers are only represented by rectangles and the special case of squares.

equall to the rectangle figures whiche are comprehended under the line undevided, and under every one of the partes of the other line."[18] The proof is done by drawing a rectangle contained by lines of the length of the given line, and dividing one of those lines into parts. From each division point of that line, a line is drawn to the opposite side parallel to the other edge of the rectangle. Thus, the original rectangle is divided into several smaller rectangles, and it is obvious from the image that the smaller rectangles together are equal to the original rectangle. (See Figure 6). Billingsley's translation of Barlaam's arithmetical version reads, "Two numbers beyng geven, if the one of them be devided into any numbers how many soever: the playne or superficiall number which is produced of the multiplication of the two numbers first geven the one into the other, shall be equall to the superficiall numbers which are produced of the multiplication of the number not devided into every part of the number devided."[19] This proof requires showing that if the two original numbers are called C and AB, and AB is divided into AD, DE, and EB, the product of C and AB is equal to the sum of the products of C and AD, C and DE, and C and EB. (See Figure 7). By the definition of multiplication, the number C measures the product of C and AB, which Billingsley calls F, by the number of unities contained in AB (i.e, F divided by C equals AB, in which F, C, and AB each represent an integer). In other words, if you take sets of C unities and arrange them in a single row, you will have a row of F

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[18] Henry Billingsley, *The Elements of Geometrie of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), 61*v.* I have used Billingsley's translation to avoid any modernization of the mathematics. The content of the proposition is identical in Clavius's and Commandino's texts. Today we express this simply as the sum of the products is equal to the product of sums, i.e. $ax + ay + az = a(x + y + z)$. It is known as the distributive property of multiplication.
[19] Ibid., 62*v.*

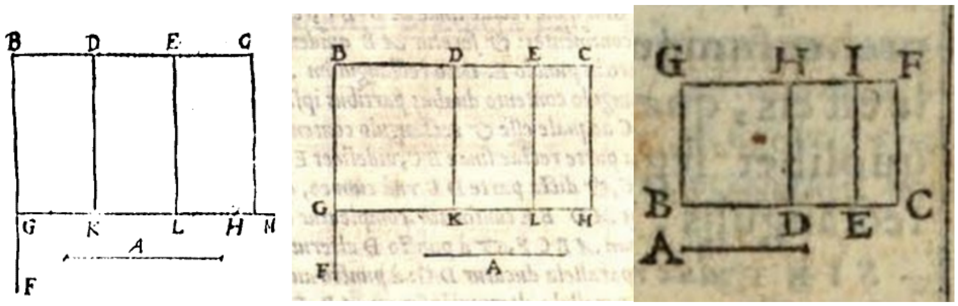

Figure 6: Diagrams for the Geometric Version of Book Two, Proposition, 1

In Billingsley's (left), Commandino's (middle), and Clavius's (right) images for Euclid's first proposition in Book Two. This proposition illustrates what we today call the distributive property of multiplication, but it does so using magnitudes, i.e. lines and rectangles.

Figure 7: Diagrams for Barlaam's Arithmetic Version of Book Two, Proposition 1

In left to right order, Billingsley's, Commandino's, and Clavius's images for Barlaam's version of the first proposition of Book Two. Note that Clavius and Commandino use dots to represent number, in keeping with their diagramming practice in the number theory books. It is not clear why Commandino used a solid line for F. Perhaps that was the printer's choice. The use of numerical labels still allows the reader to count the dots in the other numbers to verify the results. Billingsley's exclusive use of lines requires the reader to use the numerals and their knowledge of the properties of number rather than counting dots to verify equality. In all three figures the product of AB and C is F and the product of AD, DE, and EB with C are GH, HI, and IK. GH, HI, and IK sum to F.

unities once you have AB number of sets of C unities. Likewise, C measures its products with AD, DE, and EB by the unities contained in each of those numbers. And, since AD, DE, and EB, total to AB, the sum of their products with C will contain AB unities of C. Therefore, C measures the sum of its products with the segments of AB by AB, which means that the sum of the products of the segments of AB with C must be equal to the number F.

While Billingsley is the only one of the three authors to include Barlaam's text in the second book, both Clavius and Commandino included the monk's versions of the ten propositions in the ninth book as part of their development of number theory.[20] Despite the fact that all three authors included the same arithmetical versions of the ten propositions, their placements of Barlaam's work changed its significance in each text. In Billingsley's text, the arithmetical versions of the propositions are given after each of the geometric propositions and a numerical example thereof. This placement gave Barlaam's version equal importance to Euclid's. Commandino placed the ten propositions after his commentary on a proposition of the ninth book but did very little to delineate them from the rest of the commentary, suggesting that he saw the arithmetical versions as nothing more than a supplement to Euclid's text.[21] Clavius placed his ten propositions before the same proposition, but he clearly separated them from his commentary as a statement of numerical principles necessary to the

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[20] All three authors attributed the arithmetical versions to Barlaam and have very similar wordings for them. It is likely that they all used the same source, quite possibly the 1564 version.
[21] Commandino, *Euclidis Elementorum,* 114*v*.

remaining propositions in the ninth book.[22] However, since those principles had already been established geometrically, the arithmetical versions served primarily as pedagogical aids to ease the readers' understanding of Euclidean number theory by separating the arithmetic of the number theory books from the geometry of the second book.

In Billingsley's text, the arithmetic versions of the Euclidean propositions supplied by Barlaam were not merely equivalent to Euclid's propositions, they were superior. Billingsley made the mathematical equivalence of the arithmetical and geometric versions of the propositions clear in the first sentence of his commentary on the first of the propositions: "Because that all the Propositions of this second booke for the most part are true both in lines and in numbers, and may be declared by both: therefore I have added to every Proposition convenient numbers for the manifestation of the same."[23] He saw no distinction between the numerical and geometric versions of the problem. The numbers he provided, served as demonstrations of the arithmetic truth of Euclidean claims. In his treatment of the arithmetical versions, it becomes clear that Billingsley found those to be more valuable than the geometric propositions because they provided a simpler method of multiplication. To transition the reader from his numerical example of the geometric version of the first proof to Barlaam's arithmetical version, he observed that "by the aide of this Proposition is gotten a compendious way of multiplication by breaking of one of the numbers into his

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[22] Clavius, *Euclidis Elementorum,* 315*v*. Interestingly, nowhere in the ninth book does Clavius actually cite any propositions from the second book.
[23] Billingsley, *Elements of Geometrie,* 62*r*.

partes."[24] Thus, the arithmetical procedure for multiplication outlined in Barlaam's proposition was the source of the value contained in Euclid's geometric principle.

Furthermore, in Billingsley's reading, the geometric proposition was at best a special case of the arithmetic claim. This relationship between the two branches of mathematics is especially evident in the differences between the diagrams that Billingsley included for the geometric and arithmetic versions. Even though the products in Barlaam's proposition are described as "playne" numbers, Billingsley (and his Latinate contemporaries, who use the adjective "planus" to describe the products) represented the numerical proposition with lines or dots arranged linearly instead of repeating the rectangular areas drawn for the geometric version. (See Figures 6 and 7.) The meaning of the term "plane numbers" is that the units composing the number could be arranged in a planar geometric figure, so the choice to visualize the proposition linearly served to clearly differentiated number and magnitude. The visualizations all show that number, unlike magnitude, is not spatially defined. However, while Clavius and Commandino used series of dots to represent the numbers, thereby emphasizing the independence of arithmetic and geometry as, respectively, studies of discrete and continuous quantity, Billingsley represented the numbers with lines, suggesting that geometric magnitudes could be generalized to numerical quantities. Furthermore, the lines Billingsley drew are not to scale with one another, making it impossible for the image to be used to confirm the proposition.[25] Only the numerical labels could lead the readers' to assent to the truth of the claim.

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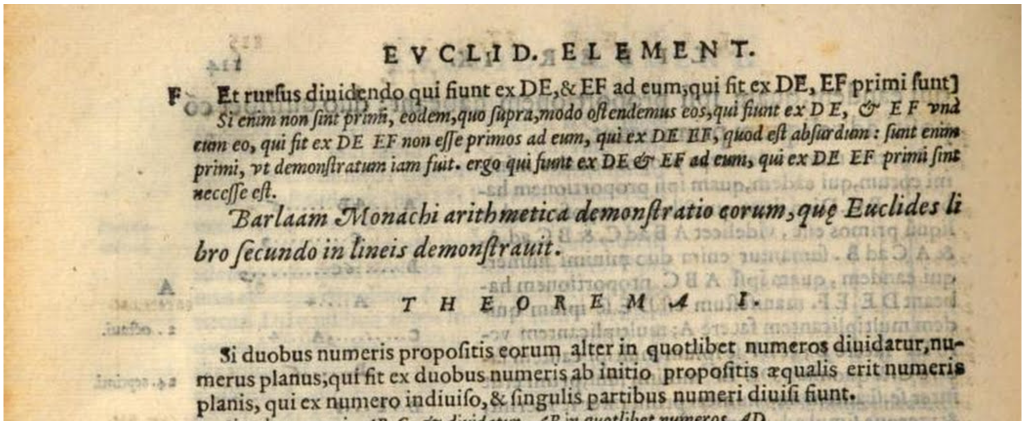
[24] Ibid., 62*v*.

[25] In the Latin texts, the reader can count the dots in order to verify the proposition.

Four (C) times six (AB) is twenty-four. Four times two (AD/DE/EB) is eight, and eight plus eight plus eight is also twenty-four. Thus the physical lines, which define the geometric version of the proof, are superfluous to the arithmetic proposition, and the latter becomes a general description of multiplication that applies to both abstract numbers and physical magnitudes.

In contrast, Commandino's placement of the arithmetical propositions minimizes their importance and treats the arithmetical formulations of geometric propositions as unnecessary supplements to Euclid's discussion of number theory. While Billingsley's juxtaposition of the geometric and arithmetical versions of the propositions allowed him to show that he valued the latter more, Commandino's complete separation of the arithmetical topics from their geometric analogs allowed the geometry of the second book to stand on its own and raised no questions about its foundational status. Furthermore, as seen in Figure 8, Commandino placed the ten theorems immediately following commentary on a proposition in the ninth book. There is no extra spacing between the last line of commentary and the first line introducing Barlaam's work. It is possible that the lack of space was an oversight on the part of the printer. The font introducing the theorems is a little larger than the font for his commentary, but since it retains the same italic setting as Commandino used for his commentary, even if there had been a space, the arithmetical propositions would appear as a continuation of the immediately preceding commentary. It is possible that he fully intended for Barlaam's demonstrations to be read as part of his commentary. The proposition that they follow (Book Nine, proposition fifteen) states



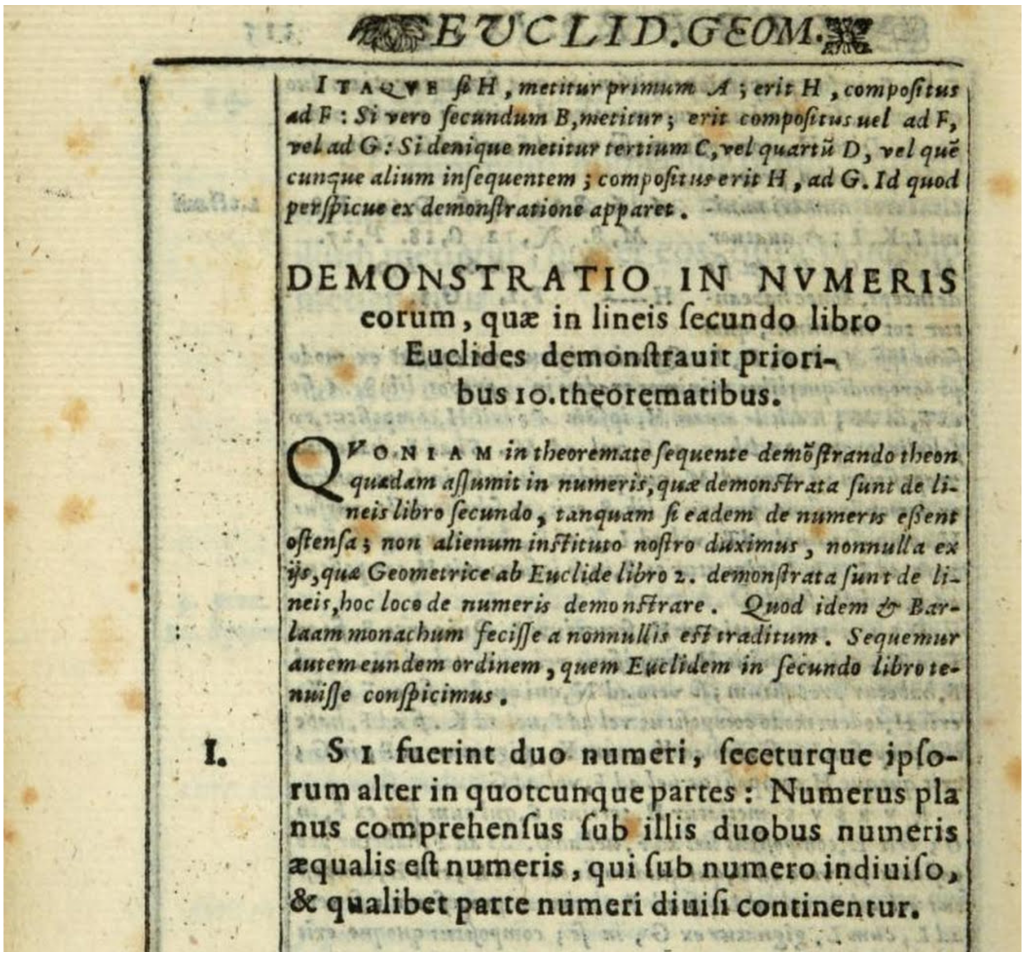
EVCLID. ELEMENT.

F Et rursus diuidendo qui fiunt ex DE, & EF ad eum, qui fit ex DE, EF primi sunt]
Si enim non sint primi, eodem, quo supra, modo ostendemus eos, qui fiunt ex D E, & E F vnà cum eo, qui fit ex DE EF non esse primos ad eum, qui ex DE EF, quod est absurdum: sunt enim primi, vt demonstratum iam fuit. ergo qui fiunt ex DE & EF ad eum, qui ex DE EF primi sint necesse est.

Barlaam Monachi arithmetica demonstratio eorum, quę Euclides libro secundo in lineis demonstrauit.

THEOREMA I.

Si duobus numeris propositis eorum alter in quotlibet numeros diuidatur, numerus planus, qui fit ex duobus numeris ab initio propositis æqualis erit numeris planis, qui ex numero indiuiso, & singulis partibus numeri diuisi fiunt.



EVCLID. GEOM.

ITAQVE si H, metitur primum A; erit H, compositus ad F: Si vero secundum B, metitur; erit compositus uel ad F, vel ad G: Si denique metitur tertium C, vel quartũ D, vel quẽcunque alium insequentem; compositus erit H, ad G. Id quod perspicue ex demonstratione apparet.

DEMONSTRATIO IN NVMERIS eorum, quæ in lineis secundo libro Euclides demonstrauit prioribus 10. theorematibus.

QVONIAM *in theoremate sequente demõstrando theon quædam assumit in numeris, quæ demonstrata sunt de lineis libro secundo, tanquam si eadem de numeris essent ostensa; non alienum instituto nostro duximus, nonnulla ex ijs, quæ Geometrice ab Euclide libro 2. demonstrata sunt de lineis, hoc loco de numeris demonstrare. Quod idem & Barlaam monachum fecisse a nonnullis est traditum. Sequemur autem eundem ordinem, quem Euclidem in secundo libro tenuisse conspicimus.*

I. SI fuerint duo numeri, seceturque ipsorum alter in quotcunque partes: Numerus planus comprehensus sub illis duobus numeris æqualis est numeris, qui sub numero indiuiso, & qualibet parte numeri diuisi continentur.

Figure 8: Commandino's and Clavius's Placements of Barlaam's Theorems

These two images show the introduction of Barlaam's theorems in Commandino's text (top) and Clavius's text (bottom). Both images are cut off immediately following the enunciation of the first theorem.

that if three numbers are the smallest sequence of numbers with a given proportion between them, the sum of any two of the numbers is relatively prime to the third.[26] The proof for that theorem relies on the ability of the reader to create and factor products, ideas developed in these demonstrations.[27] In addition, Commandino did not give any discussion of why he chose to include arithmetical renderings of propositions from the second book. Before providing the first theorem his text only says, "Barlaam of the Monks' arithmetical demonstrations of those that Euclid had demonstrated in lines in the second book."[28] It is left to the reader to determine what significance these demonstrations have, suggesting that they serve only as an interesting addition to the preceding proposition.

Clavius's approach to the same arithmetical theorems took a middle road between those of his contemporaries, treating Barlaam's collection of arithmetical demonstrations as a tool to aid the reader in his study of number theory as a whole, not just the proposition to which they were attached (again, Book Nine, proposition fifteen). Like Billingsley, Clavius treated the arithmetical demonstrations as

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[26] Two numbers are said to be relatively prime if they share no factors besides 1.

[27] The proof is set up with a total of five numbers: the three numbers in sequence, and the two smallest numbers that express the proportion between the original three. In order to make the following outline of the proof intelligible, we will call those last two numbers x and y. As the smallest numbers that express the proportion, they must be relatively prime to one another. From a previous proposition, the reader can recognize that because the three numbers in sequence are said to be the least numbers with this proportion between them, they must be, in ascending order, the square of the smaller of the two numbers in the ratio, the product of the two numbers in the ratio, and the square of the larger of the two numbers in the ratio. The proof is then based on these numbers. If x is the smaller of the two numbers making up the proportion, the three numbers of the proposition are x^2, xy, and y^2. The proof is then done by showing three cases to be true: $x^2 + xy$ is relatively prime to y^2, $xy+y^2$ is relatively prime to x^2, and x^2+y^2 is relatively prime to xy. Those cases are accomplished through factoring the terms in the sums.

[28] Commandino, *Euclidis Elementorum,* 114*v.* "Barlaam Monachi arithmetica demonstratio eorum, que Euclides libro secondo in lineis demonstravit."

foundational propositions, albeit in the limited context of number theory. However, like Commandino, he restricted the propositions to a role as supplements to the Euclidean text. By presenting the demonstrations as their own section delineated from the rest of the text by a clear heading (see Figure 8), Clavius made it obvious that the arithmetical demonstrations, as part of the foundation of number theory, were propositions supplemental to the whole body of Euclid's work rather than just commentary on a single proposition. However, when Clavius introduced the ten theorems, he justified their presence saying, "Since in the following theorem, which is about to be demonstrated, Theon assumes certain things for numbers, which are demonstrated for lines in the second book, so that if the same have been shown in numbers, we think that it is not alien to our intention to demonstrate here in numbers a few of those which are demonstrated geometrically in lines by Euclid in the second book."[29] While the numerical demonstrations may have been foundational to more than one proposition, it was the assumption of their principles in a single proposition that allowed Clavius to justify their inclusion. Their primary purpose thus appears to have been to assure the reader that the proof for that particular theorem in Euclid's text did not rely on any unproven assumptions, making them pedagogical aids to assure the reader of the certainty of geometry. Furthermore, Clavius never asserted that the numerical demonstrations were necessary to the ninth book; they were simply "not alien." This meant that the geometric proofs in the second book that demonstrated the

[29] Clavius, *Euclidis Elementorum,* 315*v*. "Quoniam in theoremate sequente demonstrando theon quaedam assumit in numeris, quae demonstrata sunt de lineis libro secundo, tanquam si eadem de numeris essent ostensa; non alienum instituto nostro duximus, nonnulla ex iis, quae Geometrice ab Euclide libro 2. demonstrata sunt de lineis, hoc loco de numeris demonstrare."

same principles through the use of lines could serve as the foundation for the number theory found in Book Nine. Thus, Clavius showed that, even though arithmetic could be independently developed, geometry was the foundational branch of all of mathematics.

Proportion: Geometry or Arithmetic?

Barlaam's arithmetical propositions were not the only opportunity for the authors to systematically make use of numbers before the number theory books of *The Elements*. The fifth and sixth books undertake a study of proportion, whose obvious connections to the study of number provided the authors with myriad opportunities to employ numbers and arithmetic in the study of geometry. Indeed, as can be seen in Table 3, all three authors accompanied several of the definitions for the fifth book with some use of numbers, suggesting that they all used arithmetic to make a geometric theory of proportion intelligible.[30] However, only Billingsley used numbers to

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[30] Sabine Rommevaux offers an analysis of the fifth book of Clavius's Euclid in her book, *Clavius une clé pour Euclide au XVIe siècle*. In her third chapter, she asks whether Clavius successfully arithmeticized the study of proportion found in the fifth book and comes to the conclusion that he did not. According to Rommevaux, there are three forms of arithmeticization that Clavius could have pursued. The first is the use of numerical examples, but in the fifth book Clavius avoided using numerical examples that required anything other than whole numbers, suggesting that he saw his numerical examples as aids to the reader rather than as the beginning of a study of arithmetic. The second was the unification of the theories of proportion for magnitudes and numbers, but, as Rommevaux shows, Clavius presented the theories of proportion for magnitudes and numbers as two parallel theories. Finally, Rommevaux argues that Clavius could have used ratios to establish a unity between magnitudes and numbers, but his use of ratios only appears to apply to magnitudes. By comparing Clavius's text to Billingsley's and Commandino's, I hope to show that in 1574 Clavius did not intend to arithmeticize geometry. Instead, his use of number in the study of proportion was designed to aid the reader's understanding of geometry. However, the pedagogical use of arithmetic to facilitate the study of geometry, did more to unite the two branches of mathematics than either Billingsley's or Commandino's uses of number did. Each of those authors kept the two entirely separate and attempted to make one branch the foundation of the other. Clavius allowed the two branches to work together, which makes Rommevaux's question a starting point for further study of the

Table 3: The Uses of Number in Books Five and Six

The descriptions note the various ways in which number appears in these books. I have also noted with an asterisk where the authors include examples that are only in magnitudes. These are line diagrams without numerical labels. In Clavius's and Billingsley's texts, some of the lines are divided by hash marks of the unit length of the shortest line. I have counted these as magnitude only diagrams because the pieces of the line cannot be assigned a number without assuming that unit length is one, which it may or may not be. When lines are divided by hash marks of unit length one, the unit length was determined either by the examples in the text that reference the diagrams or by the numerical labels on the diagrams. An empty table entry indicates that no numerical example was included for that definition. NOT IN TEXT indicates that the definition itself is not found in that commentary.

Definition	Billingsley	Clavius	Commandino
V.1	Numerical examples in text. Lines labeled with numbers and divided by hash marks of unit length 1	Numerical examples in text. *Example in magnitude	Numerical example in text
V.2	Numerical examples in text. Lines labeled with numbers and divided by hash marks of unit length 1		
V.3	Numerical examples in text. Lines labeled with numbers and divided by hash marks of unit length 1	Numerical example in text (numbers described as lines of certain lengths)	
V.4	Numerical examples in text. Lines divided by hash marks of unit length 1	Lines labeled with number and divided by hash marks of shortest lines' lengths	
V.5	Numerical examples in text, *Example in magnitude	Numerical example in text Lines labeled with number and divided by hash marks of shortest lines' lengths	Lines labeled with numbers
V.6	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Numerical example in text Numerical examples "visualized" outside of text *Example in magnitude	
V.7	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with number and divided by hash marks of shortest lines' lengths	Lines labeled with numbers
V.8	Numerical examples in text, Lines divided by hashmarks of unit length 1 Numerical examples "visualized" outside of text *Example in magnitude	Numerical example in text Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with numbers
V.9	Numerical examples in text, *Example in magnitude	Lines labeled with number and divided by hashmarks of shortest lines' lengths	Lines labeled with numbers
V.10	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with numbers Numerical examples in text	Numerical example in text Lines labeled with numbers
V.11	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with numbers *Example in magnitudes visualized	Lines labeled with numbers

Continued on next page

Table 3: The Uses of Number in Books Five and Six (continued)

Definition	Billingsley	Clavius	Commandino
V.12	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with number and divided by hash marks of non-one unit length	
V.13	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with numbers	Lines labeled with numbers
V.14	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitudes	Lines labeled with numbers	Lines labeled with numbers
V.15	Numerical examples in text, Numerical examples "visualized" outside of text *Example in magnitudes	Lines labeled with number and divided by hash marks of non-one unit length	Lines labeled with numbers
V.16	Numerical examples in text Numerical examples "visualized" outside of text *Example in magnitudes	Lines labeled with number and divided by hash marks of non-one unit length	*Example in magnitude
V.17	Numerical examples in text Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with number and divided by hash marks of non-one unit length	*Example in magnitude
V.18	Numerical examples in text Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with number and divided by hash marks of unit length one	Lines labeled with numbers
V.19	Numerical examples in text Numerical examples "visualized" outside of text *Example in magnitude	Lines labeled with number and divided by hash marks of unit length one, and other by hash marks of non-one unit length	Lines labeled with numbers
V.20			
V.21		NOT IN TEXT	NOT IN TEXT
VI.5	Numerical examples in text Lines divided by hash marks of unit length 1 Numerical examples "visualized" outside of text	Numerical examples in text Lines labeled with number and divided by hash marks of shortest lines' lengths *Example in magnitude	NO USE OF NUMBERS

develop an arithmetical theory of proportion that could serve as a foundation to the geometric version. In contrast, Commandino used numbers only as a shorthand to help the reader interpret his diagrams by providing numerical lengths as part of his visual aids, giving arithmetic no place in his text as its own discipline. Clavius fell in between his two contemporaries, developing numerical examples in his commentary to clarify the geometric theory of proportion. For him, arithmetic offered a study parallel to geometry that could be used to offer students a secondary way to understand proportions.

According to Billingsley, the use of arithmetic to aid the study of geometry meant that the theory of proportion found in the fifth book of *The Elements* was properly an arithmetical study. In his commentary on the second definition, "multiplex" or "multiple," he called the reader's attention to the foundational role of number in the study of proportion.[31] Even though Billingsley presented the definition in terms of magnitudes, he claimed that because the term multiplex is "proper to Arithmetike and number, it is easy to consider that there can be no exact knowledge of proportion and proportionalitie, so of this fifth booke wyth all the other books followyng, without the ayde and knowledge of numbers."[32] In keeping with that sentiment, he provided numerical examples of all of the definitions as a means to develop an arithmetical theory of proportion within Euclid's text. In fact,

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role pedagogical texts played in the combination of arithmetic and geometry. Sabine Rommevaux, *Clavius une cle pour Euclide au XVIe siècle,* (Paris: Librairie Philosophique J. Vrin, 2005), pp. 59-75.

[31] Billingsley, *Elements of Geometrie,* 126*v.* "Multiplex is a greater magnitude in respect of the lesse, when the lesse measureth the greater."

[32] Ibid., 126*v.*

Billingsley's geometric discussions of each definition are quite brief. Most of his commentary is devoted to his numerical examples, and, after the fifth definition, there are several purely numerical examples "visualized" outside of the main text. (See Figure 9.) In these examples, Billingsley made no attempt to link his study of proportion to geometrical magnitudes, leaving the reader with the impression that the arithmetical study of proportion was a sufficient foundation for the geometrical arguments yet to come in *The Elements*. Furthermore, while Billingsley provided numerical labels on some of his line diagrams, the lengths shown in those labels were repetitions of the numerical examples given in the prose, making the magnitudes a tool to visualize the numbers rather than using the numbers to clarify the relationships between the lines. As a result, these visual aids emphasized that the definitions expressed in geometric terms could be rendered in arithmetic terms by replacing magnitudes with numbers. In those cases when Billingsley did not provide numerical labels for the lines at all, his commentary only made brief mention of the magnitudes before turning to numerical examples, clearly suggesting that he expected his readers to rely on the numbers instead of the lines. Thus, the lines representing Euclidean magnitudes became superfluous to the study of proportion.

For Commandino, quite the opposite was true. Arithmetic had no foundational place in the study of geometry, and it was the numbers that were superfluous to the study of proportion. Indeed, Commandino never even used the word "arithmetic" in his edition of Euclid's fifth book. Numbers, in the context of geometry, were nothing more than a practical shorthand to aid the study of magnitudes. He only included them

Figure 9: Some Examples of Uses of Numbers in the Definitions of Book Five

Top Two Rows: Billingsley's visualizations from a selection of definitions in Book 5. Top Left: Book Five, definition 2, the image shows 9 as a multiple of 3 by using hashmarks of unit length one, allowing the reader to count three sets of three in line CD and one set of 3 in line AB.

Top Right: Book Five, definition five, Billingsley shows that A and B have a proportion between them because A can clearly be multiplied to be greater than B. The line above A and B includes hashmarks dividing it into 3 units of length A. Unlike his use of hashmarks in the second definition, this use of hashmarks does not rely on arithmetic since the unit of 1 is not present.

Bottom Left: Book Five, definition 6, This image is one of Billingsley's numerical examples using proportions. The interior numbers represent two equal proportions (8/6 and 4/3). The exterior numbers are the result of an operation described in the definition.

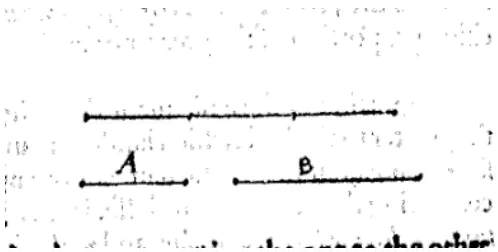
Bottom Right: Book Five, definition 8 This is another example dealing with operating on equal proportions. Again, the interior numbers represent the original proportions. Here, even though Billingsley calls it an example "in magnitudes," he includes numerical labels and hashmarks that enable counting on the original numbers.

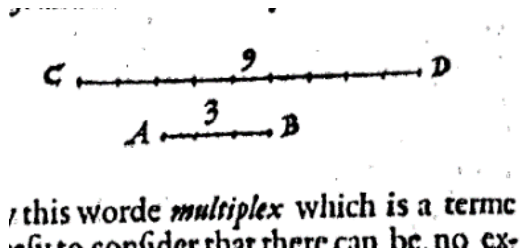
Third Row: Two of Clavius's visualizations.

Left: Book Five, definition 5 These four lines represent that equimultiples of two numbers yield two numbers with the same proportion between them as the first two. Many of Clavius's examples include lines either labeled with numbers or with hashmarks of the length of the shorter line in a pair. In this case the pairs are 6 and 12 and 8 and 16. The hashmarks make it clear that the latter line is the double of the former. Unlike Billingsley, Clavius rarely used hashmarks of unit length, which kept his focus on the relationships between the lines rather than the numbers.

Right: Book Five, definition 6 This is an example of Clavius's visualization of a numerical example. In it the numbers in the square represent the proportions being compared, in this case 3/2 and 6/4. Going out from each number are three sets of equimultiples of the antecedent of one proportion and the consequent of the other, the operation required by the definition to show that the original proportions are equal. That is 3 and 6 are multiplied by the same numbers and 3 and 4 are multiplied by the same numbers. Each set of numbers produced is connected by a curved line allowing the reader to understand which numbers to compare to verify the conditions set forth in the definition.

Bottom: An example of Commandino's numerically labeled lines. These lines go with definition 11 and show four numbers in proportion with each other. The numbers, 8, 12, 18, and 27 are a proportional series with the ratio 3/2 between each pair.


A
B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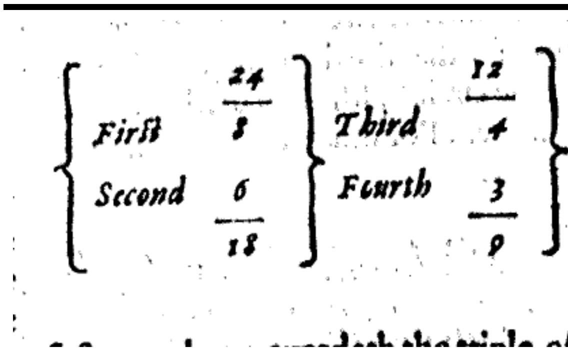
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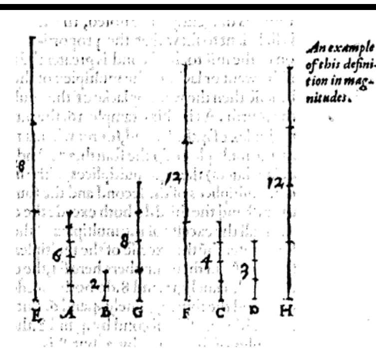
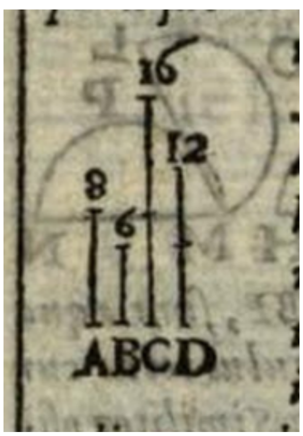

An example of this definition in magnitudes.
E A B G F C D H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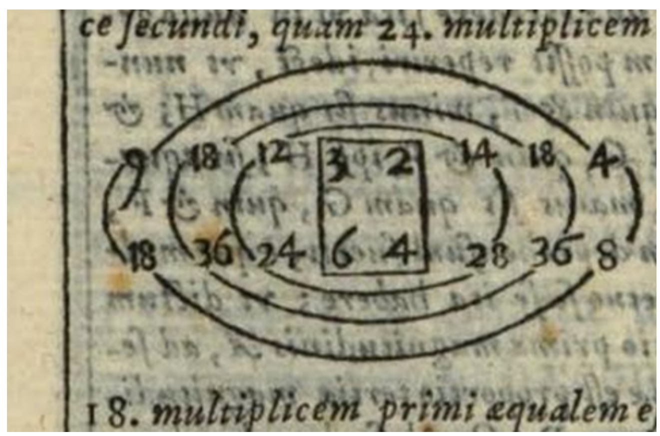
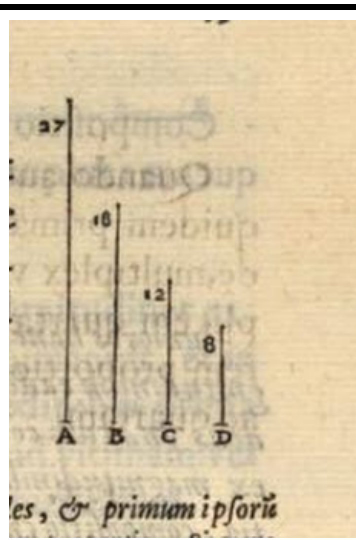

ce secundi, quam 24. multiplicem
9 18 12 3 2 14 18 4
18 36 24 6 4 28 36 8
18. multiplicem primi æqualem e


27
18
12
8
A B C D
es, & primum ipsoru

as measurements for the lines he drew as his visual aids for the definitions. As such, they offered a shortcut to the reader who need not measure each line for himself and might also have acted as a safeguard against possible printing errors that could have created lines that did not have the relationship claimed in the text. Moreover, Commandino did not acknowledge the presence of numbers in his commentary, which discussed the propositions purely in terms of the relationships between magnitudes. While numbers were tools to read the visual aids that he provided (Figure 9), they were not essential to the development of a Euclidean theory of proportions.

Like Commandino, Clavius relied on numbers as a tool to clarify the geometric theory of proportion. However, because all of the same basic principles applied to ratios of discrete and the ratios of continuous quantities, Clavius saw numerical proportions as direct analogs to those of magnitudes. Therefore, even though linear magnitudes remained the focus of his text, his numerical examples were more than measurements of the lines, and they offered arithmetical proportion as a second, usually shorter, way to conceive of proportion. The analogous relationship between arithmetic and geometry is evident in the structure of Clavius's commentary. He began his discussion of each definition with an examination of magnitudes without any mention of number. His line diagrams, often labeled with numerically designated lengths, appeared in this section of his commentary. After presenting a purely geometric explanation of a concept, Clavius included numerical examples, usually using the same numbers found in his labels, as a second way for the reader to grasp the definition. Most of Clavius's numerical examples appeared in the form of lengths assigned to lines, but, on occasion, he gave examples complete with visual aids that had no connection to a given magnitude. (See Figure 9). Still, even these purely numerical examples were intended to help the reader interpret the geometric claims, not to develop the theory of proportion arithmetically. Indeed, when Clavius introduced his first purely numerical example he said that it "was sufficient to persuade" the reader of the truth of the claim being made, suggesting that Clavius felt that a geometric example would have been more complete, but the analogy between proportion in geometry and arithmetic allowed him to supply the shorter numerical example instead.[33]

Despite his use of numerical examples to help clarify the geometric theory of proportion, Clavius did not share Billingsley's belief that arithmetic was a necessary foundation for the study of geometry. He first introduced the idea that the theory of proportion could be developed outside of geometry in a treatise titled "On Proportion" inserted between the third and fourth definitions of Book Five. He claimed that he included this treatise because a general introduction to the various kinds of proportions could be useful to Euclid's study of proportion in magnitudes, suggesting that the study of proportion could be developed in multiple branches of mathematics, and that those analogous studies could lead to insights about geometric proportions.[34] Book

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[33] Clavius, *Euclidis Elementorum,* 155*v.* "placuit unum exemplum adducere in numeris."
[34] Ibid.*,* 145*v*. "Operae pretium esse arbitror, paucis hoc loco exponere, quotnam sint genera proportionum apud Mathematicos, vel ob hanc praecipue utilitatem, ut ea, quae in duobus libris ab Euclide demonstrantur de propotionibus magnitudinum, rebus possint materialibus accommodari, quando opus fuerit." The treatise covers pages 145*v*-152*v*. For most of it, Clavius treats the values discussed as magnitudes assigning units of length to values (e.g. a line of twenty palms). However, he does on occasion allow the discussion to rely on numbers instead of lines. Sabine Rommevaux notes that Clavius's 1589 edition extended the treatise. Her analysis of Clavius's potential arithmetization of geometry in her book, *Clavius un cle pour Euclide,* relies on the later edition. The extended version

Five of *The Elements* only addressed proportions between magnitudes. Other texts were available to study proportion in the other branches of mathematics. In fact, in Book Seven of *The Elements*, Clavius included several definitions on proportion in numbers, beginning the development of an arithmetical theory of proportion.[35] Already in his commentary on the fourth definition of the fifth book, Clavius pointed the reader to Jordanus and Boethius as sources for the study of proportion in arithmetic and music, respectively.[36] However, these references are available for the interested reader without being required reading to understand the theory of proportion. Clavius was only acknowledging the presence of analogous theories in other branches of mathematics, a far cry from Billingsley's claim that proportion is "first and naturally founde" in number.[37] From Clavius's perspective, number and arithmetic could help readers to understand geometry but only as analogs to magnitude and geometry.

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furthers the treatment of arithmetic, but as Rommevaux convincingly argues, Clavius still did not arithmetize geometry. Stillman Drake criticized Clavius's inclusion of this treatise as an explanation of "medieval proportion terminology in Book V, where it was completely out of place." Drake, "Euclid Book V," 70. However, such a criticism says only that Clavius did not adhere strictly to the ancient separation of geometry and arithmetic without considering what purpose Clavius believed the treatise served.

[35] Clavius was not the only author to treat proportion in his definitions, but he treated it more extensively than the other two authors. All three authors included one definition for proportional numbers, and Commandino included a definition identical to one in Clavius's text describing a series of proportional numbers. Clavius's text contained three definitions not found in either of his contemporaries' texts: a number measured by another number, proportion, and the roots of proportions (i.e. in today's terminology, reduced fractions).

[36] Clavius, *Euclidis Elementorum,* 153*r.* "Multae autem habitudines proportionum, seu proportionalitates, (Nos enim comparationem duarum quantitatum, proportionem appellabimus; habitudinem autem propotionum, Proportionalitatem) a scriptoribus, praesertim Boetio, & Iordano, descributunr; inter quas primum semper locum obtinuerunt apud Ueteres, Proportionalitas Arithmetic, Geometrica, atque Musica, seu harmonica."

[37] Billingsley, *Elements of Geometrie,* 126*v*. "For the opening of them [proportions] in numbers (in which they are first and naturally founde) geveth a great and marveilous light to their declaration in magnitudes." Billingsley's treatise also comes between the third and fourth definitions, but he did not separate it from the text with a heading. In his text it serves as the commentary on the third definition.

Number Theory in a Geometry Book?

Books Seven through Nine of *The Elements* offer a study of number theory. While, given the ostensible division between number and magnitude in sixteenth-century descriptions of mathematics, it may seem odd that there are three books on number theory in a geometry textbook, all three authors compared in this study agreed that the number theory books were a necessary interlude between the studies of plane and solid geometry in order to introduce numerical ideas that make commensurability and incommensurability, the topics of the tenth book, intelligible.[38] Those concepts, in turn, are necessary to the books on solid geometry which rely on the relationships among the lines that compose solid figures to demonstrate the properties of the various figures. The discovery of those relationships was often begun by determining whether

[38] Commensurability is the property of lines being able to measure one another or having a common measure. It appears to be analogous to the numerical concept of factoring. However, that analogy is not quite right because commensurable lines simply have a ratio from one to another. For example, a line of length 2 and a line of length 3 are commensurable with one another because the ratio between them, 3/2, is a measurable quantity. One could produce half of the smaller line and add it to the smaller line to produce a line of equal length to the larger line. The two numbers, two and three, are both prime numbers with only one as a common factor. Even if the length three were replaced with a non-integer rational length, such as 7/3, the first line could be used to create a line equal in length to the second. Incommensurable lines share no common measure, numerical or proportional. (Note that rational numbers were not considered proper numbers, but rather proportions of numbers.) In modern terms, numerically, one line in a pair of incommensurable lines is irrational in length. Because one line has an irrational length and the other does not, it is not possible to manipulate either line to produce a new line of the same length as the other line. Commensurable and incommensurable numbers are respectively defined in propositions six and seven of the tenth book. (Billingsley, *Elements of Geometrie,* 235*v* - 236*v*; Commandino, *Euclidis Elementorum,* 127*r*-128*r*; Clavius, *Euclidis Posteriores,* 12*v* – 14*r*.) It should be noted that two irrational numbers may be commensurable to one another. The square root of two is irrational, but it is commensurable with all multiples of the square root of two. Likewise, the circumference of a circle is incommensurable with the circle's diameter, but if two circles have commensurable diameters, the circumferences of those circles are also commensurable since the factor of π cancels out. That is, if the diameter of the first circle is a, and the diameter of the second is b¸ then the circumference of the first is πa and the circumference of the second is πb. The ratio between the two circumferences is the ratio between the diameters, a/b.

two quantities were commensurable or incommensurable to one another. Since the seventh through tenth books are the part of *The Elements* in which the study of number is formally developed, they offered the authors an opportunity to provide an explicit statement of the relationships between magnitudes and numbers and their respective studies, geometry and arithmetic.

Of the three authors studied here, Billingsley provided the clearest analysis of the relationship between the two disciplines. In his introduction to the seventh book, he offered extensive praise for arithmetic, explicitly claiming that it is the true foundation of mathematics because Euclid could not complete his study of geometry without introducing arithmetical ideas. The solid geometry books require "the helpe and ayd of nombers" because the irrational quantities that are introduced in the tenth book and that appear in the solids of the remaining books cannot "be knowen and found out without number."[39] Billingsley's praise of arithmetic offers arguments frequently found in cases for the nobility of mathematics. Number is the purely abstract foundation of "all other sciences and artes. As to musicke, Astronomy, natural philosophy, perspective, with others."[40] By attributing the features of mathematics that give it nobility to arithmetic rather than to geometry (as Clavius and Commandino both did), Billingsley allowed the latter to become a useful practical application of

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[39] Billingsley, *Elements of Geometrie,* 183*r*.
[40] Billingsley, *Elements of Geometrie* 183*r*. "Now thinges sensible are farre under in degree then are thinges intellectuall: and are of nature much more grosse then they. Wherefore number, as being only intellectuall, is more pure, more immateriall, and more subtile, farre then is magnitude and exceedeth itself farther."

number to the physical world rather than a noble study of abstract ideas.[41] By making geometry a useful extension of the study of number, Billingsley reminded the reader that his motivation for translating *The Elements* was to provide his countrymen with a tool that could aid inventions.[42]

The relationship Commandino outlined between arithmetic and geometry is quite the opposite of what Billingsley presented. In his text geometry's continuous magnitudes are shown to allow a more complete study of quantity than is possible with discrete numbers. In his text, arithmetic is reduced to an inspiration for geometric study. In fact, Commandino only addressed the relationship between number and magnitude in his scholium preceding the tenth book, and even there he focused on magnitude, mentioning number only as the source of Pythagoras' inspiration for the notions of commensurability and incommensurability. While he acknowledged that legend claimed that Pythagoras developed the notion of incommensurability through a consideration of number, he argued that the fact that unity serves as a common measure for all numbers prevents a complete development of the concept of incommensurability. As he explained, one cannot find a common measure for all magnitudes, as unity is for numbers, because "All numbers divided equally into however many sections, leave some minimum part that does not allow division [unity]. However, all magnitudes divided infinitely do not leave a part, no

[41] Of course, since Billingsley was a merchant, attributing the ennobling traits of mathematics to arithmetic, he also aggrandized his own profession. If arithmetic was the most noble branch of mathematics, the arithmetic work of a business required expertise in a noble science.
[42] See Chapter 2 and Billingsley's letter to his reader.

matter how small it is, that cannot be cut."[43] The notion of irrational, and therefore incommensurable, quantities is thus tied to the geometrical construction of lines because they can only exist as lines, which need not be composed of collections of unities as numbers must be. While the idea may have historically originated in Pythagoras's contemplation of number, it was only in geometry, the true foundation of mathematics, that the notions of commensurability and incommensurability could be developed through the creation of incommensurable quantities. For Commandino, number, and thus arithmetic, were more limited in their scope than magnitude and geometry, allowing geometry to remain the foundational branch of mathematics.

Once again, Clavius found a middle ground between his contemporaries by arguing that arithmetic acted as a tool for the study of geometry. Like Billingsley, he began the seventh book with a brief comment expressing the utility of arithmetic to geometry. He noted that the studies in the seventh through ninth books "concerning the qualities and properties of numbers, insofar as they serve the interests of geometrical things, are taken up, so that then in the tenth book the demonstrations of commensurable and incommensurable lines may be more easily and clearly executed."[44] However, he stopped short of arguing that arithmetic was foundational to

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[43] Commandino, *Euclidis Elementorum,* 121*v*-122*r.* "Venerunt autem initio ad inquisitionem symmetriae, hoc est commensurabilitatis Pythagoraei primi, ipsam ex numerorum cognitione invenientes, cum unitas, sit omnium numerorum communis mensura, & in magnitudinibus communis mensura inveniri non possit. Huius caussa est, quod omnis numerus, iuxta quaslibet sectiones divisus relinquit particulam aliquam minimam, & quem sectionem non admittit. Omnis autem magnitudo in infinitum divisa non relinquit particulam, quae propterea quod minima sit, secari non possit. … Cum hoc intelligerent pythagoraei, ut fieri potuit, in magnitudinibus mensuram invenerunt. Omnes enim, quas eadem mensura metitur, commensurabiles appellarunt; eas vero, quas non metitur eadem mensura, incommensurabiles."

[44] Clavius, *Euclidis Elementorum,* 233*v.* "Quare hoc libro septimo, & duobus insequentibus, circa numerorum proprietates, affectionesque, quantum eae rei Geometricae inserviunt, occupatur, ut in

geometry or that arithmetic was essential to understanding geometric claims. In fact, at the beginning of the tenth book, Clavius included an argument much like Commandino's that claimed that the study of roots (i.e. square roots, cube roots, etc.) required knowledge of geometry.[45] However, Clavius minimized the impact of this argument by placing it in a parenthetical comment following his dismissal of the argument that arithmetic, specifically the study of roots, was foundational to geometry.[46] His focus was not on the fundamental relationship between the two branches of mathematics, but on the benefit arithmetic could provide to the student of geometry, specifically, easing the reader's understanding of the notion of commensurability. By treating arithmetic as a pedagogical tool, Clavius allowed it to be more than merely an inspiration for the geometric notion of commensurability, as it

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decimo deinde facilius, ac plenius demonstrations linearum commensurabilium, & incommensurabilium exequatur."

[45] Clavius, *Euclidis Posteriores,* 2*v.* "Immo contra persuasum mihi prorsus habeo, cognitionem perfectam illius partis Arthmetices, pendere ex hoc 10. lib. tantum abest, ut existemem, tractionem illam radicum requiri, ut facilius hic liber intelligatur. Non negarim tamen, eum qui rationem radicum atque calculum tenuerit, maiore cum voluptate hunc librum precepturum, quam qui illarum omnino sit ignarus propterea quod ille demonstrationes ad usum potest revocare, hic vero nullo modo. Hac enim de causa & nos priora decem theoremata secundi lib. numeris accommodavimus, ut oblectationem animi maiorem ex eo studiosus caperet, ac fructum, non autem ut ea, quae in illo demonstrantur, facilius aribtraremur intelligi posse ex numeris. Cur ergo (dicet aliquis) ut in eo libro, non perinde etiam in hoc exempla numerorum, quibus Algebra utitur, usurpasti, ut ea res maiori voluptati esset, & commodo legentibus? In promptu causa est, quare id omittendum putaviumus. Cum enim perpauci sint hoc tempore, quibus celeberrima illa Algebrae ars sit congita, videbantur numeri illi, si adhiberentur, tenebras potius offusuri, quam lucis aliquid maioris daturi, & perspicuitatis; quippe ita ingenia studiosorum pro adiumento, ac luce, quam his nostris commentariis afferre laboramus, plus caperent incommodi, minusque demonstrationes ipsas perciperent."

[46] Ibid., 2*r-v*. He dismisses the argument by refusing to give it a complete discussion. "Quoniam vero hic liber multis obstructus est difficultatibus, ob linearum, de quibus disserit, obscuritatem; omnes nervos industriae meae in eo contendam, ut ex his, quae hactenus ab Euclide sunt demonstrate, ita planus reddatur, ac facilis, ut sine multo labore a quovis, qui praecedentium tamen librorum demonstrations recte intellexerit, possit percipi. Neque .n. in eorum possum sententiam ire, qui putant ad eius intelligentiam esse necessariam eam partem Arithmetices, quae de radicibus numerorum, tam rationalibus quam irrationalibus, ut vocant, sermonem instituit." The quotation in the preceding note is separated from this text by a colon, indicating that it was an aside.

appears to be in Commandino's text, without elevating it to the status of an essential foundation to geometry, as Billingsley explicitly does in his text.

Developing Number Theory: Definitions in Book Seven

Because the seventh through ninth books offer a formal development of number theory, Book Seven of *The Elements* begins in much the same way as Book One, with several simple definitions. As their analogs in the first book establish the fundaments of plane geometry, these enunciations establish the fundaments of number theory.[47] Just as the presentation of the definitions in Book One sets the tone for the entire study of geometry, the presentation of the analogous enunciations in Book Seven sets the tone for the study of arithmetic within the context of geometry. Therefore, an examination of the authors' treatment of the definitions can illuminate how each author understood the relationship between arithmetic and geometry. Billingsley attempted to develop the study of arithmetic as a foundation to geometry. Commandino provided very little commentary, limiting the study of arithmetic to the minimum that the ancients felt was necessary to the remainder of Euclid's text. Clavius developed arithmetic as its own discipline, but only to the extent that its analogy to geometry could be used in the subsequent study of commensurability.

The first definition in Book Seven is "unity," the numerical analog to a point. As Billingsley translated it, the definition says, "Unitie is that, whereby everything

[47] As the reader will recall from the previous chapter, "enunciations" is used to refer to the statements of Euclid's propositions without any commentary or demonstration.

that is, is sayd to be on."[48] Billingsley's commentary on the definition of unity establishes his interest in arithmetic as the discipline at the foundation of all others. He began his commentary by recalling the definition of a point and establishing a detailed analogy between a point and unity. He writes, "As a point in magnitude, is the least thing in magnitude, and no magnitude at all, & yet the ground and beginning of all magnitudes: even so is unitie in multitude or nomber, the least thing in nomber, and no nomber at all, and yet the ground and beginning of all nombers."[49] Thus, in his first sentence of commentary he suggested the importance of unity beyond the scope of *The Elements* as the foundation of number. By clearly stating that unity is not a number, this sentence elevates the study of number beyond enumeration of objects. Unity is not the number one; it is a concept that creates an understanding of oneness as discrete, and, therefore, orderable. As Billingsley argued, unity made it possible to define individual objects and isolate one "thing" from another. Without unity nothing could be distinguished from anything else, and everything would "be in confusion. And where there is confusion, there is no order, nor any thing can be exactly knowen, either what it is, or what is the nature, and what are the properties thereof."[50] As a final demonstration of the foundational nature of arithmetic, Billingsley concluded his commentary on the definition of unity by quoting the definition for the same term from Jordanus's textbook on arithmetic and praising its clarity, ensuring that the

[48] Billingsley, *Elements of Geometrie,* 183*v*. His translation closely matches the definitions found in the Latin texts. Clavius's definition reads "Unitas est, secundum quam unumquodque eorum quae sunt, unum dicitur." (Clavius, p. 233); Commandino's: "Unitas est, qua unumquodque eorum, quae sunt unum dicitur." (Commandino, p. 87b)
[49] Ibid., 183*v.*
[50] Ibid., 183*v.*

reader would recognize unity as an arithmetical, not geometric, concept. That definition, which Billinglsey quoted in Latin, says, "Unitas est res per se discretio," identifying the discrete nature of oneness as the foundation of arithmetic.[51]

In contrast, by providing very little commentary throughout the books on number theory, Commandino presented arithmetic as a tool contained within the study of geometry. He included no commentary on the definition of unity and no discussion of arithmetic. He did not even remind the reader of the analogy drawn in Book One between unity and a point. In fact, in contrast to his definition of a point, which included a commentary that might have whet the reader's appetite for geometry with its discussion of how something as seemingly insignificant as a point could give rise to the entire field of geometry, Commandino's definition of unity was presented without any remark on its role in arithmetic that could inspire further study of number. His lack of commentary on the source of numbers indicates that he may have believed that *The Elements* did not require a complete study of number. Indeed, Commandino's treatment of the subject was minimal. Instead of fleshing out the rudiments of number theory provided by Euclid with commentary based on other ancient studies of number, Commandino only provided commentary when he felt that rephrasing a passage or providing an example would contribute to the clarity of the ancient text. For example,

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[51] Ibid., 183*v*. "According whereunto Iordane (in that most excellent and absolute worke of Arithmeticke which he wrote) defineth unitie after this manner. 'Unitas est res per se discretio:" that is, unitie is properly, and of it self the difference of any thing. That is, unitie is that whereby every thing doth properly and essentially differ, and is an other thing from all others. Certainely a very apt definition and it maketh playne the definition here set of Euclide." The book referenced is likely Jordanus of Nemora's *De elementis arithmetice artis*. It is not clear what his precise source was, but printed versions were published in 1496 and 1514 along with other treatises. It is possible Billingsley had access to one of those versions or to a manuscript of the text.

his commentary on the definition of prime numbers says only "No number measures a prime number, except insofar as it measures itself" which merely provides the converse of the definition which says, "A prime number is that which only unity measures"[52] Thus, instead of beginning a new branch of study, in Commandino's text the number theory books were the means by which the reader was given a tool to enable the subsequent study of commensurability.

Like Commandino, Clavius treated the arithmetic developed in Euclid's books on number theory as a tool for the study of geometry. However, while Commandino presented those books as part and parcel of the Euclidean elaboration of geometry, Clavius argued that the utility of arithmetic arose from its status as an analogous branch of mathematics. As he transitioned from his argument that knowledge of arithmetic would assist the reader in the remaining geometry books to his commentary on the definition for unity he said, "Beginning therefore, as is custom, at its [arithmetic's] beginning, unity is first defined…," establishing that the number theory books do indeed begin a new field of study.[53] Still, unlike Billingsley, Clavius did not frame these books as the start of a complete study of arithmetic. In contrast to Billingsley's commentary on the definition of unity, which used the analogies between arithmetic and geometry to describe the intellectual priority of number over magnitude, Clavius commentary on the same definition focused on the analogy

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[52] Commandino, *Euclidis Elementorum,* 88*v.* Definition: "Primus numerus est, quem unitas sola metitur." Commentary: "Primum numerum nullus metitur numerus, praeterquam quod ipse se ipsum metitur."

[53] Clavius, *Euclidis Elementorum,* 233*v.* "Incipiens igitur more suo a principiis, definit initio unitatem …" This phrase serves as Clavius's transition between his justification for the presences of books seven through nine and his commentary on the definition of unity.

between unity and a point insofar as it could aid the study of commensurability in the tenth book. After clarifying the definition by observing that it is "according to unity that we are accustomed to saying one rock, one animal, one body, etc.," Clavius explained that unity is analogous to a point in that neither can be divided, a much narrower analogy than Billingsley's.[54] For Clavius, it did not matter that unity was the source of all number, as a point was the source of all magnitude. It only mattered that unity could not be divided, meaning that not all numbers could measure one another, which allowed number to illuminate the notion of commensurability. In his view, while pairs of incommensurable quantities could only be found in magnitudes, it was only by studying discrete quantity that the concept of commensurability could become meaningful. If quantity can be infinitely divided, for any two quantities a third quantity by which the first quantity measures the second can be found (though it cannot necessarily be created from the given quantities).[55] The number theory books

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[54] Ibid., 233*v*. "Nam secundum unitatem unum lapidem, unum animal, unum corpus, &c. dicere solemus. Caeterum unitas in numeris nullam suscipit divisionem, quemadmodum nec punctum in magnitudinibus, ut in primo lib. docuimus." These two sentences are the only sentences that are commentary directly on the definition of unity. They are the last two sentences of a short paragraph in which Clavius justifies the presence of the number theory books in *The Elements*.

[55] It is easy to see how indivisibility allows the development of the notion of commensurability by taking a more modern understanding of number as infinitely divisible, which is how modern mathematicians understand real numbers. In the set of real numbers we can find a solution to the division of any number by any other non-zero number. In other words, every number can be measured by any other number and thus all numbers are commensurable, rendering commensurability meaningless. All real numbers are commensurable with respect to the set of real numbers. Likewise all rational numbers are commensurable with respect to the set of rational numbers, and all integers are commensurable with respect to the set of integers. Incommensurability is inconceivable in such a set up. And while, as discussed in footnote 33, the early modern notion of commensurability allowed what we call the rational numbers as proportions of true natural numbers to serve as measures of commensurable pairs of numbers (which means that 3 and 2 are commensurable by the ratio 3/2), not all continuous quantities can be expressed as a ratio of whole numbers. In modern language, 2 and the square root of 2 are incommensurable, since the latter number is irrational. In continuous magnitudes such incommensurable quantities can be found. Even though magnitudes are continuous quantities, incommensurable magnitudes are recognized as such because it is not possible to create them by multiplying the line some number of times, even if that number is a ratio of whole numbers which could

may have introduced the field of arithmetic, but Clavius limited the development of that field to its function as a tool for geometry.

From unity, Euclid's text proceeds to define number and its various kinds from even numbers to square numbers to perfect numbers. Among the categories of numbers, four have clear geometric analogs: plane numbers, solid numbers, square numbers, and cubic numbers.[56] While Commandino offered no embellishment on these definitions, Clavius and Billingsley included diagrams alongside commentary for each one (See Figures 10 and 11) in which they illustrated the relationships that they had already articulated between geometry and arithmetic. In their figures both represented units with dots. For plane numbers and square numbers (Figure 10), the dots are arranged in rectangular arrays with the number of dots on each side of the rectangles corresponding to one of the numbers being multiplied. In the special case of the square number, the array has a square shape. Billingsley connected his dots, effectively enclosing the planar shape. Thus, his diagram showed that numbers could be used to generate geometric figures. However, in his commentary, Billingsley claimed only that plane numbers "represent some superficiall form or figure geometricall."[57] While Billingsley did not explicitly say that arithmetic could be used to develop geometric figures, the juxtaposition of the word "represent" with the

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be used to define cuts of the line. (Think of multiplication as adding the line to itself. One could easily add a segment of half the length of the line.)

[56] These categories of numbers are definitions sixteen through nineteen in Clavius's and Commandino's text and seventeen through twenty in Billingsley's text. Clearly, the analogies are plane surfaces, solid bodies, squares, and cubes.

[57] Billingsley, *Elements of Geometrie,* 186*r*. "They are called plaine and superficiall numbers, because being described by their unities on a plaine superficies, they represent some superficiall forme or figure geomtricall, having length and breadth."

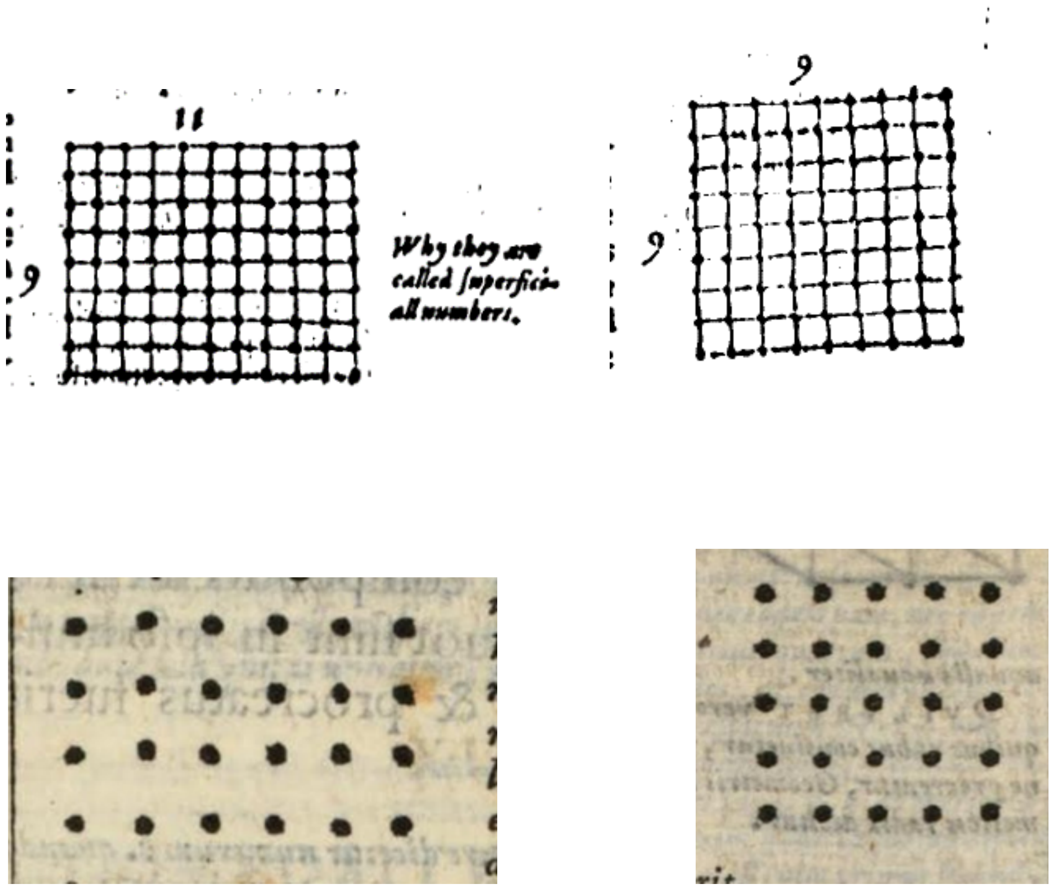

Figure 10: Billingsley's and Clavius's Diagrams of Plane and Square Numbers

Top left: Billingsley's image of a plane number. Top right: Billingsley's image of a square number. Bottom left: Clavius's image of a plane number. Bottom right: Clavius's image of a square number. Note that square numbers are special cases of plane numbers.

enclosed rectangle created the impression that the arithmetical representation of a geometric figure, the plane number, can easily be made into the geometric figure itself, meaning that arithmetic can be the source of geometric quantities. In his diagrams for the solid figures (Figure 11), Billingsley shifted his emphasis away from the geometric figure by only sketching part of it. In both cases he drew only three faces of the solids, leaving the reader to extrapolate the rest of the figure. Furthermore, the perspective is imperfect, so the shapes of the figures are not clear; the cubic number does not look like a cube. By allowing the diagrams to be imperfect, Billingsley forced his reader to focus on the numbers described, but he still provided enough of an image to allow the reader to imagine using numbers to construct geometric forms. Instead of providing the complete form, he provided the reader with the relevant numbers and an outline of how to arrange them in order to create the figure.

Where Billingsley showed that geometric forms could be generated through the manipulation of the units that compose numbers, Clavius kept the numerical diagrams distinct from the geometric figures. In his diagrams for plane and square numbers (Figure 10) he did not even connect the dots to enclose a shape. However, in his commentary, he explained that the created number "is said to contain a rectangular parallelogram."[58] Without an accompanying drawing of a rectangle, this comment

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[58] Clavius, *Euclidis Elementorum,* 237*v*. "Omnis numerus procreates ex multiplicatione mutua duorum numerorum, planus appellatur, quia secundum suas unitates in longum, & latum dispositas parallelogrammum rectangulum refert, cuius latera sunt duo numeri multiplicantes, qui idcirco latera numeri procreati vocantur, quod ipsum contineant, no secus, ac rectae lineae angulum rectum ambientes, parallelogrammum rectangulum continere dicuntur, ut latius lib. 2. explicavimus."

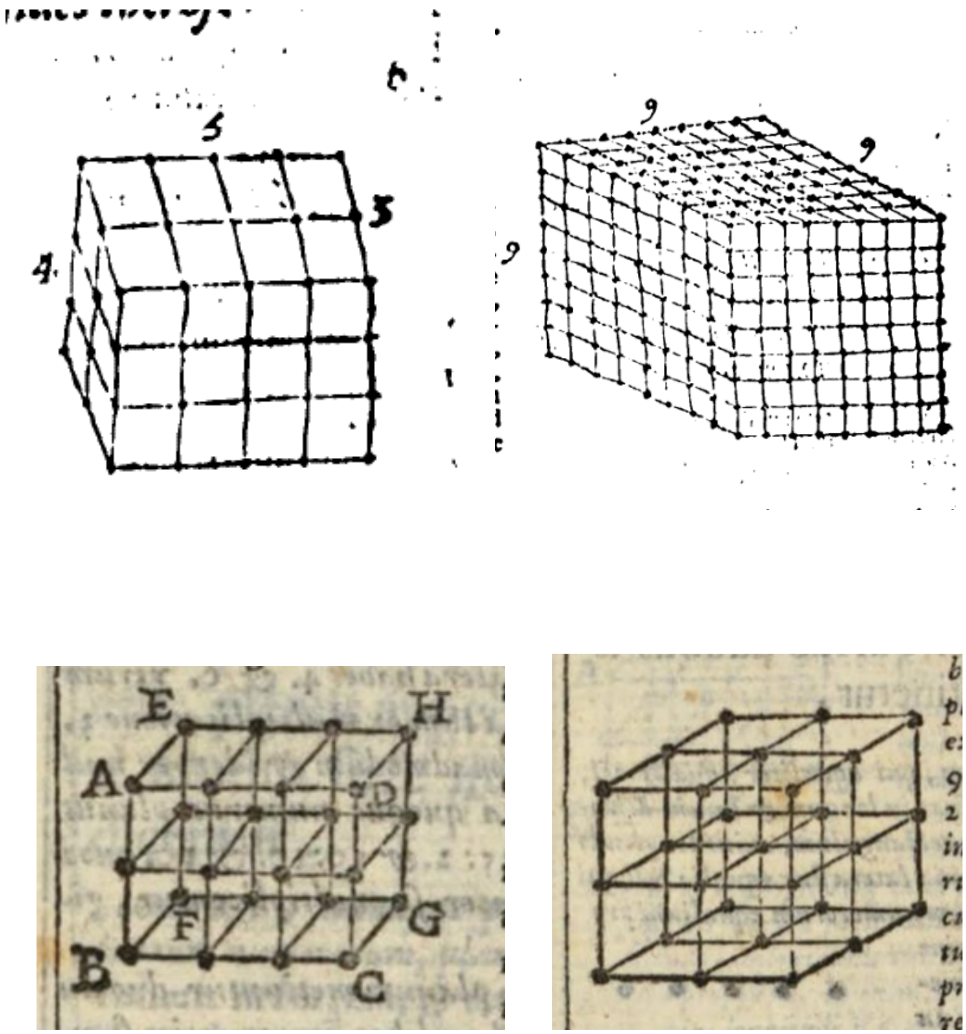

Figure 11: Billingsley's and Clavius's Diagrams of Solid and Cube Numbers

Top left: Billingsley's image of a solid number. Top right: Billingsley's image of a cube number. Bottom left: Clavius's image of a solid number. Bottom right: Clavius's image of a cube number. Note that cube numbers are special cases of solid numbers.

allowed the reader to imagine the rectangle contained by the dots but treated it as an entity separate from the array shown. Although the arrangement of the dots made it possible for the reader to identify the rectangular magnitude contained therein, the dots themselves neither composed nor, as Billingsley claimed, represented that rectangle. Likewise, in his diagrams for solid and cubic figures, Clavius kept his emphasis on the arrays of dots, or the "multitude of unities [the dots]" that compose a number as distinct entities from the geometric forms they contain.[59] Unlike Billingsley, Clavius included all of the dots to fulfill the multiplication problem he set up in his commentary to define solid numbers as the product of three terms.[60] He also included the lines connecting the dots because they allowed him to visualize the breakdown of the multiplication into two steps, the first of which multiplied two terms to create a planar array and the second of which multiplied the planar array by the third term, and provided the necessary perspective to identify each individual planar array generated in the second step. In his descriptive image for a solid number he included all three possible orders of multiplication, so that the reader could see each face of the solid as an array that was multiplied some number of times.[61] His carefully drawn images

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[59] Ibid., 233*v.* The definition for number says that a number is composed of a multitude of unities. "Numerus autem, ex unitatibus composita multitudo."

[60] Of course, there could be logistical reasons for this difference. For the cubic number Clavius chose to represent 3 cubed, which is 27. Billingsley chose to represent 9 cubed. Perhaps the latter's printer told him that including all 729 dots was not possible. Even for the solid number, Billingsley's choice of 60 was significantly larger than Clavius's 24.

[61] Clavius, *Euclidis Elementorum,* 238*v*. The number shown in his figure is 24. It is generated by multiplying 2, 3, and 4. That can be done in three orders: (2x3) x4 = (3x4) x 2 = (4x2) x 3 . By counting the number of dots on one face of the solid and then counting the number of parallel arrays, the reader could see that (2x3) x4 = (3x4) x 2 = (4x2) x 3 = 24. Clavius described all three cases in his text. "Ut quia hi numeri 2, 3, 4 mutuo sese multiplicatnes, producunt 24. Nam ex 2 in 3 procreatur numerus 6 & ex 6 in 4 fit 24. Vel ex 2 in 4 gignitur numerus 8 & ex 8 in 3 efficitur 24. Vel denique ex 3 in 4 producitur numerus 12 & ex 12 in 2 generatur 24."

show the reader how a number can be constructed to contain a figure, but his emphasis was on the arrays of unities, not on the outline of the shape. Indeed in his definitions for plane and solid numbers, Clavius noted that Euclid limited his discussion to right-angled shapes, but that Jordanus had shown that numbers could be arranged in any geometric form.[62] The shapes of the figures in those definitions were incidental.[63]

Still, the possibility of arranging unities to contain certain geometrical forms allowed number to provide a useful analogy to magnitude because the geometric forms can be understood as lengths, breadths, and depths represented by specific arrangements of units. In his commentary on square and cube numbers, Clavius made the analogy explicit by observing that the sides of the geometric figures were recognized as roots in arithmetic.[64] These quantities would take on more significance in the study of commensurability, where roots are a ready source for incommensurable quantities. Thus, where Billingsley showed how number could be used to represent geometry, Clavius built analogies that he could use in the later study of commensurability.

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[62] Ibid., 237*v* – 238*r*, "Caeterum cum infinita sint genera numerorum planorum apud Arithmeticos, quemadmodum & figurae planae apud Geometras: Euclides solum definit planum quadrangularem rectangulum, qui videlicet sub duobus mumeris, ex quorum mutua multiplicatione gignitur."; Ibid., 239*r*, "Definit autem & hic Euclides tantum numerum solidum rectangulum, cuius bases oppositae sunt parallelae, contineturque sum tribus numeris, omissis infinitis aliis, de quibus Iordanus, ob causam in praecedenti definitione datam, quia scilicet hi prorsus aequales sunt, & similes cubis, & parallelepipedis Geometricis."

[63] Square and cube numbers are special cases which, by definition, have to contain the shape described by their names.

[64] Clavius, *Euclidis Elementorum,* 239*r,* "Alteruter autem numerorum aequalium, sub quibus quadratus numerus continetur, vel ex quorum multiplicatione producitur, latus quadrati a Geometris, radix vero ab Arithmeticis plerisque appellatur."; Ibid. 239*v*, "Quilibet vero trium numerorum aequalium, sub quibus cubus continetur, vel ex quorum mutua multiplicatione procreatur, Geometris latus cubi, plerisque autem Arithmeticis radix dicitur."

Developing Number Theory: Postulates and Axioms in Book Seven

Just as the authors all included a series of postulates and axioms, necessary first principles whose truth was readily assented to without demonstration, to establish the foundation necessary to the study of geometry in the first book of *The Elements*, they all included similar principles in the seventh book to establish the foundation of the study of arithmetic.[65] Specifically, these postulates and axioms establish the rules for understanding the relationships between numbers. However, unlike those in the first book, where most of the differences between the texts were in the classification of the enunciations of the principles, the content of the postulates and axioms in the seventh book was not widely agreed upon in the sixteenth century.[66] Thus, the principles that each author chose to include define the study of arithmetic and its relationship to the geometry found in *The Elements*. Table 4 shows the number theory postulates and axioms from each text studied here. Billingsley included only axioms, and his text had far fewer than either Commandino's or Clavius's, but the axioms he included created a foundation for the study of arithmetic as its own discipline. In contrast, Commandino and Clavius both created a tool for the study of commensurability that appeared in Euclid's tenth book. However, while Commandino made sure to emphasize the more perfect nature of magnitude as compared to number in order to maintain geometry's foundational role, Clavius showed arithmetic to be a

[65] See Chapter 3 for a discussion of the definitions of postulates and axioms and how they were understood by the three commentators examined here.

[66] See Chapter 3 for a discussion of the postulates and axioms in the first book. It should also be noted that modern versions of *The Elements* do not include postulates and axioms at all in the seventh book. That exclusion is indicative of the modern equation of number and magnitude that would not have made sense to sixteenth-century authors.

Table 4: Postulates and Axioms in Book Seven

Note that Billingsley's second and third axioms are found in the first two axioms of the Latin authors. His fourth through sixth axioms are in reversed order in the last three axioms of the two Latin texts. Clavius and Commandino also have most of the same axioms, though the order is sometimes different. Only Commandino's fifth axiom is not found in Clavius's text.

	Billingsley	**Clavius**	**Commandino**
Postulate 1		It is postulated that for any number, there can be found any number of equal numbers or multiples.	For any number, there can be found any number of equal numbers or multiples.
Postulate 2		For any number there can be found a greater number.	For any number there can be found a greater number.
Postulate 3			Number can be infinitely augmented, but not infinitely diminished.
Axiom 1	The lesse part is that which hath the greater denomination: and the greater part is that, which hath the lesse denomination.	Numbers that are equal to the same number or are equimultiples of the same number, are equal to each other.	Whatever numbers are equal or equimultiples of the same number, are equal to each other.
Axiom 2	Whatsoever numbers are equemultiplices to one & the selfe same number, or to equall numbers, are also equall the one to the other.	Numbers to which the same number is an equimultiple, or for which equimultiples are equal are equal to each other.	Numbers to which the same number is an equimultiples or for which equimultiples create equal numbers, are equal to each other.
Axiom 3	Those numbers to whome one and the selfe same number is equimultiplex, or whose euqemultiplices are equall: are also equall the on to the other.	For any number of equal numbers, the factors and the non-factors of those numbers are equal.	Whatever numbers are equal, then both their factors and non-factors are equal.

Table 4: Postulates and Axioms in Book Seven (continued)

	Billingsley	**Clavius**	**Commandino**
Axiom 4	If a number measure the whole, and a part taken away: it shall also measure the residue.	If the factors and the non-factors of numbers are equal, then those numbers are equal to each other.	If both the factors and non-factors of numbers are equal, then those numbers are equal to each other.
Axiom 5	If a number measure any number: it also measureth every number that the sayd number measureth.	Unity measures all numbers by the number of unities they contain, that is unity measures the number by the number itself.	Every number is a factor of unity by its denomination, two is a factor of unity by the second denomination, which is called half, three is a factor of unity by the third denomination, called a third, four by a fourth, and so on in others
Axiom 6	If a number measure two numbers, it shall also measure any number composed of them.	All numbers measure themselves by unity.	Unity measure every number by the unities that are in it.
Axiom 7	If in numbers there be proportions how manysoever equall or the selfe same to one proportion: they shall also be equall or the selfe same the one to the other.	If a number multiplying a number will have produced another number, the multiplying number will measure the product by the multiplied number and the multiplied number will measure the product by the multiplying number.	Every number measures itself.
Axiom 8		If a number measures a number, then that number by which it is measured, measures the same number by the unities which are in the measuring number, that is by the measuring number.	If a number measures a number, then that number by which it is measured, measures the same number by the unities which are in the measuring number.

Table 4: Postulates and Axioms in Book Seven (continued)

	Billingsley	Clavius	Commandino
Axiom 9		If a number measuring a number multiplies that number by which it measures, or is multiplied by that number, it will produce the number that it measures.	If a number that measures another, multiplying, or being multiplied by, that number by which it measures, will produce the number that is measures.
Axiom 10		If a number measures however so many numbers, then it measure a number composed of those numbers.	If a number multiplying a number will have produced another number, the multiplying number will measure the product by the unities in the multiplied number and the multiplied number will measure the product by the unities that are in multiplying number.
Axiom 11		If a number measures some number, it measures all numbers that that number measures.	If a number measures two or more numbers, then it measure a number composed of those numbers
Axiom 12		If a number measures a whole and a part taken from it, then it measures the residue.	If a number measures some number, it measures all numbers that that number measures.
Axiom 13			If a number measures a whole and a part taken from it, then it measures the residue.

study whose analogies to geometry allowed it to inform that study through the notion of commensurability.

In the postulates it becomes clear that all three authors used the same distinctions between postulates and axioms that they had drawn in the first book. Those distinctions became instrumental in defining the relationship between geometry and arithmetic. For Billingsley, the difference between a postulate and an axiom was that a postulate was particular to its field of study while an axiom expressed a general truth. In the case of geometry, he took that to mean that postulates could be demonstrated through construction. However, the study of number does not rely on constructions, and Billingsley did not include any postulates. Thus, he made it clear that arithmetic was a study of general truths, which earned it the status of the foundational branch of mathematics.

In contrast, Commandino, who had distinguished between postulates and axioms based on the ability of a novice to understand the claim, included three postulates each of which alluded to some sort of infinite process; this suggests that only the infinite took arithmetic beyond common knowledge. The first postulate allowed for the infinite production of multiples of any number. The second permitted the infinite augmentation of a number. The third prohibited the infinite division of a number. However, where Billingsley treated arithmetic as the foundation of geometry because its knowledge was general, for Commandino it was precisely the failure of the infinite process described in the third postulate, a denial of the feature of number that he used to separate it from common knowledge, that showed that number lacked the versatility of magnitude. The fact that number could not be infinitely divided allowed geometry to be the more complete foundation for mathematics. Commandino made this point clear in the wording of his postulates. The third postulate reads, "Number can be infinitely augmented, but not diminished." Because the preceding postulate said, "For any number, there can be found a greater number," the emphasis falls on the negative second clause.[67] Thus, Commandino called attention to the impossibility of an infinite division of numbers, something which was possible for magnitudes. In so doing, he showed that arithmetical quantities were more restricted than geometric quantities, granting geometry the status as the foundational mathematical study.

While Clavius supplied the same information as Commandino in his section of postulates, he emphasized the analogy between arithmetic and geometry, presenting the former as the key to understanding commensurability. This emphasis arose from his definition of postulates as tasks to be done. Because postulates had to be formulated as tasks, Commandino's third postulate appeared only in the commentary. The two postulates Clavius provided, which made the same claims as Commandino's first two postulates, both had a direct analog among the postulates of his first book and contributed to the study of commensurability. Since the first number theory postulate allows for the infinite increase of a number through multiplication, it can be seen as analogous to the postulate that allows for the infinite extension of a line (Book One, Postulate 2). However, this postulate does not allow for the creation of any number. Instead, it generates multiples of a given number, meaning that each number created

[67] Commandino, *Euclidis Elementorum,* 89*v.* Postulate 2: "Quolibet numero sumi posse maiorem." Postulate 3: "Numerus infinite augetur, sed non infinite diminuitur."

is, by the definition of a multiple, measured by the original number, and, thus increasing a number according to this postulate, creates a series of obviously commensurable numbers. The second postulate, which allows for the creation of a number greater than any given number, is analogous to the first half of the postulate in the first book that allows for the creation of magnitudes greater or smaller than any given magnitude (Book One, Postulate 4). In this case, it is the breakdown of the analogy that allows arithmetic to prepare the reader for a geometric discussion of commensurability. By imagining magnitudes as indivisible blocks, like numbers, commensurability is easily understood.[68] However, by placing the failed segment of the analogy into his commentary instead of giving it is own postulate, Clavius emphasized the similarities between arithmetic and geometry that enabled readers to apply arithmetical concepts to magnitudes that they could imagine as discrete blocks.

Because all three authors took the axioms to be general truths which were necessary to the development of number theory, these principles contained many of the same claims in all three texts. Indeed, most of the axioms either make a claim about the equality of two or more numbers or explain conditions in which one number can be said to measure another number, thereby providing the necessary foundation to discuss commensurability. For example, all three authors include an axiom explaining that if one number measures another, then the first number will also measure all multiples of the second (e.g. 3 measures 9, and 9 measures 18; therefore 3 measures 18).[69] However, despite their similarities, the axioms in these three texts differ in

[68] See note 55.
[69] This is Billingsley's fifth axiom, Clavius's eleventh, and Commandino's twelfth.

ways which reveal how each commentator understood the purpose of the number theory books within *The Elements.*

In Billingsley's text the inclusion of two axioms not found in his contemporaries' texts, and the exclusion of seven others serve to highlight his efforts to create a complete foundation for arithmetic within these books. The two axioms Billingsley added to the text focus on proportion between numbers. His first axiom establishes a general rule for numbers that the greater the denominator of a number, the smaller the number, and his last axiom establishes the transitive property for proportions.[70] Since Euclid had already completed his study of proportion of magnitudes in Books Five and Six, their presence suggests that Billingsley strove to establish a complete foundation for arithmetic in Book Seven, rather than merely including what was necessary to the remaining Euclidean study of geometry. The axioms Billingsley excluded were relevant to the study of commensurability and were already clearly expressed in the definitions. For example, he eschewed two axioms that address unity (Commandino's 6 and 7, and Clavius's 5 and 6): unity measures all numbers, and every number measures itself. Since a number was defined as a multitude of unities, it was obvious that any number be measured by unity and that the measuring number would be itself. Thus, while these axioms defined the extreme cases of commensurability in which the smallest possible factor (unity) is involved,

[70] Billingsley, *Elements of Geometrie,* 187*v*-188*r*. First Axiom: "The lesse part is that which hath the greater denomination: and the greater part is that, which hath the lesse denomination." E.g. One half (denominator of 2) is greater than one third (denominator of 3) since 3 is greater than 2. The last axiom states, "If in numbers there be proportions how manysoever equall or the selfe same to one proportion: they shall also be equall or the selfe same the one to the other." In modern terms, it is the transitive property, which says that if a = b and b = c, then a = c.

they do not extend the definition of number to build a foundation for arithmetic. By eliminating these axioms, Billingsley diminished the focus on commensurability as the most significant feature of arithmetic within *The Elements.*

While the content of Commandino's and Clavius's axioms showed that both developed arithmetic as a tool for the understanding of commensurability, the differences between their axioms illustrate that Commandino sought to present arithmetic as a subsidiary to geometry while Clavius saw them as analogous branches of mathematics. One axiom in particular, Commandino's fifth, which is not found in Clavius's text, allowed arithmetic to be seen as simply a specific interpretation of geometrical quantities. The axiom says, "Every number is a factor of unity by its denomination, two is a factor of unity by the second denomination, which is called half, three is a factor of unity by the third denomination, called a third, four by a fourth, and so on in others."[71] Thus, by dividing unity, it creates the possibility of fractional quantities such as one-half or one-third. Although unity was defined as indivisible, this axiom shows that smaller quantities (albeit not technically numbers) can be created. These quantities are easily seen in geometry, in which magnitudes are readily bisected or otherwise cut into segments. Therefore, arithmetic is shown to be part and parcel of geometry as numbers are applied to measurement with any magnitude taking on the role of unity. By leaving this axiom out, Clavius did not give the reader reason to question the indivisibility of unity, preserving the primary

[71] Commandino, *Euclidis Elementorum,* 89*v.* "Omnis numeri pars est unitas ab eo denominata, binarii enim numeri unitas pars est ab ipso binario denominaata, quae dimidia dicitur, ternarii vero unitas est pars, quae a ternario denominata tertia dicitur, quaternarii quarta, & ita in aliis."

distinction between number (discrete quantity) and magnitude (continuous quantity), thereby allowing arithmetic to remain a separate field of study.

Even when Clavius and Commandino included the same axioms, small changes to the presentation thereof created differing notions of the relationship between arithmetic and geometry that allowed the former to aid the latter. For example, as part of their efforts to position arithmetic as a tool for the study of commensurability, both Commandino and Clavius included three axioms that established measurement as a geometric, rather than an arithmetic concept by exploring the connection of multiplication and measurement. All three of these axioms (Clavius's 7, 8, and 9; Commandino's 8, 9, and 10) address what the modern reader will recognize as the commutative property of multiplication. Each axiom says that if one number is a factor of another number, the factor multiplied by the dividend (the result of the factors division into the number of which it is a factor) will yield the original number, and that if the dividend is multiplied by the factor, the same number will be produced. (For example, 3 is a factor of 18, with the dividend 6. Both 3 times 6 and 6 times 3 yield 18.) Each axiom presents this information in a slightly different fashion. One version (Clavius's seventh axiom and Commandino's tenth axiom) presents it in terms of multiplication. Clavius's version says, "If a number multiplying a number will have produced another number, the multiplying number will measure the product by the multiplied number and the multiplied number will measure the product by the multiplying number."[72] Another version (the eighth axiom in both

[72] Clavius, *Euclidis Elementorum,* 244*r.* "Si numerus numerum multiplicans, aliquem produxerit, metietur multiplicans productum per multiplicatum, multiplicatus autem eundem per multiplicantem."

texts) presents it in terms of measurement. Clavius's version for this one reads, "If a number measures a number, then that number by which it is measured, measures the same number by the unities which are in the measuring number, that is, by the measuring number."[73] Finally, both texts present yet another version (the ninth axiom in both texts), which combines multiplication and measurement by setting the relationship up in terms of multiplying a number which measures another number by the number by which it measures the other number. Clavius's rendering of this combination reads, "If a number measuring a number multiplies that number by which it measures [the second number], or is multiplied by that number, it will produce the number that it measures."[74] By presenting the same relationship separately in terms of multiplying and measuring, Clavius and Commandino created a distinction between the two actions, one of which, measuring, is closely tied to geometry.

However, the order in which the authors present these axioms changes the significance of that distinction. Commandino used them to show that arithmetic was dependent on geometry, and Clavius, allowed arithmetic to stand as its own branch of mathematics that was analogous to geometry. Commandino began with the geometric

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Commandino's version is very similar, but he expresses the multiplicands in terms of the number of unities contained in each term. Commadino, *Euclidis Elementorum,* 89*v,* "Si numerus numerum alium multiplicans aliquem produxerit, multiplicans quidem productum metitur per unitates, quae sunt in multipicato; multiplicatus vero metitur eundem per unitates, quae sunt in multiplicante."

[73] Clavius, *Euclidis Elementorum,* 244*r – v.* "Si numerus numerum metiatur, & ille per quem metitur, eundem metietur per eas, quae in metiente sunt, unitates, hoc est, per ipsum numerum metientem." Commandino's is identical except that it leaves off the last phrase (beginning with "hoc est"). (Commandino, *Euclidis Elementorum,* 89*v*.)

[74] Clavius, *Euclidis Elementorum,* 244*v*. "Si numerus numerum metiens, multiplicet eum, per quem metitur, vel ab eo multiplicetur, illum quem metitur, producet." Again, Commandino's is quite similar. "Si numerus metitur numerum, & ille, per quem metitur, eundem metietur per eas, quae sunt in metiente, unitates." (Commandino, *Euclidis Elementorum,* 89*v*.)

notion of measuring, then offered the combination, and concluded with the arithmetic construction based on multiplication. In so doing, he showed his reader that the arithmetic notion of multiplication grew out of the geometric notion of measuring. In contrast, Clavius began with the arithmetic construction of multiplication, which he followed with the geometric construction of measuring, and concluded with the combined formulation. In his case, the first two axioms are clearly analogous to one another; geometric measuring is seen as analogous to arithmetic multiplication. The final axiom then brings multiplication into geometry, showing that the analogy present between geometry and arithmetic allows the latter to inform geometric reasoning and permits the geometry student to turn to arithmetic for clarification.

Conclusion: Applying Number Theory to the Study of Commensurability

While the three commentators studied here disagreed on the exact relationship between geometry and arithmetic, they all agreed that the pieces of arithmetic developed in the number theory books were necessary to Euclid's study of commensurability in the tenth book. Thus, the uses of number prior to Book Ten can be seen as preparatory efforts to grapple with the relationship between number and magnitude in that book. Therefore, there is no more fitting way to conclude this chapter than to provide an example of how each author translated his understanding of the branches of mathematics into the combination of geometry and arithmetic in the final proof in the tenth book: the proof that the a square's diagonal is incommensurable with its side. It should be noted that this particular proposition is not found in modern versions of *The Elements* because it is no longer believed to be original to Euclid.[75] However, prior to in the sixteenth century, it was a key demonstration in understanding the relationship between geometry and arithmetic because it presented a commonly found pair of magnitudes (the side and the diagonal of a square) that could not be related to one another through numbers. But, to understand the nature of incommensurability, one first had to understand number theory.

In all three texts the contents of the final proposition in Book Ten follow the same outline. All three authors present two proofs for the incommensurability of the side and diagonal of a square, each of which assumes that there exists a numerical proportion with the ratio between the diagonal and the side of the square. The proofs are then done by showing that such numbers cannot exist because the requirements imposed on them by the relationship between the square of the diagonal and the square of the side (namely that the former is double the latter), lead to contradictions of basic principles of numbers. In the first proof it is shown that for the proportion between the side and the diagonal of the square to be a rational number, one of two relatively prime integers composing that proportion would have to be both even and odd, which is impossible. In the second proof it is shown that numbers that were assumed to share

[75] The proposition disappeared from versions of Euclid in the nineteenth century, which suggests that the objection to its non-original status may have been paired with the development of the real numbers in the decision to remove it. Because real numbers include irrational numbers, they allow for a purely numerical demonstration of the irrationality of the square root of two. Thus, the geometric structure to the demonstration found in *The Elements,* is unnecessary to a modern demonstration of the concept. Of course, such a shift requires the understanding of a continuous number line, a fundamental break with ancient mathematics.

only one as a common factor, share the smaller of the two numbers, which cannot be one, as a factor. Since such numbers cannot exist, there is no numerical proportion between the side and the diagonal of the square, and the two lines must be incommensurable.[76] Thus, while geometry provides the scenario by raising the question of the possibility of identifying a proportion between the sides and the diagonal of a square, the demonstration itself is done purely with numbers. Even the figures provided by all three authors (Figure 12), illustrate the use of number by including images of squares and series of dots or lines to represent the relevant numbers. Finally, after the proofs for this proposition each of the commentators included a brief discussion of the extension of the study of incommensurability from lines and figures to the study of solids.[77]

While all three authors provided similar proofs to demonstrate the incommensurability of the sides and diagonal of a square, each of their established relationships between arithmetic and geometry remained clear. In the first of the proofs discussed above, both Billingsley and Commandino isolated the two branches of mathematics from one another by placing the relevant geometric constraint (that the square of the diagonal is double the square of the side) in the first sentence of their demonstrations.[78] While the constraint is presented again towards the end of the

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[76] See Appendix C for Clavius's version of both of these proofs. The proofs found in Commandino's and Billingsley's texts are very similar.

[77] All three authors have extremely similar discussions, suggesting that they were taken from some other source. Billingsley claimed that it may have been written by Theon, so it may be present in the 1533 printing of Theon's Greek text.

[78] Billingsley, *Elements of Geometrie,* 309*v*; Commandino, *Euclidis Elementorum,* 187*r*. Billingsley's proof begins, "It is manifest (by the 47 of the first) that the square of the line AC is double to the square of the line AB." Commandino's begins the same way. "Itaque manifestum est quadratum ex AC

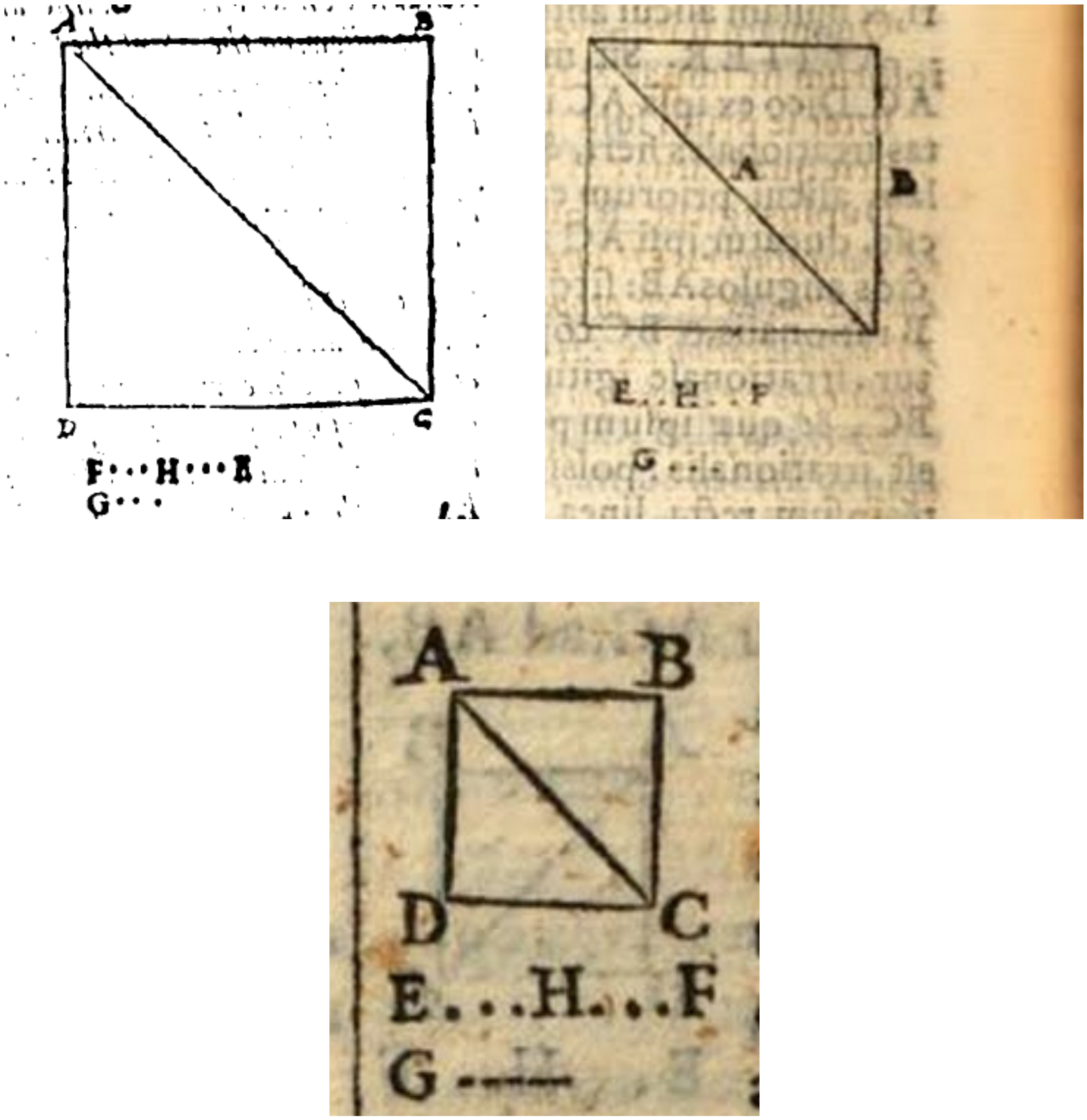

Figure 12: Diagrams for the Demonstration of the Incommensurability of the Side and Diagonal of a Square

Billingsley's (top left), Commandino's (top right) and Clavius's (bottom) images for the proof that diagonal of a square is incommensurable with its side. Each author included labeled series of dots to represent the numbers used in the demonstration. EF and G are the hypothesized integers with a proportion between them equivalent to the proportion between the side and the diagonal of the square.

proof, when it becomes relevant, beginning the proof with the geometric claim separates the arithmetic demonstration from the geometric problem. However, the context of the preceding books changes how that isolation is seen. In Billingsley's work, the geometric constraint is presented as a description of the square under study and is restricted to setting up a problem about the relationship between roots of numbers, one of which is double the other. Indeed, in his commentary, he provided a third demonstration (credited to Candalla) in which the original square is hardly mentioned, and the geometric constraint that the square on the side is half of the square on the diagonal is only discussed as the proportion between two square numbers, independent of their origins as the areas of particular squares. Billingsley wrote that he thought this demonstration was "good to adde, for that the former demonstrations seme not so full," revealing his own preference for the arithmetical demonstration.[79]

In contrast, in Commandino's work, the presentation of the geometric constraint at the start of the proof served to call attention to the geometrical origin of the problem, establishing that the arithmetical demonstration only became meaningful in the context provided by the geometric problem. Instead of separating square numbers from their physical sources, Commandino allowed the geometric shapes to give the numbers meaning. Only because the squares are constructed on continuous lines could the impossible proportion be imagined. And thus, the numerical values

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duplum esse quadrati ex AB." It is not until the end of the demonstration that the significance of this relationship becomes clear.

[79] Billingsley, *Elements of Geometrie,* 310*v*.

were only descriptions of the geometric shapes to which they were attached. Furthermore, since this is a proof by contradiction, those numerical descriptions were shown to be unable to fully explain the relationships between magnitudes. Commandino's emphasis on the status of arithmetic as a tool to geometry is especially pronounced in his placement of the aforementioned discussion of the application of the study of incommensurability to solid geometry. Instead of treating it as an addition to the whole of Book Ten, as Billingsley had done, he appended it directly to the end of the proof (before his commentary), only using a paragraph break to denote that it was distinct from the demonstration. Thus, he showed that even though the proof relied on arithmetic, its function was to develop geometry, the only branch of mathematics that could make sense of incommensurable quantities.

Unlike either of his contemporaries, Clavius did not attempt to keep arithmetic and geometry distinct within the study of commensurability. Instead of beginning with the geometrical portion of the proof, he opened his demonstration with the analogy between geometry and arithmetic that allowed the proof to be developed numerically, namely that if the side and the diagonal of the square were commensurable the proportion between them was the same as a proportion between some pair of numbers.[80] He only introduced the geometric constraint of the relationship between the squares on the diagonal and the side of the square where it became relevant in the middle of the proof. After the two demonstrations, Clavius included a scholion composed of the same additional proof found in Billingsley's text

[80] Clavius, *Euclidis Posteriores*, 115*r.* "Si enim non est incommensurabilis, commmensurabilis erit longitudine; ac propterea AC, AB proportionem habebunt, quam numerus ad numerum."

and the discussion on using incommensurability in solid geometry that the other two authors provided. However, unlike Billingsley, he presented the arithmetical proof as merely another viable demonstration of the theorem, making no comment about its relationship to the two demonstrations already included. Furthermore, the inclusion of the discussion on the uses of incommensurable lines in the study of solid geometry in the same scholion prevented the interpretation of the arithmetical proof as superior to those that made more use of the underlying geometry. As in Commandino's text, the placement of that discussion served to remind the reader that even the arithmetical demonstration of the claim was a tool to the further study of geometry. However, Clavius introduced that discussion with a brief note that its role was an appendix to Book Ten, not just the final proof.[81] Thus, commensurability, and through it, arithmetic, were allowed to stand as studies independent of the plane and solid geometry found in the rest of *The Elements*.

Thus, in the culmination of the study of commensurability, Clavius maintained a relationship between arithmetic and geometry that was between the relationships established by his contemporaries. On the one extreme, Billingsley saw arithmetic as the true foundation of all of mathematics. In his eyes, geometry, along with the rest of mathematics, depended upon the study of number. Even as early as the second book of *The Elements*, the geometric theorems could be recast as arithmetical theorems.

[81] Ibid., 116*r*. "Caeterum in exemplaribus Graecis reperitur hoc loco appendix quaedam, cuius intelligentia ex sequentibus Stereometriae libris pendet, ut merito omitti posset. Verum quia in ea continetur doctrina non contemnenda ad commensurabilitatem omnium magnitudinum, & incommensurabilitatem pertinens visum est, eam paucis explicare, assignando more nostro solito in margine loca Stereometriae, quae ad demonstrationem eorum quae hic dicuntur, necessaria sunt. Est igitur appendix haec."

Once Euclid reached the study of proportion, arithmetic was necessary to make the geometric demonstrations intelligible. According to Billingsley, Euclid himself recognized the inability of geometry to stand on its own, so he included three books on number theory before he introduced commensurability. In contrast, Commandino saw arithmetic as a tool for geometry. While number could be used to provide specific measurements for generalizable geometric examples, the study of number could never be as complete as the study of magnitude. In Commandino's view, the discrete nature of number that made it useful for examples, especially in the study of proportion, prevented it from examining a complete range of quantities. As was evident in the existence of incommensurable pairs of magnitudes, geometric quantities did not have that limitation. For Clavius, geometry and arithmetic were separate branches of mathematics, whose similarities created informative analogies. Through number's analogy to magnitude, arithmetic served as a pedagogical tool because it provided another way to examine quantities that could provide insight into geometry. This union of arithmetic and geometry was picked up by Jesuit students, including Descartes and Gregory of Saint-Vincent (1584-1667), and other seventeenth-century mathematicians who began to use arithmetic approaches in their studies of geometry, leading to the "algebraization of geometry" and the development of mathematics clearly distinct from the classical quadrivium, including its rigid divide between number and magnitude.[82]

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[82] Mancosu, 34. As noted at the beginning of this chapter Mancosu considered the "algebraization of geometry" to be one of the two significant shifts in mathematical practice that allowed seventeenth-century mathematical scholars to break with classical mathematics.

Chapter Five

Mathematics and the Visualization of Space: The Use of Diagrams in *The Elements*

"And it shall be very necessary for you to have some of this pasted paper by you, for so shal you upon it describe the formes of other bodies as Prismes and Parallelipopedons, and such like set forth in these five bookes following, and see the very formes of these bodies there mencioned: which will make these bokes concerning bodies, as easy unto you as were the other books, whose figures you might plainly see upon a playne superficies."[1]

Sir Henry Billingsley, 1570

In his 1570 edition of *The Elements*, Sir Henry Billingsley went to great lengths to ensure that his readers would be able to visualize all of the shapes and relationships between shapes that were described by Euclid. As the above quotation notes, he even provided templates for his readers to use to create three-dimensional figures, so that those could be as fully accessible as the more easily represented two-dimensional figures. While Billingsley's extensive use of three-dimensional images was unique, in the sixteenth century, no one questioned that images were an essential part of

[1] Henry Billingsley, *The Elements of Geometry of the most auncient Philospher Euclide of Megara,* (London: John Daye, 1570), 340. (N.B. The folio number on what should be 320 is printed as 340. The next folio shows 341; the following is numbered 327. After that, the number is 323, which is what it should be. From there the folios count up as one expects.)

geometric demonstrations.[2] As the adage says, a picture is worth a thousand words. However, the words that are conveyed by each picture vary from book to book, and need not be found in the written text itself. In the case of *The Elements,* a comparison of three sixteenth-century editions shows that authors' conceptions of the discipline of mathematics, the reasons it was valuable, and consequently the place they each assigned to it within the hierarchy of disciplines, are revealed by their uses of diagrams.

As the reader will recall from Chapter One, in the sixteenth century the status of mathematics was under debate. At the start of the sixteenth century, mathematics had been considered a lower discipline, with its four branches, geometry, arithmetic, astronomy and music, making up the quadrivium, half of the seven liberal arts. Consequently, mathematics professors had lower social status than their colleagues in the higher faculties of medicine, law, and theology.[3] In the middle of the sixteenth century, inspired by the rediscovery of ancient Greek texts, mathematicians challenged their subordinate status. They argued that mathematics was able to make sure claims about the world, and this ability granted it epistemological status comparable to that of

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[2] During the late nineteenth and early twentieth centuries, Bertrand Russell, David Hilbert, and other mathematicians developed a standard of proof that required all mathematical demonstrations to be purely sentential, which reduced images to mere depictions of the words in the text. For discussions of Hilbert and the potential roles of diagrams in geometry, see Jesse Norman, *After Euclid: Visual Reasoning and the Epistemology of Diagrams* (Stanford, CA: CSLI Publications, 2006), 2-19; and Mark Greaves, *The Philosophical Status of Diagrams* (Stanford, CA: CSLI Publications, 2002), 7-9.

[3] See Robert Westman, "The Astronomer's Role in the Sixteenth Century: A Preliminary Study," *History of Science* 18 (1980): 117-119. Westman notes that the mathematics professors were often still studying medicine or theology and treated the less lucrative mathematics chair as a stepping stone to the more prestigious positions in the higher faculties.

natural philosophy.[4] However, mathematicians themselves did not agree on exactly what it was that enabled mathematics to make claims about the world. Some argued for its value based on its practical ability to study physical bodies, placing it alongside physics within philosophy. These arguments centered on the utility of the "mixed" sciences, which combined abstract mathematical entities with physical objects. Others argued for mathematics' ability to make certain, universal claims based on abstract principles, placing it alongside metaphysics. These arguments emphasized the purely mathematical studies of magnitude and number as perfect, abstract quantities. In this chapter, through a comparison of their uses of diagrams in commentaries on *The Elements,* I will show that Billingsley fell into the former category, while his humanist contemporary Federico Commandino fell into the latter. In his commentary, Christopher Clavius, the Jesuit mathematics professor in Rome, created a view of mathematics that combined those of his contemporaries and positioned mathematics as a bridge between physics and metaphysics.

The ability of diagrams to reveal an author's approach to his discipline is not inherently surprising. Art historians have long connected the content of images with significances for their contexts. In his *Painting and Experience in Fifteenth-Century Italy,* Michael Baxandall argued that the style of Renaissance paintings was a product of the artists' social context and, as such, could reveal the artist's culture by showing what had recognizable value in numerous areas, including economics, religion, and

[4] Of course, these arguments were not entirely new in the mid-sixteenth century. As discussed in chapter one, Regiomontanus, a fifteenth-century humanist, had made them in his famous Padua Oration of 1464. During the Middle Ages, mathematically inclined scholars such as Roger Bacon (1214-1292) and Nicole Oresme (1320-1382) had made similar arguments for mathematics.

even mathematics.[5] Surely, then, the style of diagrams in a natural philosophy text could reveal the author's understanding of his own discipline. Recently, historians of science have begun to explore the visual arguments that go beyond the texts of their sources. Sachiko Kusakawa has argued that, in the context of a debate about the value of medical images, sixteenth-century physicians who used images in their works did so to make arguments about their discipline and how it should be studied.[6] Historians of early modern astronomy have also begun to examine the use of diagrams in their sources. A collection of articles in the *Journal for the History of Astronomy* shows that astronomers used images for a variety of purposes, showing the myriad values that could be ascribed to mathematics (in the form of astronomy, a branch of mixed mathematics). Some, notably John Blagrave's *Mathematical Jewel,* provided practical aids that emphasized the physicality of mathematics and its utility to various tasks of everyday life. Others, notably Giordano Bruno's schematic diagrams of the structure of the universe, conveyed theories that connected mathematics, especially astronomy, to a philosophical search for the universal truths governing the world.[7]

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[5] Michael Baxandall, *Painting and Experience in Fifteenth-Century Italy, Second Edition,* (Oxford: Oxford University Press, 1988), 29-40. On page 30, Baxandall alludes to Euclidean geometry as an example of one part of training architects and artists would have had in the fifteenth century. His allusion suggests that it was through Euclidean studies that such artisans learned to make sense of the shapes they used in their own work, particularly in schematic sketches. J.V. Field has shown that the mathematics of the Renaissance can be understood through examinations of artwork, especially because the development of perspective and the use of vanishing points were mathematical endeavors. Indeed, many artists, notably Piero della Francesca, were also mathematicians. See J.V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance.* (Oxford: Oxford University Press, 1997.)

[6] Sachiko Kusakawa, *Picturing the Book of Nature: Image, Text, and Argument in Sixteenth-Century Human Anatomy and Medical Botany,* (Chicago: The University of Chicago Press, 2012).

[7] *Journal for the History of Astronomy*, August 2010; See Katie Taylor, "A "Practique Discipline"? Mathematical Arts in John Blagrave's *The Mathematical Jewel* (1585)", pp. 329-353 for Balgrave. See Christoph Lüthy, "Centre, Circle, Circumference: Giordano Bruno's Astronomical Woodcuts," pp. 311-327 for Bruno.

Of course, the diagrams in the astronomy texts are not the only means through which the arguments about mathematics are made. The topics of the texts themselves suggest broader arguments about mathematics. Blagrave's treatise is about a physical tool he had created, namely, an astrolabe, pushing him towards a physical interpretation of mathematics. Bruno's texts are about the structure of the heavens, leading him to show mathematics as the source for truths about the universe. Given the right topic, it was easy for images to show certain arguments for the status of mathematics. For example, in his *Geometria practica,* which sought to apply Euclidean geometry to everyday problems, Clavius often sketched his abstract diagrams with images of towers and mountains in problems about measurements involving such entities (Figure 13), showing that mathematics could combine the abstract and the concrete.[8] While the claim that images served a variety of purposes across texts that explored different aspects of mathematics is fundamentally uninteresting, differences in images in versions of the same text, such as commentaries on *The Elements,* can reveal differing conceptions of mathematics as a discipline. Indeed, although the prescriptions within the Euclidean text heavily constrain images such that the variations between diagrams across commentaries must be minor (e.g., an equilateral triangle is always composed of three equal sides meeting at 60 degree angles), the small differences that exist in the presentations of the diagrams can and do drastically change how the image affects the interpretation of the value of mathematics.

[8] Christopher Clavius, *Geometria practica* (Rome: Aloisyius Zannettus, 1604), 54.

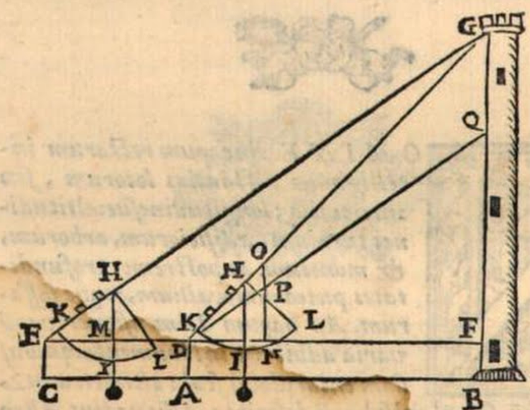
Figure 13: A Problem from the *Geometria practica* with a Diagram Superimposed on a Sketch

Page 54 from Clavius's *Geometria practica*. This diagram accompanies the first problem in the second book, which instructs the reader on how to measure a distance in a plane using a quadrant when there is an object of known height, such as a tower, at one end of the distance to be measured. In it the reader can easily see the tower, and the two necessary iterations of the quadrant, all with labels typically of points and lines on a diagram. The measurements to be taken and the plane itself are drawn simply with the relevant lines. The tower even has one edge that clearly serves as a side of a several triangles.

Astronomicum Dimensio linearum rectarum perficiatur, docebimus. Quæ ratio vt intelligatur, in promptu, & ad manum esse debent tabulæ sinuum, Tangentium, atque secantiū in nostro Theodosio, & in aliorum auctorum libris descriptæ, vnde petendæ erunt. Superuacaneum enim esse duximus, easdem hic repetere, ne opus in maiorem formam excrescat.

DISTANTIAM in plano, siue accessibilis ea sit, siue inaccessibilis, per duas stationes in eodem plano factas per Quadrantem metiri, quando in eius extremo erecta est altitudo aliqua perpendicularis, etiamsi infimum eius extremum non cernatur. Atque hinc altitudinem quoque ipsam elicere.

PROBLEMA I.



SIT distantia, siue longitudo inuestiganda, A B, in plano CB, erectaque sit in extremo B, altitudo quæpiam perpendicularis BG, licet extremum B, non appareat. Statura mensoris sit D A, ab oculo ad pedes vsque. Ne autē hæc statura mutetur, sed eadē sēper maneat, recte feceris, si pro ea statura baculum eidem æqualem accipias, ad cuius extremum oculum applices. Ducta autem cogitatione per D, ipsi C B, parallela E F, fiat prima statio in D, Secunda vero in E, puncto remotiore: sitque recta D E, quæ differentia stationum dicitur, nota secundum aliquam mensuram vulgarem. Deinde dirigatur latus quadrantis H K, in quo sunt pinnacidia, versus fastigium G, ita vt oculus in D, positus per vtriusque pinnacidij foramina, fastigium G, videat, libere pendente perpendiculo H I: diligenterque per ea, quæ cap. 2. libr. 1. Num. 7. & 10. tradita sunt, notetur in gradibus, ac minutis angulus G D F, quem arcus I L, in Quadrante manifestabit, complemen-

Differentia stationum.

In this chapter, I will compare the use of diagrams in Billingsley's, Commandino's, and Clavius's commentaries on *The Elements*. I will examine the use of diagrams in the first book as an example of planar geometry diagrams. The definitions provide valuable source material for a comparative study because, unlike most propositions, they do not require diagrams, giving the authors flexibility in their use of images. However, the diagrams for the propositions are not entirely uniform. Therefore, I will discuss the first two propositions of the first book to give the reader a sense of how the small differences between the authors' diagrams can affect the interpretation of the images. These early definitions and propositions set the tone for each author's book. In addition, the challenges presented by representing three-dimensions on paper makes a study of the solid geometry books worthwhile. I will examine the definitions in the eleventh book and one proposition from the twelfth book to show how the authors approached three-dimensional objects and to demonstrate the continuity of their early patterns in the last section of *The Elements*.

The Basics: Definitions in Book One

When studying various editions of *The Elements*, it almost immediately becomes clear that most authors used images to do more than provide necessary illustrations of proofs. Commentators usually included illustrations for nearly all of the definitions in the first book. While these images provide the reader with visual training that would enable them to identify the elemental forms of geometry in subsequent diagrams, most of those concepts are so simple that images might seem superfluous.[9] Nevertheless, even in 1482, in the first printed edition of *The Elements,* Erhard Ratdolt placed diagrams in the margin for most of the Book One definitions, no mean feat in the early days of printing (Figure 14). However, because the diagrams of these definitions are non-essential to the understanding of the Euclidean enunciations, commentators had more freedom in their presentation of these images than they did in diagrams for the propositions. Therefore, the relationships authors created between the text of the definitions and the illustrations provide a particularly interesting lens into each author's use of visualizations and, in turn, into his vision of mathematics.

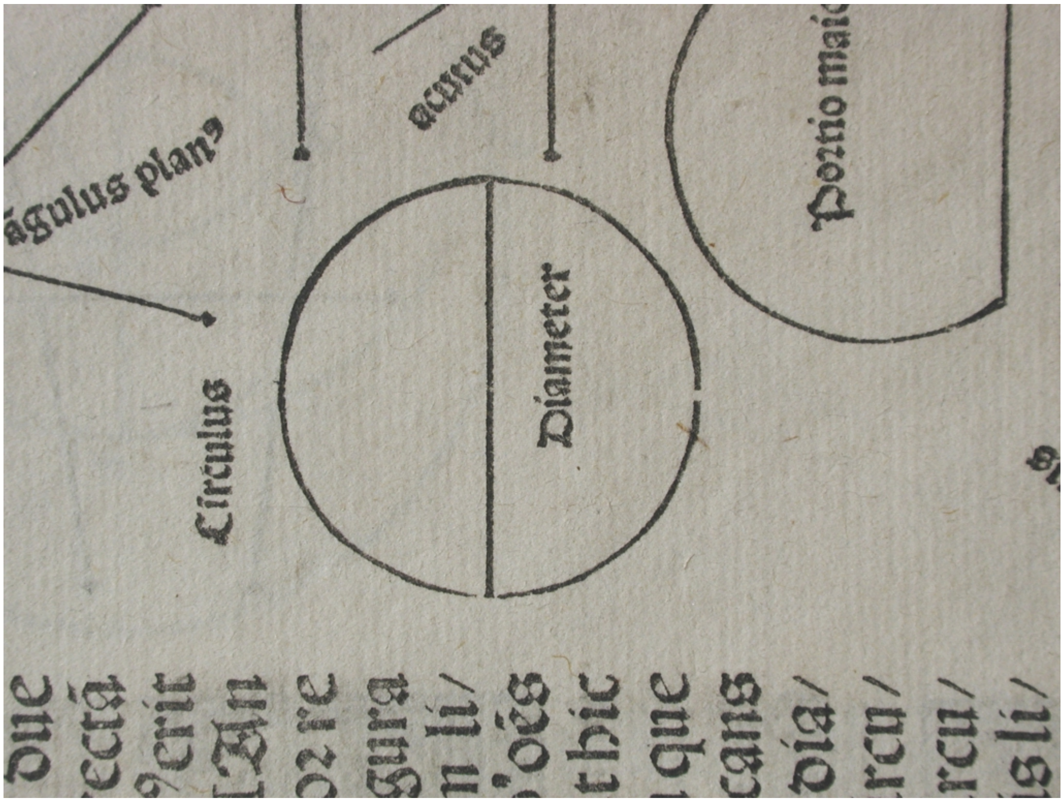
In Billingsley's text, the images were often the subject of the commentary, establishing the importance of the physical nature of mathematics. In contrast, Commandino rarely referred to the diagrams, allowing the physical particularities of the drawn mathematical objects to remain incidental to the universal principles the images were intended to illustrate. As usual, Clavius struck a balance between his two contemporaries by using diagrams as vital pieces of his explanations of the properties of defined entities. As a result, he placed mathematics at the intersection of the study of the physical objects presented by Billingsley and the universal ideas emphasized by Commandino. In this section, I will examine each author's treatment of illustrations in the definitions through general patterns that emerge in each text and a comparison of the three texts on two definitions: an equilateral triangle and a plane.

[9] Baxandall, *Painting and Experience,* 30-31. Baxandall notes that familiarity with the forms found in the definitions of Euclid's first book could provide scholars with visual tools to identify and make sense of similar forms in other images.

Figure 14: Erhard Ratdolt's Diagrams for the Definitions of Book One

The definitions for the first book and their diagrams in Erhard Ratdolt's 1482 edition of *The Elements* fit almost entirely on one page. Most definitions are represented by an image, but the labels must be read to attach any image to its definition. The close-up image below the full page shows the diagram for the definitions of a circle and the diameter of a circle.

Source: http://www.math.ubc.ca/~cass/euclid/ratdolt/ratdolt.html


Circulus
Diameter
Portio maio

The general patterns of diagram usage in each text are most clearly seen in their placements of diagrams, which reveal the value each author attached to the physical nature of mathematics. Unlike Ratdolt, the sixteenth-century commentators studied here did not place all of their diagrams for the definitions in the margin of a single page. In part, this could be due to advances in printing, but it is more likely that the authors saw the images as themselves part of their commentary, and so placed them alongside the individual definitions as their commentary demanded. In none of the texts did the commentary require that every definition be accompanied by its diagram. So, while all three commentators included images for nearly all of the definitions, their placement of the images gave them meanings beyond their role as physical instances of the concepts.[10] Table 5 gives a complete list of the definitions and the diagram placement for all three texts.

The table clearly shows that Billingsley consistently provided images alongside his definitions. Thirty-one of his thirty-five definitions have an accompanying diagram, and the commentary for two of the remaining definitions explicitly refers readers to diagrams for nearby definitions. His placement of diagrams enabled his emphasis on the physical instances of mathematical objects by providing his reader with examples of each definition. Indeed, much of his commentary is simply a discussion of the visual aids he had created. For example, in his definition of a semicircle, he identified the two semicircles in the diagram he provided (Figure 15).

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[10] As discussed in the previous chapter both Billingsley and Clavius included an image of a point with their commentary on the first definition. Commandino did not. Clavius left out an image of a multilateral figure. That lacuna will be addressed shortly.

Table 5: The Definitions and Diagrams of Book One

	Billingsley	Clavius	Commandino
1. Point	Diagram	Diagram	No diagram
2. Line	Diagram	Diagram	No diagram
3. Limits of a Line	Diagram	Refers to diagram for def. 2	Diagram
4. Straight Line	2 diagrams	Diagram	Diagram
5. Surface	Diagram	Diagram	No diagram
6. Limits of a Surface	Refers to diagram for def. 5	Refers to diagram for def. 5	Diagram*
7. Plane	Diagram	2 diagrams	Diagram*
8. Plane Angle	Diagram	2 diagrams	No diagram
9. Straight-line Angles	Refers to diagram for def. 8	Diagram	Diagram (includes non-straight line angles)
10. Right Angle	Diagram	Diagram	No Diagram
11. Obtuse Angle	Diagram*	Diagram	No Diagram
12. Acute Angle	Diagram*	Refers to diagram for def. 11	Diagram covering defs. 10-12
13. Limit	No diagram	No diagram	No diagram
14. Figure	Diagram	No diagram	No diagram
15. Circle	Diagram	Diagram	No diagram
16. Center of a Circle	Diagram	Refers to diagram for def. 15	Diagram
17. Diameter	Diagram	2 Diagrams	Diagram
18. Semi-Circle	Diagram	Refers to 1st diagram for def. 17	No diagram
			Continued on next page

*These pairs of definitions are represented by a single diagram placed such that it clearly accompanies both definitions.

Table 5: The Definitions and Diagrams of Book One (continued)

	Billingsley	Clavius	Commandino
19. Portion of a Circle	Diagram	N/A**	Diagram covering defs. 18-19
20. Straight-Line Figures	No diagram	No diagram	No diagram
21. Three-Sided Figure	Diagram	No diagram	No diagram
22. Four-Sided Figure	Diagram	No Diagram	No diagram
23. Many-Sided Figure	Diagram	No Diagram	Diagram of covering defs. 20-23
24. Equilateral Triangle	Diagram	Diagram	No diagram
25. Isosceles Triangle	Diagram	Diagram	No diagram
26. Scalene Triangle	Diagram	Diagram	No diagram
27. Right Triangle	Diagram	Diagram	No diagram
28. Obtuse Triangle	Diagram	Diagram	No diagram
29. Acute Triangle	Diagram	Refers to diagrams for defs. 24-26	2 Diagrams covering defs. 24-29
30. Square	Diagram	Diagram	No diagram
31. Rectangle	Diagram	Diagram	Diagram covering defs. 30-31
32. Rhombus	Diagram	Diagram	No diagram
33. Rhomboides	Diagram	Diagram	Diagram coving defs. 32-33
34. Trapezia	Diagram	Diagram	Diagram
35. Parallel Lines	Diagram	Diagram	Diagram
36. Parallelogram	N/A**	Diagram	N/A**
37. Complement	N/A**	Diagram	N/A**

*These pairs of definitions are represented by a single diagram placed such that it clearly accompanies both definitions.

** These definitions are not present in the author's text.

Definition of a ſemicircle.

18 *A ſemicircle, is a figure which is contayned vnder the diameter, and vnder that part of the circumference which is cut of by the diametre.*

As in the circle *ABCD* the figure *BAC* is a ſemicircle, becauſe it is contained of the right line *BGC*, which is the diametre, and of the crooked line *BAC*, being that part of the circumference, which is cut of by the diametre *BGC*. So likewiſe the other part of the circle, namely *BDC*, is a ſemicircle as the other was.

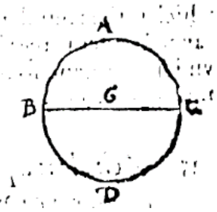


Figure 15: Billingsley’s Definition of a Semicircle

The commentary merely observes that because *BGC* is a diameter of the circle, *ABC* and *BDC* are both semicircles.

Two definitions – numbers thirteen and twenty, respectively, of limits and straight-lined objects – neither have their own image nor directly refer to another related definition's image. These are definitions of broad concepts, so, by leaving out images for only these definitions, Billingsley downplayed the importance of general concepts in favor of studying their particular cases, for which he included diagrams. Furthermore, the commentary on these definitions directed readers to a study of the particulars rather than the general concepts. In the case of the definition of limits, his commentary was just a recitation of the earlier definitions of the limits of lines (points) and the limits of surfaces (lines), though it gave no direct reference to their corresponding diagrams (Figure 16). Similarly, Billingsley's commentary on his definition for a straight-lined figure says only that a straight-lined figure may be contained under three, four, or more lines. The next three definitions, which include diagrams, are for the specific cases of three-, four-, and more-sided figures (Figure 17).

In contrast, Commandino used visuals to call the reader's attention to the universal ideas found in the definitions by grouping images for the definitions belonging to a general category into a single diagram accompanying the last definition of that category. As a result, only fifteen of his thirty-five definitions have accompanying diagrams, even though he still included images for all but one of the definitions (a point). Despite the fact that most diagrams included an image for multiple definitions, it would be a mistake to see the groups of images as multiple diagrams. Instead, they are best understood as single entities because each diagram

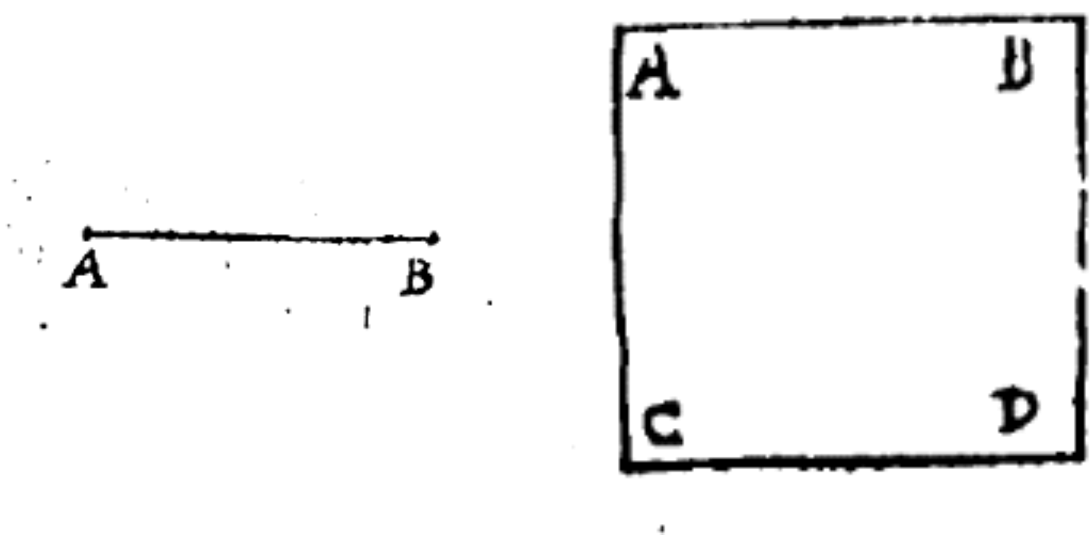
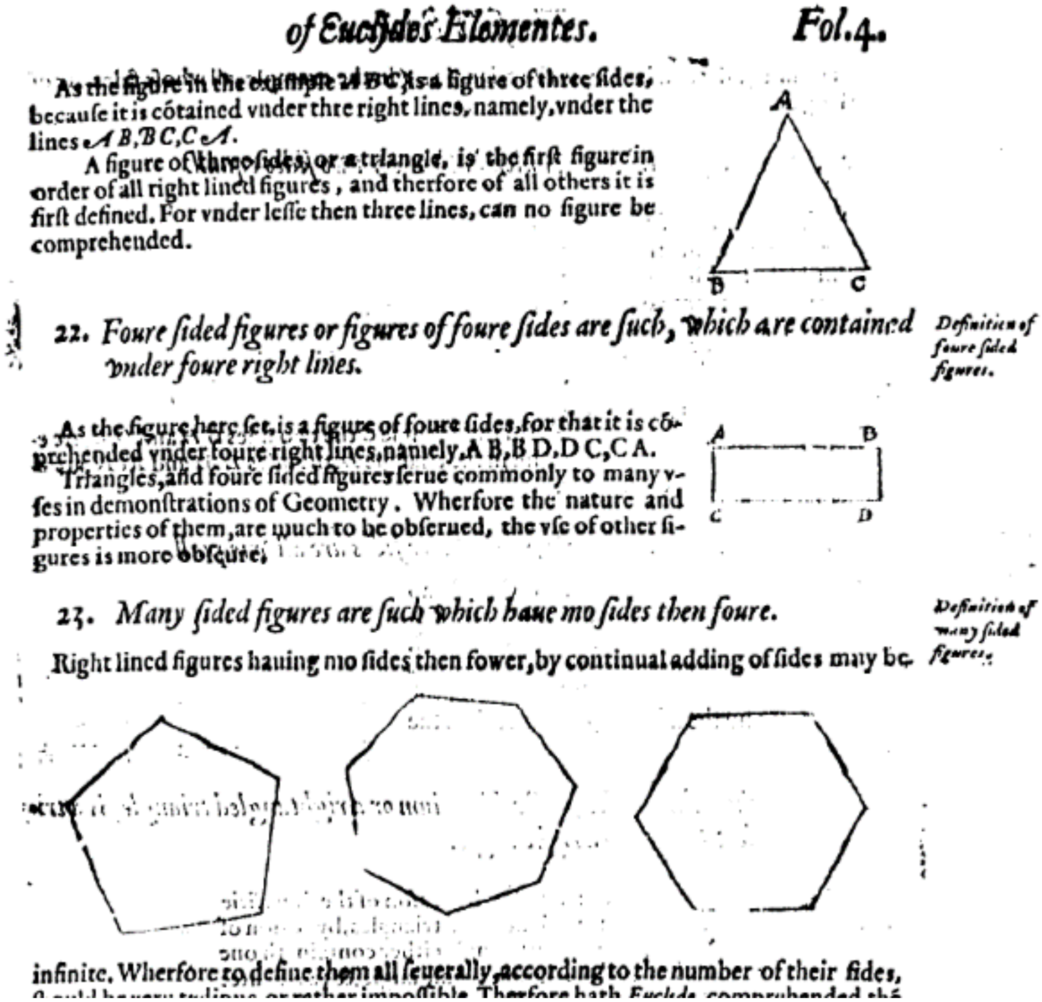


Figure 16: Billingsley's Images for a Line and a Surface

In his image for the limits of a line (left), Billingsley accentuated the endpoints A and B to call attention to them as the termini of the line. His image for a surface (right) doubles as his image for the limits, or extremes of a surface. In his commentary to the definition for the extremes of a surface he merely points out that line AB, BD, DC, and CA are the limits of the surface ABCD.


of Euclides Elementes. Fol.4.

As the figure in the example ABC, is a figure of three sides, because it is cōtained vnder thre right lines, namely, vnder the lines AB, BC, CA.

A figure of three sides, or a triangle, is the first figure in order of all right lined figures, and therfore of all others it is first defined. For vnder lesse then three lines, can no figure be comprehended.

22. *Foure sided figures or figures of foure sides are such, which are contained vnder foure right lines.* Definition of foure sided figures.

As the figure here set, is a figure of foure sides, for that it is cōprehended vnder foure right lines, namely, AB, BD, DC, CA.

Triangles, and foure sided figures serue commonly to many vses in demonstrations of Geometry. Wherfore the nature and properties of them, are much to be obserued, the vse of other figures is more obscure.

23. *Many sided figures are such which haue mo sides then foure.* Definition of many sided figures.

Right lined figures hauing mo sides then fower, by continual adding of sides may be infinite. Wherfore to define them all seuerally, according to the number of their sides,

Figure 17: Billingsley's Images for Three-, Four- and More-Sided Figures

Billingsley's images for each definition appear alongside their accompanying text.

shows the reader the relationships between the definitions illustrated, and sometimes can even uncover assumptions within definitions.

For example, the diagram accompanying the definition of an acute angle (Figure 18) has an image of a right angle (definition 10), an obtuse angle (definition 11), and an acute angle (definition 12), all three kinds of straight-lined plane angles. By placing all three images next to one another, Commandino's diagram made it easy to see the comparisons outlined by the definitions. An obtuse angle is an angle larger than a right angle, and an acute angle is smaller than a right angle. However, the definitions for obtuse and acute angles did not specify that they are straight-lined angles. Commandino's image shows three straight-lined angles, showing the only situation in which the comparisons defining obtuse and acute angles make sense. On this point, the diagram provided reinforcement for the argument found in the commentary, but that argument relied primarily on the description of counterexamples and never referenced the image.[11] In fact, Commandino rarely referenced his visuals in his text, minimizing the physical nature of mathematics embodied by the diagrams.

Clavius struck a balance between Billingsley and Commandino in his treatment of diagrams, showing mathematics to be the study of both the abstract

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[11] Federico Commandino, *Euclidis Elementorum Libri XV,* (Pisa: Jacobus Chriegher German, 1572), 2*v*. "In diffinitione anguli obtusi, & acuti genus subintelligi oportet, est enim uterque ipsorum rectilineus; hic quidem minor recto, ille autem maior. Sed non simpliciter quicumque minor est recto, is est acutus; neque quicumque maior recto est obtusus. Nam qui grece κεζατοειδμς dicitur, hoc est cornicularis, qui continetur recta linea circulum contingente, & circumferentia ipsa, non tantum recto, sed etiam omni acuto est minor, acutus autem non est. & semicirculi angulus omni recto est minor, sed tamen non est acutus, quorum quidem causa est, quod sunt mixti, & non rectilinei. & eorum, qui lineis circularibus, aut alioqui curvis continentur multi recto maiores apparent, non tamen sunt obtusi. Cum igitur rectum angulum diffinire proposuisset Euclides rectam assumpsit lineam super aliam rectam insistentem; & angulos qui ex utraque parte sunt, quos angulos deinceps appellat, inter se aequales facientem."

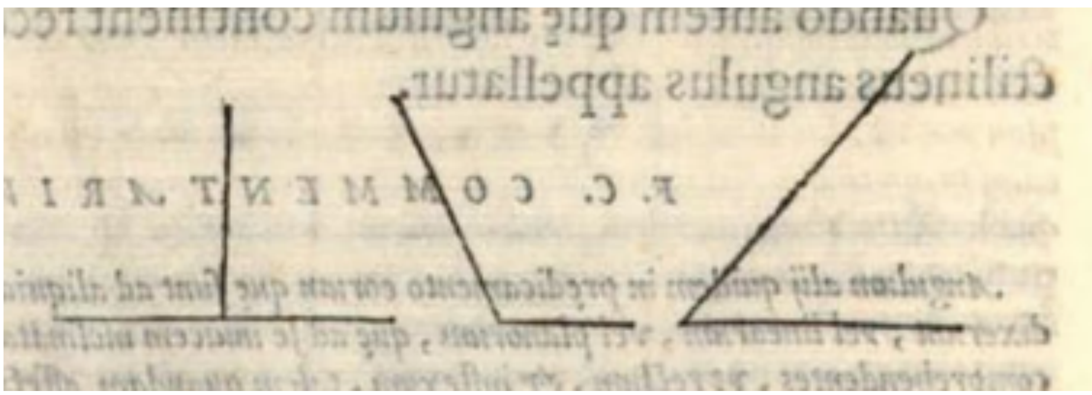

Figure 18: Commandino's Diagram Accompanying the Definition of an Acute Angle

This diagram shows all three kinds of straight-lined plane angles, allowing easy comparison, but it separates images from their definitions.

concepts expressed in the definitions and the physical objects found in the diagrams. Like Billingsley, Clavius made the connection between most definitions and their diagrams explicit. Twenty-four of his thirty-six definitions had accompanying diagrams, and the commentaries of six more definitions referred the reader to diagrams for nearby definitions. However, in cases in which his diagram served simply as an illustration of the concept, Clavius, unlike Billingsley, usually did not refer to the diagram in his text. Therefore, the image, rather than acting as a particular instance of the defined concept, served as a generic form of the concept. Still, because the images are attached to individual definitions, they lack the emphasis on universality found in Commandino's diagrams of categories of objects. Furthermore, when no one image could fully represent a broad class of objects, Clavius refrained from including diagrams. The six definitions that do not have clearly associated diagrams are all larger classes of objects, such as multilateral figures, the only definition for which Clavius does not provide any image.[12] By leaving out diagrams for the more general definitions, Clavius implied that visuals were indeed physical instances of specific objects, despite their ability to represent sets of visually similar objects.

Clavius's own understanding of the relationship between the physical and the abstract aspects of mathematics as intertwined becomes clear in a few definitions in which he allowed the image to define the concept by asking the reader to imagine

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[12] The other five definitions without explicit connections to diagrams are represented in the images accompanying cases of the concepts they define. For example, the definition of a limit is diagrammatically represented in the images for the definition of the limits of a line. No specific cases of a multilateral figure are defined, so no illustrations are included.

some physical manipulation of the diagram in order to explain the concept. By using the image to create a functional definition for concepts, he united the physical diagrams with the metaphysical notions of geometric elements, requiring the two aspects of mathematics to work together. For example, Clavius's diagram of a line appears next to the section of commentary in which he explains how a line can be understood as "a point in motion."[13] The explanation relies on the ability of the reader to imagine point *A,* which Clavius had insisted was immaterial in his commentary on the definition of a point, traveling to point *B*, leaving a trail in its wake. The diagram (Figure 19) assists the reader by presenting two possible paths for point *A* to follow.[14] The commentary's hypothetical moving point is made real by the diagram, and the line becomes its physical path. Thus, even though neither a point nor a line can properly exist in the physical world, the metaphysical dimensionless point is used to create a physical definition of a metaphysical line's breadthless length.[15]

The general patterns just described show that each author approached mathematics with a unique understanding of its value. These same patterns manifest themselves in the individual visualizations for each definition, making diagrams for specific definitions valuable sources for studying the authors' perceptions of the importance of mathematics. Even in heavily constrained definitions, for which all

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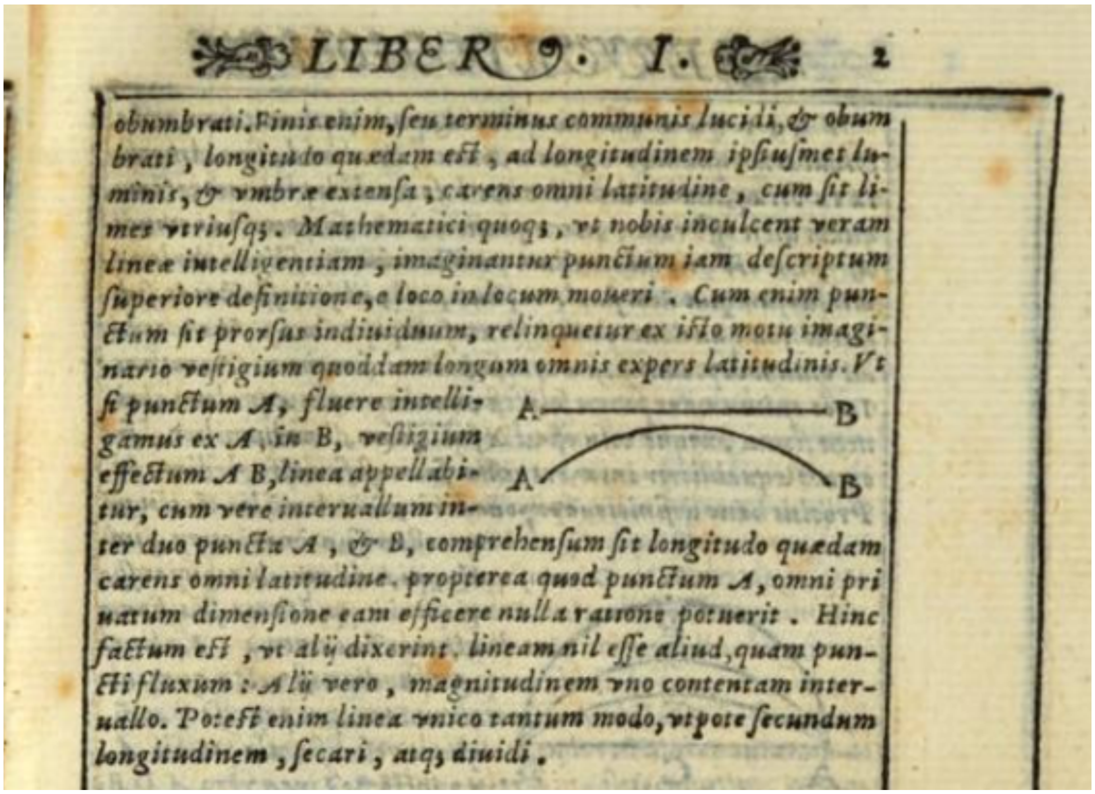
[13] Christopher Clavius, *Euclidis Elementorum Libri XV. Accessit XVI de Solidorum Regularium comparatione,* (Rome: Vincentium Accoltum, 1574), 2*r*. "Hinc factum est, ut alii dixerint, lineam nil esse aliud, quam puncti fluxum."
[14] Ibid., 2*r.* "Ut si punctum A, fluere intelligamus ex *A* in *B*, vestigium effectum *AB*, linea appeliabitur…"
[15] Ibid, 1*v*. Clavius's definition of a line is "Linea vero, longitudo latitudinis expers." That is, "A line, properly named, is length free from breadth."



LIBER I. 2

obumbrati. Finis enim, ſeu terminus communis lucidi, & obum brati, longitudo quædam eſt, ad longitudinem ipſiuſmet luminis, & vmbræ extenſa; carens omni latitudine, cum ſit limes vtriuſq;. Mathematici quoq;, vt nobis inculcent veram lineæ intelligentiam, imaginantur punctum iam deſcriptum ſuperiore definitione, e loco in locum moueri. Cum enim punctum ſit prorſus indiuiduum, relinquetur ex iſto motu imaginario veſtigium quoddam longum omnis expers latitudinis. Vt ſi punctum A, fluere intelligamus ex A, in B, veſtigium effectum A B, linea appellabitur, cum vere interuallum inter duo puncta A, & B, comprehenſum ſit longitudo quædam carens omni latitudine, propterea quod punctum A, omni priuatum dimenſione eam efficere nulla ratione potuerit. Hinc factum eſt, vt alij dixerint, lineam nil eſſe aliud, quam puncti fluxum: Alij vero, magnitudinem vno contentam interuallo. Poteſt enim linea vnico tantum modo, vtpote ſecundum longitudinem, ſecari, atq; diuidi.

Figure 19: Clavius's Diagram of a Line as a Point in Motion

In the text, the reader is instructed to imagine point A moving to point B. The contrast of the straight and the curved lines shows that any path taken by a point is a line.

three authors necessarily had very similar images, small differences in the presentation of the diagrams can expose significant variations in the authors' understandings of mathematics. For example, because equilateral triangles can only differ from each other by a scale factor, if the illustrations for the definition of an equilateral triangle are isolated from each text (Figure 20), they look virtually identical. Besides slight differences in their sizes, the only distinction between these three images is the label Billingsley included for his triangle. However, that label is not an insignificant difference. It reflects Billingsley's emphasis on the physical nature of mathematics because its presence calls attention to the specificity of "triangle A," rather than a generic "equilateral triangle." Indeed, in his commentary he referred to triangle A as an example of an equilateral triangle, not a representation of all such triangles. In contrast, Clavius and Commandino maintained their images' representative quality by neither labeling nor referring to the diagrams. Without anything to render them specific instances of an equilateral triangle, the diagrams clearly serve to represent the abstract concept.

Besides the use of labels and references to the image in their texts, authors could change the meaning of an image by changing its placement. In this example, Billingsley and Clavius presented their images alongside their definitions for an equilateral triangle (Figure 21), emphasizing the physical attributes of that specific kind of triangle. Commandino placed his in a group of images that represented all seven possible kinds of triangles (Figure 22). By grouping the triangles into a single diagram, Commandino emphasized the universal concept of the triangle both by not

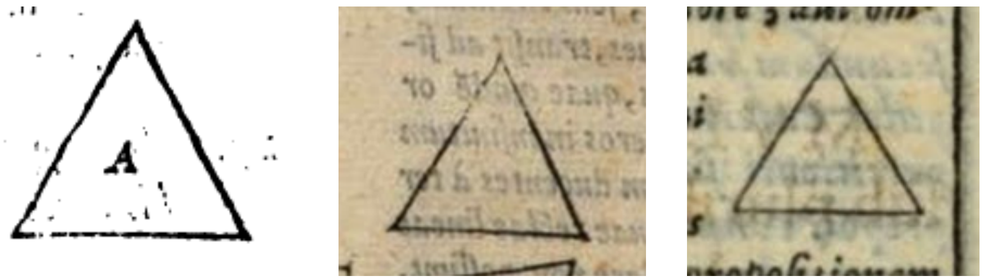
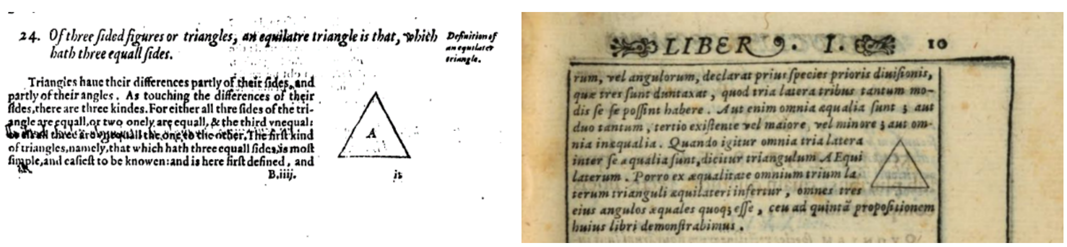


Figure 20: Diagrams of an Equilateral Triangle

From left to right: Billingsley's, Commandino's, and Clavius's images of an equilateral triangle.



24. *Of three ſided figures or triangles, an equilatre triangle is that, which hath three equall ſides.*

Definition of an equilater triangle.

Triangles haue their differences partly of their ſides, and partly of their angles. As touching the differences of their ſides, there are three kindes. For either all thre ſides of the triangle are equall, or two onely are equall, & the third vnequal: or els all three are vnequall the one to the other. The first kind of triangles, namely, that which hath three equall ſides, is moſt ſimple, and eaſieſt to be knowen: and is here firſt defined, and

B.iiij. is

LIBER. I. 10

rum, vel angulorum, declarat prius ſpecies prioris diuiſionis, quæ tres ſunt duntaxat, quod tria latera tribus tantum modis ſe ſe poſſint habere. Aut enim omnia æqualia ſunt; aut duo tantum, tertio exiſtente vel maiore, vel minore; aut omnia inæqualia. Quando igitur omnia tria latera inter ſe æqualia ſunt, dicitur triangulum Æquilaterum. Porro ex æqualitate omnium trium laterum trianguli æquilateri infertur, omnes tres eius angulos æquales quoq; eſſe, ceu ad quintã propoſitionem huius libri demonſtrabimus.

Figure 21: Billingsley's and Clavius's Diagrams for an Equilateral Triangle in Context

XXIX.

Acutiangulum vero, quod tres acutos angulos habet.

F. C. COMMENTARIVS.

Triangulorum diuiſio interdum quidem à lateribus, interdum vero ab angulis ortum habet: & præcedit ea, quae à lateribus tamquam nota, ſequitur autem ea quae ab angulis tamquam propria, quoniam & tres ipſi anguli, videlicet rectus, obtuſus, & acutus ſolis rectilineis figuris conueniunt: aequalitas verò laterum, & inequalitas inuenitur etiam in iis, quae rectilineae non ſunt. dicit igitur triangulorũ alia eſſe aequilatera, alia aequicruria, alia ſcalena, vel enim omnia latera aequalia ſunt, vel omnia inæqualia, vel duo tantũ aequalia. Rurſus triangulorum alia rectangula, alia obtuſiangula, alia acutiangula, & rectangulum quidem diffinit, quod vnum angulum rectum habet, quemadmodum obtuſiangulum, quod vnum habet

Figure 22: Commandino's Diagram for an Equilateral Triangle in Context

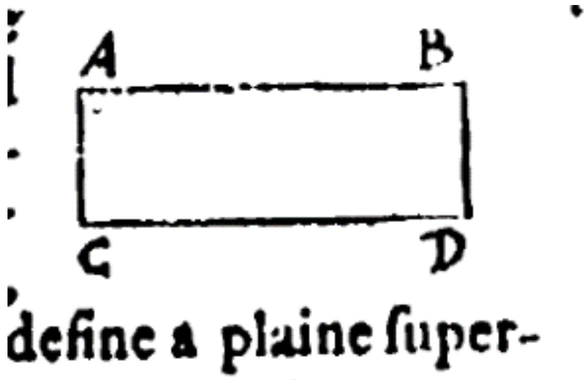
verbally distinguishing the various kinds of triangles with clear labels and by making it possible for readers to see relationships between kinds of triangles. His is the only diagram that makes it obvious that, because triangles are defined both by angles (right, acute, and obtuse) and by sides (equilateral, isosceles, and scalene), every triangle fits two of Euclid's definitions. The diagram accomplishes this by arranging the triangles in tabular form. The columns show right, acute, and obtuse triangles. The rows show equilateral, isosceles and scalene triangles. Thus, it becomes immediately obvious that equilateral triangles are necessarily acute, but isosceles and scalene triangles may also be right or obtuse.

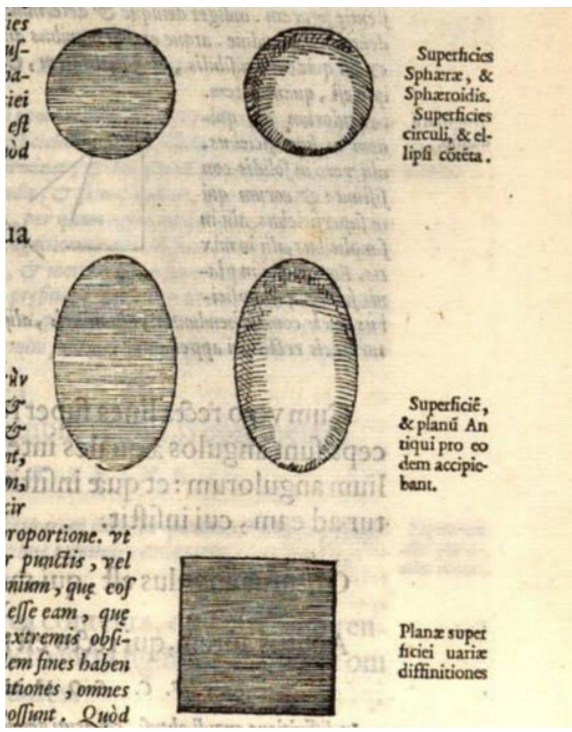
Other definitions offered more room for visual interpretation, leading to more varied diagrams. For example, the definition of a plane states, "A plane surface is that which lies equally between its lines."[16] From the previous two definitions which define a surface as that which has only length and breadth, and declare the edges of a surface to be lines, the reader could recognize that the lines of a plane surface are its edges, but he would not have any indication of how he should visually represent a surface lying equally between its edges. The freedom afforded the authors by the abstract definition of a plane allows the patterns visible in the section of definitions to manifest themselves in the diagrams for this definition (Figure 23). Billingsley offered a physical example of the concept; Commandino used his diagram in service of a conceptual argument about the nature of surfaces; and Clavius used the physical

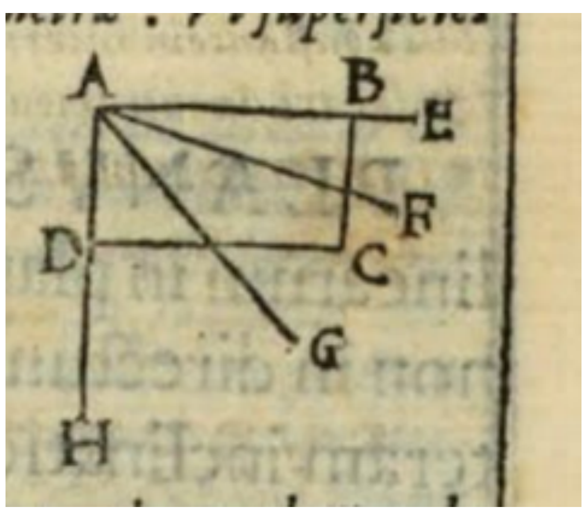
[16] Clavius, *Euclidis Elementorum,* 4*r.* "Plana superficies est, quae ex aequo suas interiacet lineas." Commandino, *Euclidis Elementorum,* 2*r*. "Plana superficies est quae ex aequali suis interijcitur lineis." Billingsley, *Elements of Geometrie,* 2*r*. "A plaine superficies is that, which lieth equally between his lines."

Figure 23: Diagrams for the Definition of a Plane

From top to bottom: Billingsely's, Commandino's, and Clavius's diagrams. Note that in Clavius's diagram, lines AE, AF, AG, and AH are meant to be iterations of the same line.


A
B
C
D
define a plaine super-


Superficies Sphæræ, & Sphæroidis. Superficies circuli, & ellipsi cõtẽta.
Superficiẽ, & planũ Antiqui pro eodem accipiebant.
Planæ superficiei uariæ diffinitiones


A
B
E
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D
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H

diagram to create a physical definition for a plane, thereby uniting the physical and the conceptual in one diagram.

Billingsley's diagram was the simplest of three. He drew a rectangle, and described for the reader that it "lyeth equally and smoothe betwene the two lines *AB* and *CD:* or between the two lines *AC* and *BD* so that no part thereof eyther swelleth upward or is depressed downward."[17] His visual representation relied on the flatness of the paper to convey the flatness of the plane, effectively making the enclosed portion of the page a physical example of a plane. Commandino also represented a plane with an enclosed rectangular area, but he relied on even shading, not the flatness of the page, to indicate to the reader that the rectangle should be seen as flat. But Commandino's rectangle was not just an example of a plane. It was also part of a collection of images that served to make arguments about the broader nature of a surface. First, by including three planar figures – a circle, an ellipse, and a rectangle - Commandino showed his reader that a plane need not be any one shape, rectangle or otherwise. The diagram also offered a much larger argument that Euclid's definition for the boundaries of a surface effectively limited surfaces to planes and things that could be easily projected onto planes, by excluding surfaces without linear boundaries. At the time Commandino was writing, the impossibility of projecting a sphere or an ellipsoid onto a plane was a much studied problem due to the contemporary interest in cartography. [18] It is possible that Commandino had this problem in mind when he

[17] Billingsley, *Elements of Geometrie,* 2*r*.
[18] Knowing which shapes could be projected onto Euclidean planes mattered. It allowed one to consider when Euclidean geometry could be meaningfully applied. It can't on the surface of a sphere. Thus, cartography relied on developing projections of a sphere that allowed the Euclidean geometry of

created his diagram for a plane. Rather than focusing on the definition of a plane itself, Commandino's image centered on the linear boundaries of planes (and surfaces that can be projected onto planes) as the defining feature that allows Euclidean geometry to be applied. It included a sphere and an ellipsoid, which lack linear boundaries, as contrasts to the planar circle and ellipse, which are each bounded by their circumferences, single, curved lines.

Clavius's diagram for a plane provided a physical means to define and identify a planar surface, uniting the abstract concept with the physical tool he created. Like his contemporaries, he started with a rectangle, but instead of using his description of the diagram or shading to convey its flatness, he devised a theoretical manipulation of the diagram to allow the reader to "check" whether or not a surface was a plane. The diagram thus becomes the tool with which the concept of a plane is defined. In it (see Figure 23), Clavius presented the rectangle ABCD as the surface under study. One edge, AB is extended to point E. In the commentary, Clavius instructed the reader to imagine rotating the straight line AE around the point A such that it passes over the surface ABCD. In order to convey this rotation he drew lines AF, AG, and AH as iterations of AE passing through its rotation. If every point in the surface touches line AE as it passes over, then the surface is a plane.[19] Later in his commentary, in order

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a flat map to approximate the globe. Several such projections were developed in the sixteenth century. Mercator's famous projection was published in 1569. For a discussion of the variety of map projections known and produced by the sixteenth century, see John P. Snyder, *Flattening the Earth: Two Thousand Years of Map Projections* (Chicago: The University of Chicago Press, 1993), 1-54.

[19] Clavius, *Euclidis Elementorum,* 4*r*. "Ut superficies *ABCD*, tunc demum plana dici debet, quando linea recta *AE*, circa punctum *A,* immobile circunducta, ita ut nunc eadem sit, quae *AB*, nunc eadem, quae *AF*, nunc eadem, quae *AG*, & nunc eadem quae *AH*, nihil in superficie offendit depressum, aut sublatum, sed omnia puncta superficiei a linea recta tanguntur, & quodammodo raduntur."

to cement the validity of his physical definition, he provided several examples of curved shapes that would fail the test of the rotating line. He even included one image that showed such a surface, though he did not provide a line to conduct the imaginary test because the surface "could not in all parts fasten on a straight line."[20]

The variation in these diagrams demonstrates that while the definitions of the first book may not have required diagrams to be understood, the authors' visualizations for these definitions showed the reader what each author took to be the significance of mathematics and, therefore, were essential sources for establishing the value of the discipline. For Billingsley, diagrams served to ground mathematics on the physical page. They gave him a way to create concrete mathematical objects for his reader. For Commandino, visualizations offered a means to make relationships between the definitions clear and to emphasize the abstract, universal concepts of mathematics. Bridging the approaches of his two contemporaries, Clavius used his images, especially those that provided mechanical definitions, to establish the mutually informative relationship between the physical instances of mathematical objects and the abstract concepts that defined them.

The definitions to the first book set the tone for the rest of *The Elements,* but, because the simplicity of the concepts meant that most readers would have had a firm grasp of most definitions even without the visual aids, the authors had a great deal of freedom in choosing how to illustrate them. In the next section, I will turn to an

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[20] Ibid., 4*v*. "Caeterae omnes superficies, quibus non omni ex parte accommodari potest linea recta, quails est superficies interior alicuius fornicis, vel exterior alicuius globi, columnaeuve rotundae, vel etiam coni etc. appellantur curvae, & non planae."

examination of visual aids in two propositions to understand how the visions of mathematics that emerged from the authors' visualizations in the definitions carried into the more constrained environment of the propositions.

Diagrams for Planar Propositions: How to Make an Image Represent Big Ideas

When preparing the diagrams to accompany the propositions in *The Elements*, commentators had much less freedom than they had for the definitions. As the reader will recall from Chapter Three, a complete proof of a proposition usually included a construction. In other words, the demonstration usually included some description of what the diagram should look like and how to produce it. For those propositions classified as theorems, which make a claim about some geometric entity, the diagram is used to advance the arguments necessary to justify the claim. For the propositions classified as problems, whose goal is a construction of a specific entity, the diagram is the solution to the problem and serves as an example of the construction process described in the text. In both cases, the diagram offers a physical entity with which to verify arguments made in the text.[21] In some cases, the diagram also serves as the source of necessary arguments that are not present in the text. The diagram is thus an integral part of the textual arguments, so it is not surprising that, at first glance, most diagrams accompanying propositions show very little variation across numerous editions of *The Elements.* However, because the diagrams are both individual

[21] See Norman, *After Euclid* for elaboration on the ways in which diagrams can be used to justify a claim. His work relies on an analysis of what a reasoner does as she reads and works through a Euclidean proposition. He suggests a "neo-Kantian" approach to diagrams which allows diagrams to make justificatory contributions to knowledge through a priori reasoning.

instances of the objects described by the proposition and representations of the universal claims made by the propositions, even small changes could shift an author's focus to the role of diagrams either as individual instances of the objects described or as representations of the universal claims made by the propositions.

The first proposition, which requires the reader to construct an equilateral triangle on a given line, offers a clear example of the various roles played by diagrams. As discussed in Chapter 3, the construction is done by drawing two circles with radii the length of the given line and centers at either end point. The point of intersection of the circles is then used as the third vertex of the triangle. The argument of the demonstration relies on the diagram to be intelligible. Even before the discussion of the triangle begins, the diagram is essential to verify the assumption that the circles will intersect one another. *The Elements* does not offer a verbal argument on that point, so the image is the only source for a demonstration of that assumption.[22] Once the triangle is drawn, the image makes the relationships between the sides of triangle developed by the demonstration's argument immediately clear. The argument works by observing that each constructed side of the triangle is a radius of one of the circles and the original line is a radius of both circles with centers of the circles at opposite endpoints. Since all radii in any circle are equal to each other, each drawn side is equal to the original line, and, therefore, the three lines are equal to each other.

[22] None of the sixteenth-century authors I have read make note that the diagrams fill in assumptions left implicit in the text. That became a concern to philosophers such as Hilbert in the late nineteenth and early twentieth centuries who wanted to eliminate diagrams from geometrical reasoning.

Hence, the triangle is equilateral. The reader can use the diagram to verify the text's assertion that the sides of the triangle are indeed radii.

The diagrams most authors provided aligned closely with the construction described in the proof. They showed two circles with a triangle drawn on their shared radius, which is the given line. The other sides of the triangle meet at an intersection of the circles. Most mid-sixteenth-century diagrams seem to come from one of two earlier commentaries. The first of these commentaries could be the first edition printed in Greek, Simon Gyrnaeus's 1533 edition, which is believed to have been the text on which both Commandino and Billingsley based their translations.[23] Indeed, Commandino's and Billingsley's diagrams share all of the same features.[24] The same diagram can be found in a 1573 Parisian edition of *The Elements* that only included the enunciations and diagrams.[25] Another edition from the early sixteenth century, Bartolomeo Zamberti's 1516 text which was based on Campanus's medieval work, shows the same diagram with slight variation in the labeling: the second point of the circles' intersection is also labeled. His diagram seems to have been the source for

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[23] For Commandino's source see Paul Rose, *The Italian Renaissance of Mathematics,* (Geneva: Librairie Droz, 1975), 206. For Billingsley's source see George Bruce Halsetd, "Note on the First English Euclid," *American Journal of Mathematics,* Vol. 2, No. 1 (Mar. 1879), 46-48.

[24] There is a difference in the placement of the labels for the third point defining the diameter of each circle, points D and E in the diagrams. Billingsley placed his labels inside the circle, whereas Commandino's are on the outside. It is likely that Billingsley used his placement of the labels to save space.

[25] Euclid, *Euclidis elementorum, libiri XV* (Paris: Gulielmum Cavellat and Hieronymum de Marnes, 1573), 47. The reader will recall from previous chapters that the enunciations in Euclid's text are the sentences that compose the definitions, axioms, postulates, and propositions without any further commentary or proof. For example, the enunciation of the first proposition is "To draw an equilateral triangle on a given line."

Clavius's work, among others, including the French commentator, Jacques Peletier (1517-1582). These images can be seen in Figure 24.

While the differences in the labels of the diagrams may appear to be a minor discrepancy arising only from the reliance on different sources, they can reveal how the diagrams helped to define the status of mathematics. The second label found in Clavius's text is the key to a significant argument for mathematics as a discipline intermediate between physics and metaphysics. By labeling both points of intersection, he gave the reader a choice of two possible equilateral triangles that can be constructed on a given line. As Clavius and the other authors who label the second intersection point observe in their commentary, either point of intersection will do as the third vertex of the triangle.[26] This choice calls attention to the diagram's role as a representation of the abstract idea of an equilateral triangle in addition to its role as a physical instance of a triangle and solution to the problem. The reader can see two distinct physical solutions to the problem, namely, the drawn tringle (*ABC* in Clavius's image) and the potential triangle using the other point of intersection (*D* in Clavius's image) as the third vertex. But, since the only difference between the two possible triangles is their orientation, either triangle can represent both options for any further discussion of an equilateral triangle drawn on the given line. Furthermore, the given

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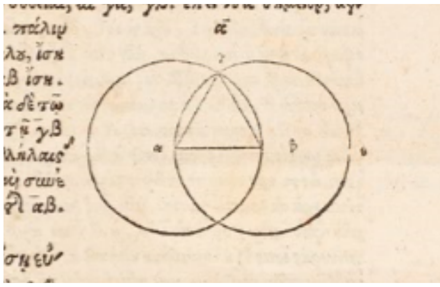
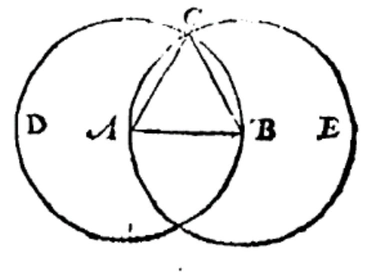
[26] Clavius, *Euclidis Elementorum,* 21*v.* "…secans priorem in punctis C & D. Ex quorum utrovis, nempe ex C, ducantur duae rectae lineae…;" Jacques Peletier, *In Euclidis Elementa Geometrica Demonstrationum Libri sex* (Lyon: Ioan. Tornaesium et Gul. Gazium, 1557), 13. "Atque hi duo Circuli se mutuo secabunt in duobus punctis, ut in E &C: quum utriusque communis sit Semidiameter AB. A duobus igitur terminis A& B, ad alteram intersectionum ut ad C, duco AC & BC lineas…;" Bartolomeo Zamberti, *Euclidis Geometricorum elementorum libri XV* (Paris: Henrici Stephani, 1516), 5*v.* "…qui circuli intersecabunt se in duobus punctis quae sint c, d. Et alteram duarum sectionum sicut sectionem d, contiuabo cum ambabus extremitatibus datae lineae…."

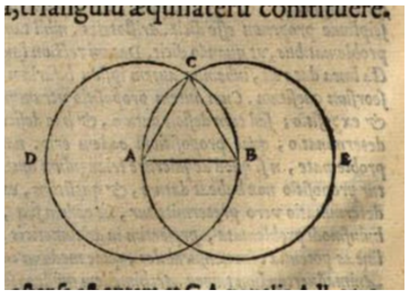
Figure 24: Diagrams for Book One, Proposition 1

The diagrams for the first proposition of *The Elements* from a selection of sixteenth century editions and commentaries. Grynaeus's 1533 commentary was the source used by both Billingsley and Commandino. Clavius had certainly read both Zamberti's and Peletier's commentaries.

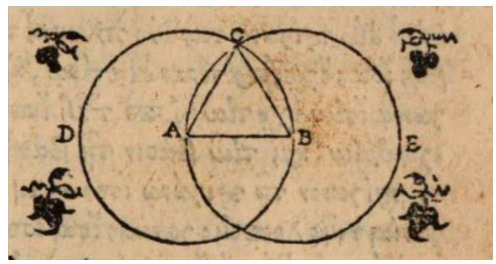

Grynaeus, 1533
(Source: Huntington Library)



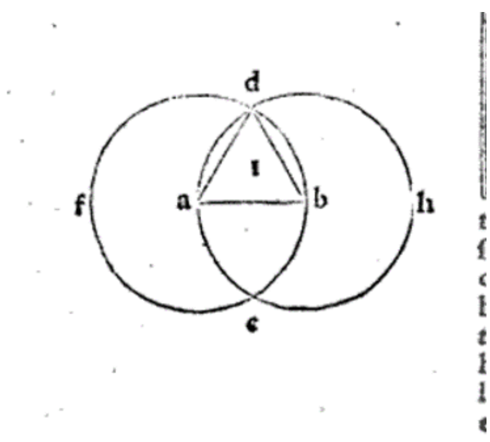
Billingsley, 1570



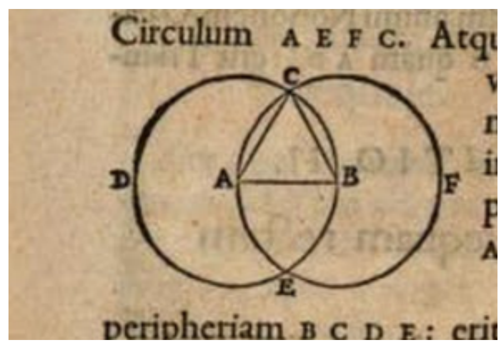
Commandino, 1572



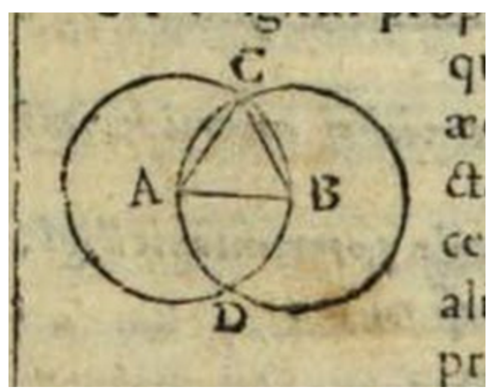
Cavellat and de Marnes, 1573



Zamberti, 1516



Peletier, 1557



Clavius, 1574

line is arbitrary, so, once it has been accepted that the drawn triangle represents both options for that line, it is easy to see that it also represents all equilateral triangles, since any other line could have been the starting point.

While Clavius's diagram allows the reader to identify visually two interpretations of the triangle, Billingsley and Commandino used their commentaries to enable the reader verbally to recognize both interpretations of the diagram. However, because their arguments are purely verbal, they keep the two interpretations of the diagram separate from each other. In their commentaries, Billingsley and Commandino identify a "particular conclusion" – that triangle ABC is equilateral - and a "universal conclusion" - that the construction process described can generate an equilateral triangle on any line. If the goal of the problem is understood simply as constructing an equilateral triangle, then the particular conclusion is a sufficient solution to that problem. The universal conclusion shows that the problem is not about drawing a specific triangle, but about providing a method to construct an equilateral triangle on any finite line.[27] For that conclusion to hold, the triangle in the diagram must represent all equilateral triangles. Showing that ABC is equilateral is a

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[27] Billingsley, *Elements of Geometrie,* 8*v*. "For there are commonly in every proposition two conclusions: the one perticuler, the other universal: and from the first you go to the last. And this is the first and perticuler conclusion, for that it concludeth, that upon the lyne AB is described an equilater triangle, which is according to the exposition. After it, followeth the last and universal *conclusion, wherefore upon a right line geven not being infinite is described an equilater triangle*. For whether the line geven be greater or lesse then thys line, the same constructions and demonstrations prove the same conclusion." (Italics serve as Billingsley's way to represent the text copied from the demonstration); Commandino, *Euclidis Elementorum,* 8*r.* "Unde colligitur triangulum ABC aequilaterum esse, atque haec est prima conclusio, quae expositionem consequitur; post hanc est ipsa unversalis. [*In data igitur recta linea triangulum aequilaterum constitutum est*.] Sive enim duplam eius, quae nunc exposita est, feceris datam, sive triplam, sive aliam quamlibet maiorem, vel minorem; aedem constructiones, & demonstrationes congruent." (Commandino used the change in font, here indicated by italics, and brackets to differentiate the text taken from the proof of the proposition from the commentary.)

step towards the universal conclusion, but the two conclusions, and the corresponding versions of mathematics, as a study of physical bodies or a study of universal truths, remain separate from each other.

There was one sixteenth-century commentator who made substantial changes to the ancient text, including the diagram for the first proposition. Francois Flussas Candalla, who claimed to have eliminated "unskilled repartees," "stammering writing," and anything that would obscure the clarity of the geometric principles, replaced Greek proofs with his own and added a sixteenth book to *The Elements* in his 1566 edition.[28] For the first proposition, he provided a diagram that includes only portions of the circles showing the intersection of arcs that defines the third vertex of the triangle. It gives just enough visual information to convince the reader that each edge of the triangle is a radius of one or both circles, and it can be recognized as an enlargement of the relevant portion of the image found in the other versions of *The Elements* (Figure 25). Candalla's diagram separated the particular solution of the drawn triangle from the universal conclusion of the technique for creating a traingle. By showing arcs instead of circles, the diagram offered a truncated procedure for the construction that allowed the physical solution to the problem to supersede the

[28] Francois Flussas Candalla, *Euclidis Megarensis Mathematici Clarissimi Elementa Geometria, Libris XV* (Paris: Iannem Royerius, 1566), aiii*r*. "In hoc autem operis tuae celsitudini consecrandi votum me rapuit (candide Princeps) non imperitae arguitiae, non balbutientia scripta, non harum inventionum obtusi conatus, tanti principis dignitate alieni." Commandino summarized Candalla's arguments for replacing the ancient proofs by saying that the French Duke saw the Greek proofs available to sixteenth-century authors as less elegant than his own. Commandino, *Euclidis Elementorum*, *2*v* -*3*r,* "At Candalla vir & generis nobilitate, & rerum congitione insignis, licet omnes Elementorum libros, qui postulari a latinis videbatur, latinos fecerit, locupletaueritque, parum tamen (ut audio) eo nomine commendatur, quod longius iter ab Euclide averterit, & demonstrationes, quae in graecis codicibus habentur, velut inelegantes, & mancas suis appositis reiecerit." According to Commandino, his own edition of *The Elements* was superior to Candalla's precisely because he provided the ancient proofs.

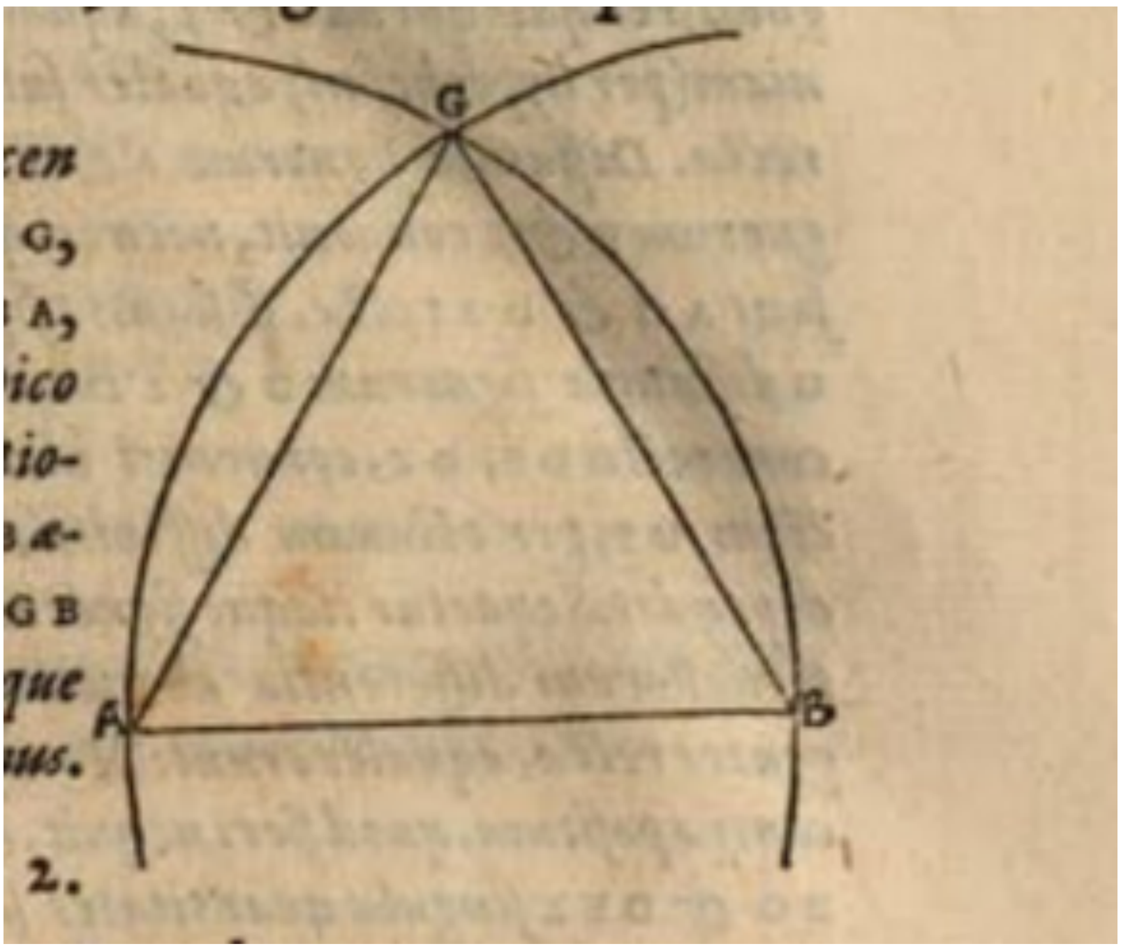

Figure 25: **Candalla's Diagram for Book One, Proposition 1**

theoretical requirement of using complete circles to identify the triangles' third vertex. Because the proof that the triangle is equilateral depends on a property of circles, namely that all radii of any one circle are equal, including the full circles in the construction made the proof intelligible. Indeed, Candalla's *text* maintains the dependence on the full circles, and he, like all the other authors, begins the construction with instructions to draw *circles*, not arcs, of radius length AB centered on A and centered on B.[29] However, his diagram deviates from the text and shows that the circles are not required for the construction, drawing the reader's attention to the particular rather than the universal conclusion.

While Candalla was the only author to give such precedence to the particular conclusion in the diagram for his proof, Clavius and Billingsley highlighted the physical nature of that conclusion in their commentaries by introducing the use of arcs as a shorter, less cumbersome technique to produce triangles, including isosceles and scalene triangles. In so doing, they separated the physical result of construction from the theoretical process of demonstration, and the particular conclusion of a single equilateral triangle from the universal conclusion of a procedure that could be shown always to generate an equilateral triangle. The diagrams they presented as guides for these shorter techniques have very small arcs showing little more than the intersection of the two circles (Figure 26). Unlike the diagrams for the proposition, these procedural diagrams relay very little information about the triangle to the reader. The

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[29] Candalla, *Euclidis Megarensis*, 5*r*. "Si proposita recta AB terminata in A&B, centro A: intervallo autem AB circulus ducatur BG, per. 3 postulatum, centro item B, intervallo BA, circulus ducatur AG…"

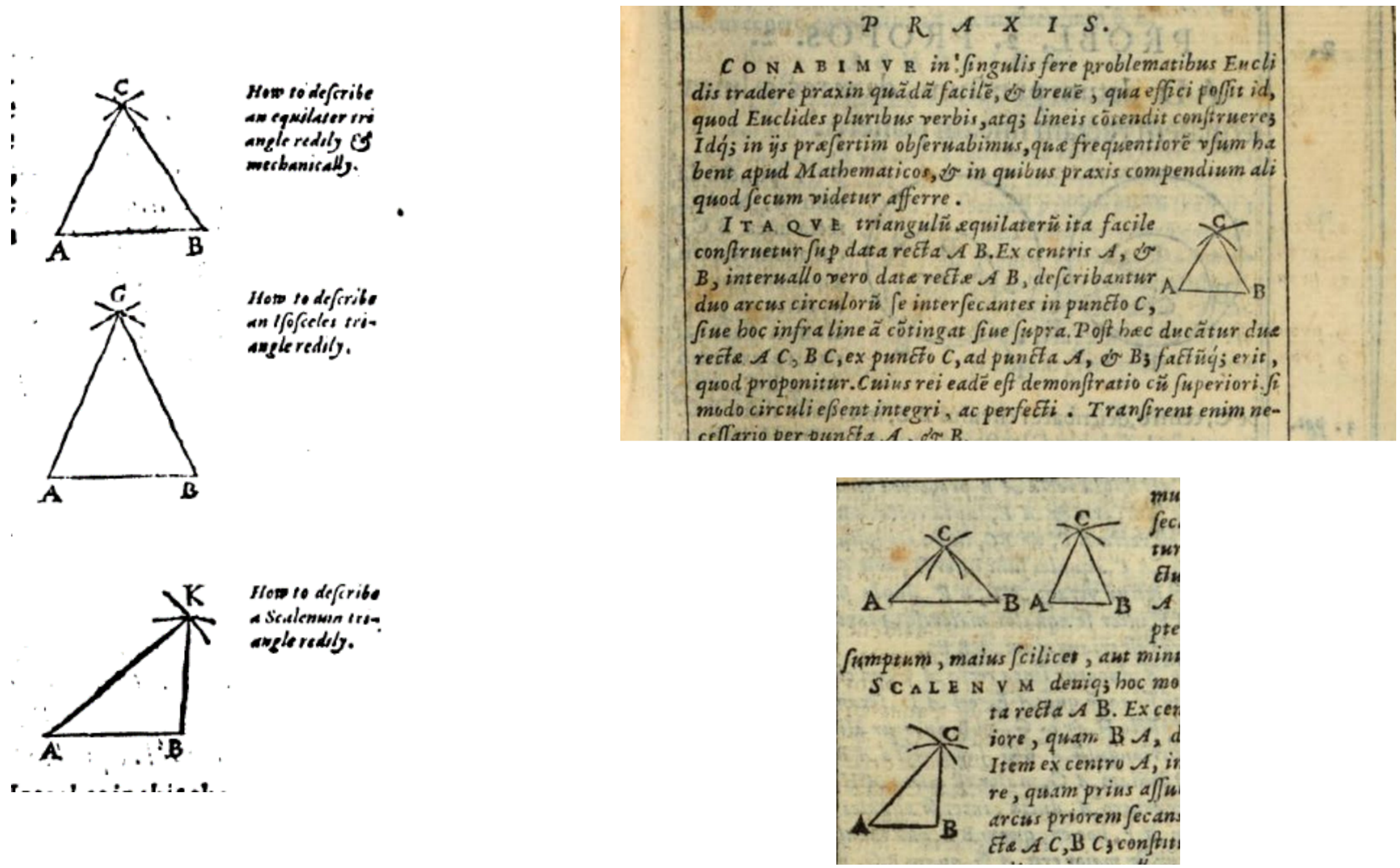

Figure 26: Billingsley's and Clavius's Procedural Diagrams for Book One, Proposition 1

Billingsley's diagrams are the three on the left. Clavius's, which are found under the heading "Praxis" are on the right.

arcs are so small that the reader cannot rely on them to infer anything concrete about the relative lengths of the sides of the triangle. If the reader wanted to verify that the procedure described was in fact what the diagram represented, he would have to measure the triangles and arcs on the printed page and trust that nowhere in the printing process had an error been introduced. Thus, the triangles and the arcs used to define them are nothing more than physical examples of a construction process made possible by the reader's knowledge of the missing circles. While both Clavius and Billingsley embraced the physical nature of mathematics by including a discussion of practical shortcuts, it should be noted that Clavius was far more careful than Billingsley to separate the use of such shortcuts from any theoretical meaning. Billingsley explicitly observed that the techniques were only useful when no demonstration of the properties of the drawn triangle was required, but that note appeared simply as a transition from one part of his commentary to another, giving the physical shortcuts just as much weight as the theoretical discussion preceding it.[30] In contrast, Clavius's constructions appeared under their own heading, "Praxis," which suggests an active, rather than a contemplative, approach to mathematics. Indeed, he justified their presence by offering them as a means to speed up the reader's construction of triangles as they were frequently necessary in the rest of the Euclidean text, reducing the value of the physical conclusion to that of a tool.[31]

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[30] Billingsley, *Elements of Geometrie,* 10*r*. "This is to be noted, that if a man will mechanically and redely, not regarding demonstration upon a line geven describe a triangle of three equall sides, he needeth not to describe the whole forsayd circle, but onely a little part of eche; namely, where they cut the one the other, and so from the point of the section to draw the lines to the ends of the line geven."
[31] Clavius, *Euclidis Elementorum,* 23*r*. "Conabimur in singulis fere problematibus Euclidis tradere praxin quamdam facilem, & brevem, qua effici possit id, quod Euclides pluribus verbis, atque lineis

The distinction between universal and particular conclusions may not appear terribly significant in the first proposition. The conclusion that *ABC* is an equilateral triangle is of little interest beyond the first proposition, but the certain method for drawing equilateral triangles is often required in later propositions for which an equilateral triangle is part of the construction. Thus, it is essential that triangle *ABC* does represent all equilateral triangles. It is easy to assent to that claim because *ABC* is one of only two possible equilateral triangles on line *AB,* and those only differ from one another by orientation. Furthermore, any other equilateral triangle could be found by scaling the original line up or down. However, in some propositions the universal nature of the claim leaves the exact configuration of the diagram ambiguous in more ways than allowing for multiple possible orientations. When multiple cases arise, the particularity of any one diagram threatens the ability of the proposition to convey a universal claim. Why should a reader agree that the proof of the proposition would be valid for multiple possible configurations, especially if one does not closely resemble another? The problem of multiple cases actually arises quite early, first appearing in the second proposition of the first book. The proposition asks the reader to draw a line from a given point that is of equal length to a given line without specifying the relationship of the point and the line. The point could be separate from the line, either in line with it or off to one side, or it could be part of the line, either an endpoint or in

contendit construere; Idque in iis praesertim observabimus, quae frequentiorem usum habent apud Mathematicos, & in quibus praxis compendium aliquod secum videtur afferre." Note that this justification implies that Clavius expected his reader to be drawing his own diagrams as he worked through the Euclidean demonstrations. Learning mathematics was therefore both active and contemplative.

the middle of the line (Figure 27). It is not immediately clear that any one construction and demonstration could be a valid solution for all four cases. Thus, the authors were faced with the challenge of devising ways to represent the universality of the proposed conclusion across the possible arrangements of the given entities.

In order to make sense of the differences between the diagrams, it will be helpful to understand how the problem is solved for each of the four cases. The first case, in which the point is removed from the given line and off to one side of it, begins by connecting point A to the endpoint B of the given line BC. The reader is then instructed to draw an equilateral triangle on line AB. The two new legs of the triangle, DA and DB are extended to points E and F. The reader then is told to draw a circle of radius BC and center B. That circle is GCH. Then the reader is asked to draw another circle, this time with D as the center and DG as the radius. That circle is GKL. In this process, the required line has been drawn. Line AL is a line from point A of the same length as line BC. The remainder of the demonstration shows the congruence of lines AL and BC by observing that since DG and DL are the same length, and DB and DA are the same length, BG and AL are the same lengths. BG is also the same length as BC, since they are radii of the same circle. Thus, BC and AL are the same length. Two of the other three cases follow nearly identical procedures. For the case of point A being on an extension of line BC, the diagram looks different, but the procedure is the same as the first case. The only possible change is that B be replaced with point C

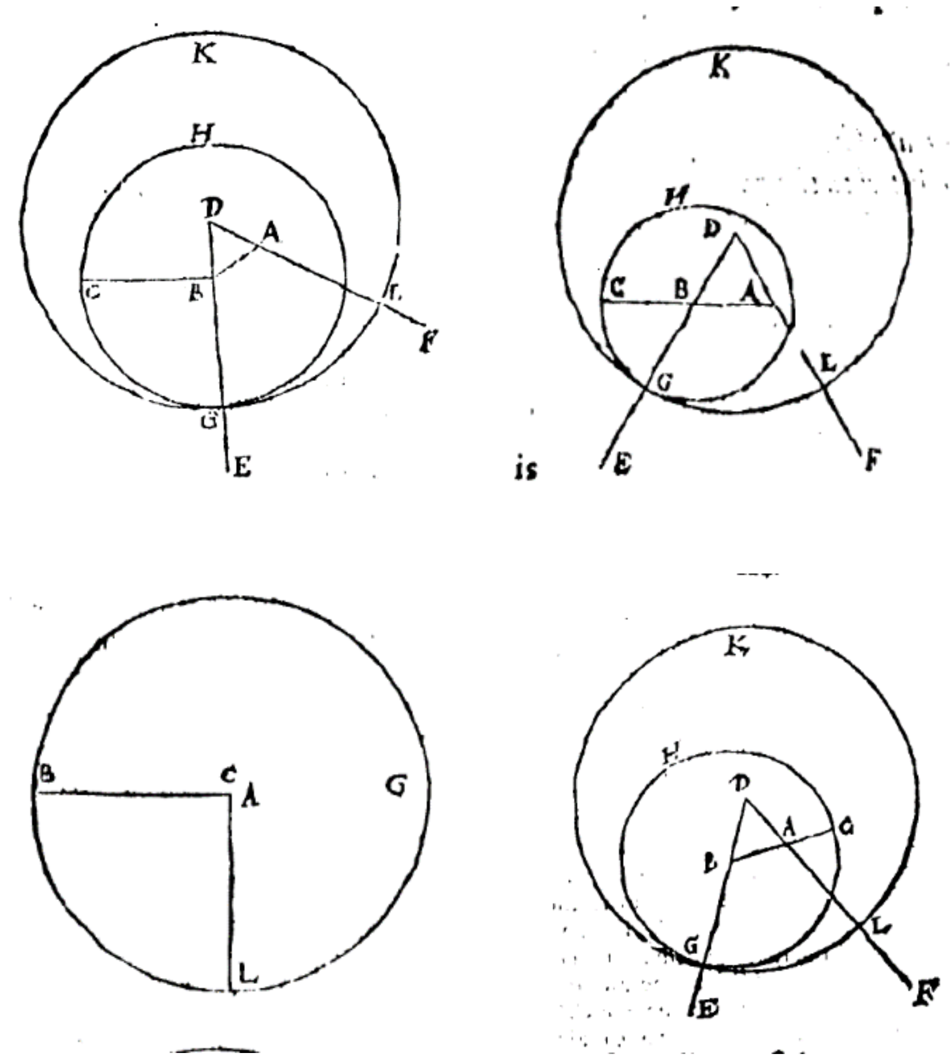


Figure 27: **Billingsley's Diagrams for the Four Cases of Book One, Proposition 2**

In all four diagrams the given point is point *A*, and the given line is line *BC*. The desired line is line *AL*.

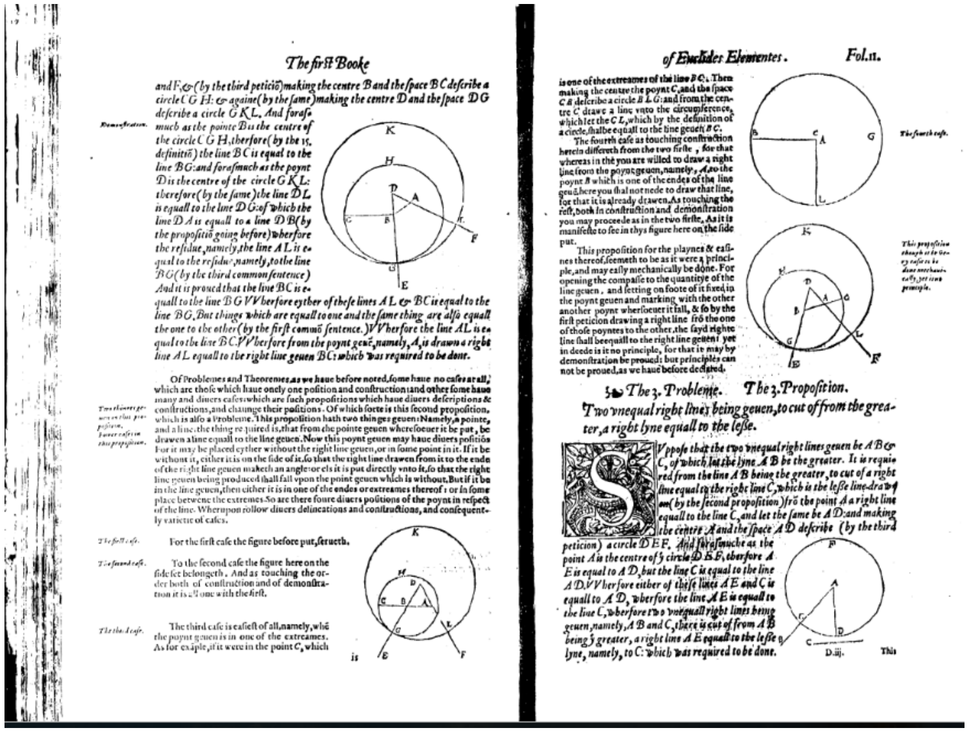
Top left: The case of the point removed from the line and off to one side,
Top right: The case of the point removed from the line and in-line with it
Bottom left: The case of the point as an endpoint of the line
Bottom right: The case of the point in the middle of the line

in the first step if C is closer to point A.[32] In the case in which point A is part of the line, no line needs to be drawn to connect A and B since that connection already exists within BC. Otherwise the procedure is the same as the first case. For the last case, when A is an endpoint of the line, drawing a line of length equal to that of BC becomes much easier. Just draw a circle with point A as the center and a radius of BC. Any radius of that circle fills the requirements of the problem.

When, as in this proposition, there were only a handful of possible cases, the simplest way to use diagrams to demonstrate the universality of a proposition was to include multiple diagrams such that every possible case had a visual representation and some accompanying verbal explanation. That is exactly how Billingsley and Commandino approached the second proposition.[33] Both authors presented the demonstration with the diagram for the case in which the point was off to one side of the given line as the demonstration that was original to Euclid's text. The then both included diagrams for the remaining three cases in their commentary. By presenting a diagram for each case, Billingsley and Commandino separated the particular conclusions of each case from the universal conclusion that a line equal to a given line could be drawn from any point. However, the relationship of the diagrams to the text determined which kind of conclusion received the most emphasis. Figure 28 shows

[32] Commandino is the only author to raise this point, but since his diagram still shows the triangle constructed on AB, the discrepancy is little more than a semantic point, noting that the line AC is all that is needed to connect A and B since BC composes the rest of the line AB.

[33] If there were too many cases to diagram each one, verbal explanations for the universality of the given images would have to suffice. Such explanations could involve an inductive approach in which the truth of one case could be used to argue for the validity of all other cases, but in Euclid's text the justifications for universality usually appealed to the universality of the entities in question and the immutability of the properties of those entities that were relevant to the demonstration.



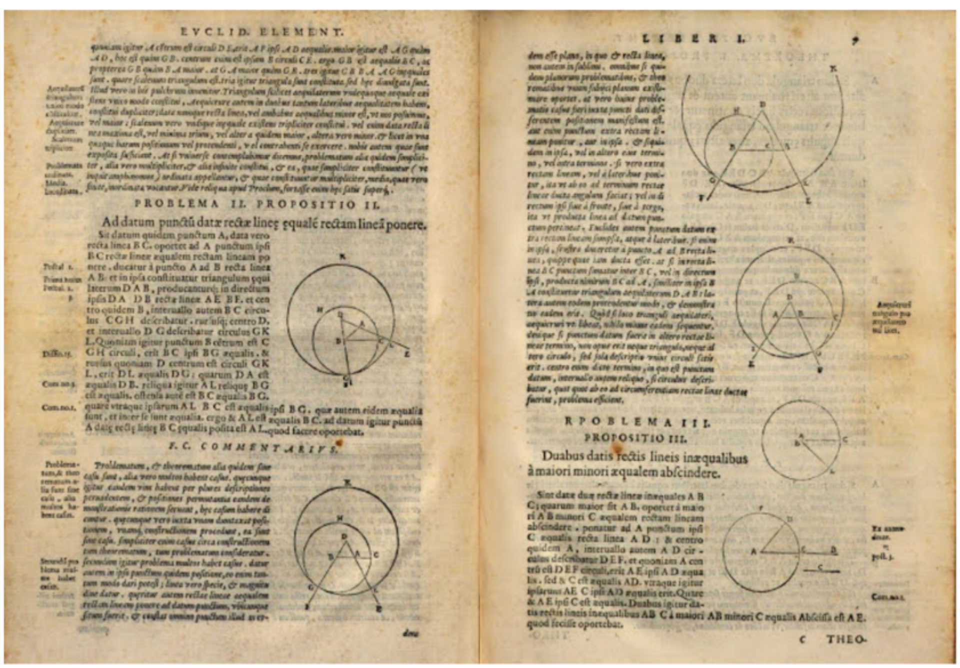


Figure 28: Billingsley's and Commandino's Diagrams for Book One, Proposition 2 – In Context

Billingsley's and Commandino's diagrams in their contexts. On the one hand, Billingsley emphasized the particular conclusions, and thus the physical nature of mathematics by treating the four cases as independent entities. He provided a separate section of commentary alongside each diagram, in effect offering four separate proofs. Of course, two of the proofs in his commentary made such minor changes to the original that his commentary needed only to tell the reader that the construction and demonstration are the same or nearly the same as the ancient proof, but for the case in which the point is an endpoint of the line, Billingsley offered an entirely new proof in which he made no reference to the other cases.[34] On the other hand, Commandino emphasized the universal conclusion by treating each case as a variation of the ancient proof. He provided only a single paragraph of commentary in which he addressed only the changes to the construction outlined in the ancient proof for each remaining case. Even in the case in which the point coincides with an endpoint of the line, Commandino focused on the differences between that case and the original rather than treating it as its own proof. He introduced that case by saying, "If the given point is in either of the endpoints of the line, neither the triangle nor the second circle will be useful, but the description of only one circle will be enough."[35]

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[34] Billingsley, *Elements of Geometrie,* 10*v*-11*r*. "The third case is the easiest of all, namely, when the poynt geven is in one of the extreames. As for example, if it were in the point *C*, which is one of the extreames of the line *BC*. Then making the center the poynt *C*, and the space *CB* describe a circle *BLG*: and from the centre *C* drawe a line unto the circumference, which let the *CL*, which by the definition of a circle, shalbe equall to the line geven, *BC*."

[35] Commandino, *Euclidis Elementorum,* 9*r*. "Denique si punctum datum fuerit in altero rectae lineae termino, non opus erit neque triangulo, neque altero circulo, sed sola drscriptio unius circuli satis erit. Centro enim dicto termino, in quo est punctum datum, intervallo autem reliquo, si circulus describatur, quot quot ab eo ad circumferentiam rectae linae ductae fuerint, problema efficient."

Unlike his contemporaries, Clavius attempted to unite the universality of the propositions' claim with the physical reality of multiple cases by minimizing the number of diagrams he included. Furthermore, he placed his diagrams next to each other so that it was easy for readers to identify their similarities. He even claimed in his commentary that in all four cases the "construction and demonstration is always the same."[36] Reflecting that claim, Clavius only provided two diagrams, each representing one of the two broad cases – the point removed from the line and the point as part of the line – with no separate diagrams given for the further gradations of those cases (Figure 29). He also placed both of his diagrams alongside his original demonstration making it natural for the reader to examine both diagrams as he read the proof so that it was immediately clear that the procedure was nearly identical for the both cases represented by the diagrams. Within the text of his demonstration Clavius noted the slight difference in the procedures: if the point was part of the line, the step connecting the two is unnecessary. However, he made this comment in parentheses, illustrating the insignificance of the difference to the overall procedure and argument.[37] Furthermore, when he discussed the other two cases, including the special case of point A as an endpoint of the line, he relied on the existing diagrams. Thus,

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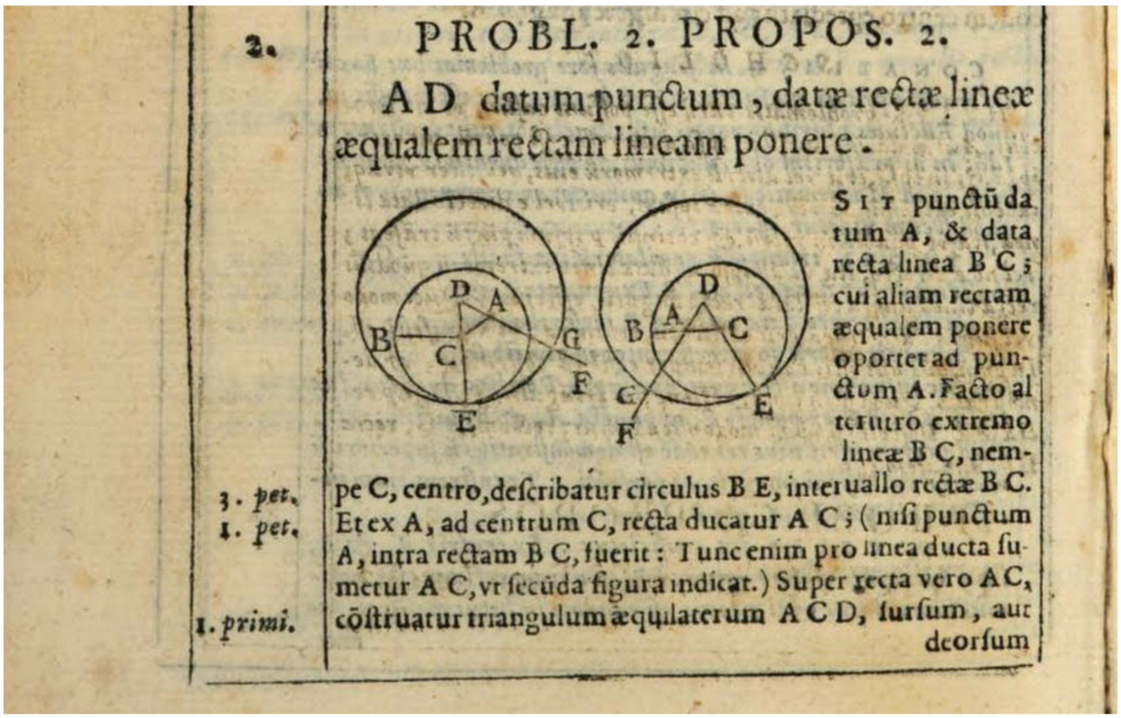
[36] Clavius, *Euclidis Elementorum,* 24*r*. "Huius problematis varii esse possunt casus, ut ait Proclus. Autem datum punctum in ipsa data recta est positum, aut extra ipsam: Si in ipsa, erit vel alterum extremorum eius, vel inter utrumque iacebit extremum. Si vero extra ipsam, erit vel e directo datae lineae, ita ut producta in rectum, & continuum ipsum punctum transeat vel non e directo, ita ut ab ipso ad datae lineae extremorum quodvis recta linea ducta cum data recta angulum efficiat; Quo modo vel supra datam lineam erit constitutum, vel infra, ut manifestum est. In omnibus aut istis casibus semper eadem est constructio, & demonstratio."

[37] Ibid., 23*v*. "Et ex A, ad centrum C, recta ducatur AC; (nisi punctum A, intra rectam BC, fuerit: Tunc enim pro linea ducta sumetur AC, ut secunda figura indicat.) Super recta vero AC…"



2. PROBL. 2. PROPOS. 2.

AD datum punctum, datæ rectæ lineæ æqualem rectam lineam ponere.

Sit punctū datum A, & data recta linea B C; cui aliam rectam æqualem ponere oportet ad punctum A. Facto ab alterutro extremo lineæ B C, nempe C, centro, describatur circulus B E, interuallo rectæ B C.
3. pet. Et ex A, ad centrum C, recta ducatur A C; (nisi punctum
1. pet. A, intra rectam B C, fuerit: Tunc enim pro linea ducta sumetur A C, vt secūda figura indicat.) Super recta vero A C,
1. primi. cōstruatur triangulum æquilaterum A C D, sursum, aut deorsum

Figure 29: **Clavius's Diagrams for Book One, Proposition 2**

The diagram on the left shows point *A* removed from the line *BC*. The diagram on the right shows *A* as part of line *BC*. Clavius used these diagrams to describe all four possible cases.

even though his proof for the special case did not refer to the proofs for the other cases, the diagrams subsumed it into those cases.

In this section, we have seen how the authors balanced the particular and universal conclusions of each proposition in the relationships they established between their diagrams and commentaries. Billingsley's visual emphasis on particular conclusions made manifest his understanding of mathematics as a physical discipline. Commandino's focus on universal conclusions illustrated his belief that mathematics was the study of universal truths. As he had done in the diagrams for the definitions, Clavius struck a middle road between his contemporaries and attempted to unite the particular and universal conclusions. In so doing, he situated mathematical demonstrations as a bridge between the concrete objects of physics and the abstract concepts of metaphysics. In the next section, we will turn to the solid geometry books to examine how the authors responded to the challenge of representing three-dimensional objects in a two-dimensional medium.

Solid Geometry: Depicting Three-Dimensional Space on Two-Dimensional Paper

While multiple cases made it possible for a diagram in plane geometry to represent a situation that did not exactly line up with the case depicted, two-dimensional diagrams, assuming that they were accurately drawn, were always instances of a general case described. For example, a diagram of an equilateral triangle is in fact an equilateral triangle. A diagram showing how to create a line of equal length to a given line from a fixed point, actually contains the desired line. But geometry is not always two-dimensional. When the authors wrote their commentaries on the last several books of *The Elements*, which treat solid geometry, they had to contend with the difficulty of representing three-dimensional objects in two-dimensions. By the sixteenth century, perspective was the most obvious solution to this challenge. In figures for which interior lines did not need to be visible, authors could also employ shading to indicate a third dimension. However, even a well-drawn diagram, has its limitations. Because perspective relies on foreshortening, the lines and angles drawn in perspectival diagrams will not all have the actual measurements of the solid objects they portray. A skeptical student could not measure the diagrams to prove equality, as is possible with one and two-dimensional figures.[38] Faced with these challenges, the authors were forced to decide just how much they valued the ability of mathematics to describe concrete physical objects.

In this section, we will see that Billingsley emphasized the importance of mathematics' ability to study physical bodies, and, consequently, he devoted his efforts to providing the most accurate possible representation of the entities discussed. Commandino's efforts focused instead on the universal truths of mathematics could discover based on abstract forms, and he reduced the physical bodies and their images to tools in the search for such truths. As he did throughout *The Elements,* Clavius combined the particular physical bodies and the universal forms by entwining his

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[38] Of course even for one and two-dimensional figures the student who sought to rely on measuring the diagram needed to count on the accuracy of the printed image. If the compositor let an inaccuracy slip through, measurements of the image would no longer work.

physical representations of mathematical bodies and the universal ideas in his commentary such that each informed the other.

As in the plane geometry books, the definitions for the solid geometry books afforded the authors more freedom to create their own visualizations than the propositions did. Of the three original solid geometry books, only the eleventh book has definitions. I will examine the definitions from that book in order to show how each commentator used visualizations of the definitions to establish the place of mathematics within the hierarchy of disciplines. As was the case in the plane geometry books, the authors' visions of mathematics can be seen in their patterns of diagram usage in the definitions. Of these three authors, Commandino used diagrams most sparingly in the eleventh book. He only included images to clarify and develop any supplementary concepts that were introduced in the definition. All seven of his diagrams are for definitions that rely on the use of lines or figures beyond those being defined. In contrast, Billingsley used diagrams whenever possible to render the definitions as physical entities using three-dimensional models. He only excluded diagrams for definitions with clear analogs in plane geometry. As a result, more than two-thirds (twenty-four out of thirty) of his concepts are diagrammed, and nearly half of them (fourteen, to be precise) have diagrams that offer three-dimensional renderings of the concepts either in the form of pop-up diagrams or in templates for three-dimensional models. For example, the definition of a line perpendicular to a plane, which will be discussed shortly, relies on lines contained in the plane. Clavius's use of diagrams is more varied, reflecting that he saw diagrams as a versatile tool that could be useful both in the study of physical mathematical objects and in the study of the universal ideas found in mathematics. Some of his sixteen images are designed to be simple physical expressions of the defined concepts. Others extend the definitions with supplementary ideas. Table 6 provides a summary of the diagrams included by each author for each definition.

In both Billingsley's and Clavius's texts, an examination of the definitions without diagrams reveals that they are all either definitions with clear analogs in plane geometry or are so closely related to preceding definitions that they needed little explication, visual or otherwise. In fact, they are consistently the definitions for which Clavius included the least commentary. In Billingsley's text all of the definitions without diagrams have clear analogs in plane geometry. They are those for a solid, the edges of solids, parallel planes, similar solid figures, equal solid figures, and the diameter of a sphere. A solid is analogous to a plane, and just as a plane's borders are lines, a solid's borders are planes. Parallel planes are analogous to parallel lines. Similar solid figures and equal solid figures are analogous to similar planar figures and equal planar figures in that the requirements for similarity and equality are the same in plane and solid geometry: for two figures to be similar to one another, they must have all of their angles equal to one another, and equality demands that all angles and edges be equal to one another. The diameter of a sphere is analogous to the diameter of a circle. Clavius excluded a few additional diagrams, but each of those definitions could be easily identified elsewhere. For example, he did not include diagrams for a sphere,

Table 6: The Definitions and Diagrams of Book Eleven

	Billingsley	Clavius	Commandino
1. Solid	No diagram*	No diagram	No diagram
2. The limits of a solid	No diagram*	No diagram	No diagram
3. Line perpendicular to a plane	Pop-up	2 diagrams w/o perspective	Diagram w/ perspective
4. Plane perpendicular to a plane	Pop-up*	Diagram w/o perspective	Diagram w/ perspective
5. Inclination of a straight line to a plane	Pop-up*	Diagram w/o perspective	Diagram w/ perspective
6. Inclination of a plane to a plane	Pop-up	Diagram w/ perspective	Diagram w/ perspective
7. Similarly inclined planes	Pop-up	No diagram	Diagram w/ perspective
8. Parallel planes	No diagram	Diagram w/ perspective	No diagram
9. Similar solid figures	No diagram	No diagram	No diagram
10. Equal solid figures	No diagram	No diagram	No diagram
11. Solid angle	Pop-up	Diagram w/o perspective	No diagram

Continued on next page

*These pairs of definitions were each combined into single definitions in Billingsley's text.
**N/A indicates definitions that were not present in that author's text.

Table 6: The Definitions and Diagrams of Book Eleven (continued)

	Billingsley	Clavius	Commandino
12. Pyramid	Diagram w/ perspective 2 pop-ups, 2 Templates	3 diagrams w/ perspective	No diagram
13. Prism	2 diagrams w/ perspective and 1 pop-up Template	3 diagrams w/ perspective	No diagram
14. Sphere	2 diagrams w/ perspective, 1 diagram of semicircle	No diagram	No diagram
15. Axis of a sphere	Refers to diagrams for def. 14	No diagram	No diagram
16. Center of a sphere	Refers to diagrams for def. 14	No diagram	No diagram
17. Diameter of a sphere	No diagram	2 Diagrams	No diagram
18. Cone	3 diagrams w/ perspective	3 diagrams w/ perspective	No diagram
19. Axis of a cone	Diagram w/ perspective*	Refers to diagram for def. 18	No diagram
20. Base of a cone	Refers to diagram for def. 19*	Refers to diagram for def. 18 Diagram for right vs. scalene cones w/ perspective	Diagram w/perspective of orthogonal cone Diagram for right vs. scalene cones w/ perspective

Continued on next page

*These pairs of definitions were each combined into single definitions in Billingsley's text.
**N/A indicates definitions that were not present in that author's text.

Table 6: The Definitions and Diagrams of Book Eleven (continued)

	Billingsley	Clavius	Commandino
21. Cylinder	2 diagrams w/ perspective	Diagram w/ perspective	No diagram
22. Axis of a cylinder	Refers to diagrams for def. 21*	Refers to diagram for def. 21	No diagram
23. Base of a cylinder	Refers to diagrams for def. 21*	Refers to diagram for def. 21 Diagram for right vs. scalene cylinders w/ perspective	Diagram w/ perspective
24. Similar cones and cylinders	Diagram	Diagram	No diagram
25. Cube	2 diagrams w/ perspective Template	No diagram	No diagram
26. Tetrahedron	2 diagrams w/ perspective Template	No diagram	No diagram
27. Octahedron	2 diagrams w/ perspective and w/o perspective Template	No diagram	No diagram
28. Dodecahedron	2 diagrams w/ perspective and w/o perspective Template	No diagram	No diagram
29. Icosahedron	2 diagrams w/ perspective and w/o perspective Template	No diagram	2 Diagrams covering defs. 24-29
30. Parallelepiped	3 diagrams w/ perspective Template	4 diagrams w/ perspective	No diagram
31. Inscription	N/A**	No diagram	N/A**
32. Circumscription	N/A**	No diagram	N/A**

*These pairs of definitions were each combined into single definitions in Billingsley's text.
**N/A indicates definitions that were not present in that author's text.

the axis of a sphere, or the center of the sphere, but all three of those concepts could be identified in the diagram for the diameter of a sphere.[39]

The only diagram for which Billingsley excluded a diagram and Clavius provided one is the definition of parallel planes. In that case the analogy Billingsley drew does not hold in Clavius's text because of a difference in the wording of the definitions. Clavius's definition only says "Parallel planes are those which never come together."[40] Billingsley's definition notes that parallel planes never intersect even when they are "produced or extended any way."[41] Because the definition of parallel lines notes that the lines must not intersect when they are extended infinitely in either direction, without that phrase, the analogy breaks down. What Billingsley inserted into his definition to ensure an analogy to a previously diagrammed concept, Clavius sought to reveal through the use of a diagram. Indeed, Clavius's diagram is clearly intended to reveal the information left implicit in his definition rather than to show parallel planes because his commentary actually describes the intersecting planes in the diagram rather than the planes that appear to be parallel. The diagram offers a negative example to explain that parallel planes must be able to be infinitely extended in all directions without touching.

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[39] Clavius also excluded diagrams the five regular solids, but he added templates for each of those in his commentary on relevant propositions. Nor did he provide diagrams for his definitions for inscription and circumscription, which neither Billingsley nor Commandino included. Those definitions apply equally as well to planar geometry as they do to solid geometry.

[40] Christopher Clavius, *Euclidis Posteriores libri sex ad XV. Accessit XVI de solidorum rgularium comparatione*, (Rome: Vincentium Accoltum, 1574), 120*r*. "Parallela plana sunt, quae inter se non conveniunt."

[41] Billingsley, *Elements of Geometrie,* 313*v*.

As can be seen in the table, there are fewer differences between the authors on which definitions received diagrams in the first eleven definitions than in the remaining definitions. This is likely because the early definitions are the most general introduction to solid geometry, describing relationships of one- and two-dimensional figures that allow the formation of three-dimensional figures. As such, they present both individual physical entities and universal claims about the nature of three-dimensional space. However, the diagrams themselves offer quite different visual presentations of each authors' version of mathematics. For these definitions, in order to give the readers three-dimensional models, Billingsley used pop-up diagrams exclusively. His commentary usually only describes the diagram and how to manipulate the pop-up feature to create a specific instance of the definition in question. Commandino's diagrams in this section rely on perspective to convey three-dimensions as they draw attention to the various lines and angles each definition uses to identify its particular relationship between a line and a plane or two planes. In contrast, Clavius provided two-dimensional figures, often without perspective, to show how these definitions could be developed from the concepts of planar geometry.[42]

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[42] While Clavius's lack of perspective can make it difficult to recognize their three-dimensional forms, there does appear to have been a method to his madness. He only used perspective when he had two or more intersecting planes in a diagram, possibly suggesting that he believed that only solid bodies created by intersecting planes required perspective to be recognizable in a two-dimensional drawing. For other diagrams, he relied on his commentary to instruct the reader in how to interpret the image. Since a line could fairly easily be imagined to be at some angle to the plane of the page, such explanations were simple and allowed for the possibility of offering more than one way of seeing the diagram. For example, as will be discussed shortly, in his second diagram for a plane, he asked the reader to imagine rotating line AB, effectively asking the reader to see multiple configurations for the diagram. By forcing his reader to rely on the text to interpret the image, Clavius prevented possible

To explain the significance of the differences between the images, I will examine the definition of a line perpendicular to a plane, the first definition to receive a diagram in all three texts. The definition states that a line is perpendicular to a plane when it is perpendicular to all of the lines it touches in the plane. Figure 30 shows all three authors' diagrams. Each diagram shows a plane with several lines contained in it and another line that intersects all of those lines, but beyond that general description the diagrams diverge.

The most obvious difference is that Billingsley created a three-dimensional model of the definition through the use of a pop-up diagram while Commandino and Clavius presented two-dimensional diagrams. For Billingsley, the diagram was a model of the concept that the readers could use to familiarize themselves with the rules of three-dimensional space without the added difficulty of imagining three-dimensions on two-dimensional paper. To do so his diagram provided a pasted in piece of paper on which the required line was drawn. The reader could manually lift a flap to raise it above the page until the line was perpendicular to the drawn plane. As he explained, he was concerned that "these five books following are somewhat hard for young beginners, by reason they must in the figures described in a plaine imagine lines and superfieces to be elevated and erected, the one to the other, and also conceave solides or bodies, which, for that they have not hitherto bene acquainted with, will at the first light be somewhat straunge unto them."[43] This explanation

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misinterpretations of a perspectival diagram and eliminated the temptation of treating the diagram as an accurate depiction of the defined concept.

[43] Billingsley, *Elements of Geometrie,* 322*v*.

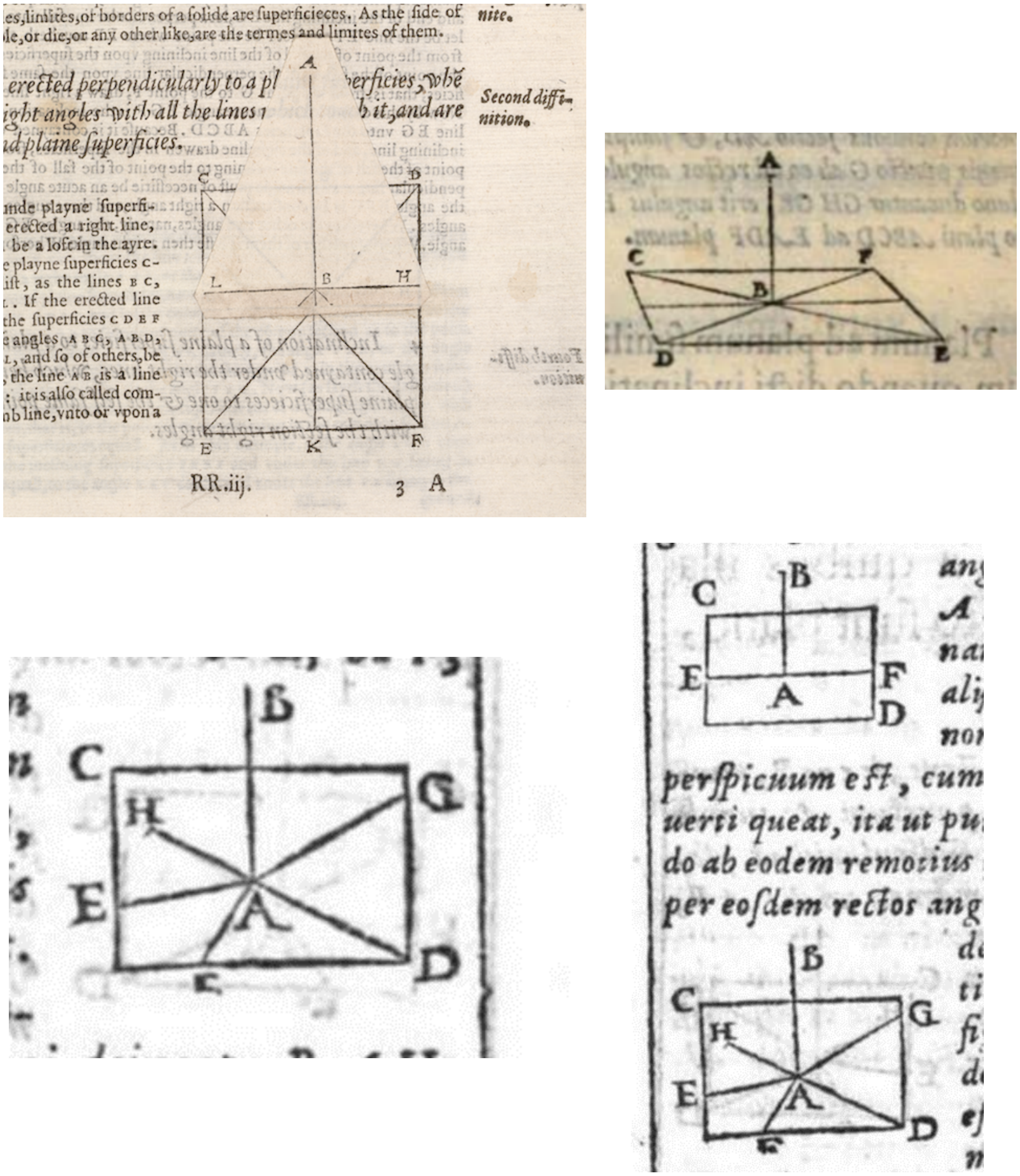

Figure 30: Diagrams for the Definition of a Line Perpendicular to a Plane

Top left: Billingsley's pop-up diagram (Source: Huntington Library)

Top right: Commandino's diagram

Bottom left: Clavius's first diagram

Bottom right: Clavius's second diagram

applies equally as well to the diagrams in the definitions as it does to those in the propositions which it describes. In this particular definition, he further appealed to his ideal merchant or artisan reader by noting that the line *AB* was commonly called a plumb line, relying on his potential reader's past experience with plumb lines to recognize what the pop-up should look like when *AB* was perpendicular to the plane.[44]

Commandino's diagram focused on the role of the lines in the plane, uncovering the abstract principles that remain implicit in the definition and cannot be represented physically. The principle that remains unexpressed in this definition is that any lines used to check the perpendicularity of a given line are simply representatives of an infinite set of lines that compose the plane. His image uses perspective to show a plane, *CDEF*, and intersecting line, *AB,* and three lines in the plane drawn from one edge to the other through point B. Commandino emphasized the arbitrariness of these lines by leaving one line unlabeled. The other two lines are only labeled because their endpoints coincide with the labeled corners of the plane. As the commentary in the scholium explains, the lines in the plane represent the process of breaking the plane up into an infinite number of non-parallel straight lines to which the intersecting line must be perpendicular.[45] Thus, the definition of a line

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[44] Ibid., 312v. 'It is also called commonly a perpendicular line or a plumb line, unto or upon a superficies." The *Oxford English Dictionary* places the origins of "plumb line" in the mid-fifteenth century, and notes a 1538 dictionary in which a plumb line was defined as a carpenter's tool. Thus, it seems probable that Billingsley's hoped-for artisan readers, who would have been familiar with trade tools, would have recognized the plumb line as a way to understand perpendicularity between a line and a plane. *Oxford English Dictionary,* 3rd ed. (online version, updated 2006), s.v "plumb line." http://www.oed.com/view/Entry/146079?rskey=kEPw02&result=1#eid.

[45] Commandino, *Euclidis Elementorum,* 189*r*. "Si posset planum in rectas lineas resolui, ita dixisset. Quando ad omnes rectas lineas, ex quibus planum constat, rectos facit angulos, tunc & ad ipsum recta erit. Sed quoniam planum etiam infinite rectis lineis sectum in ipsas non resoluitur, contentus fuit linearum infinitate pro toto plano. Contingentes autem addit, ut non parallelae sint."

perpendicular to a plane rests on the assumption that a plane is composed of an infinite number of lines, something which cannot be completely represented in any physical model.

Clavius used two diagrams (found in Figure 30) to make both the physical nature of the definition and its abstract grounding clear. The first diagram was dedicated to illustrating the physical features of the definition, but Clavius made no attempt to present an accurate representation. He did not even use perspective, relying on his commentary to make the three-dimensional shape of the diagram clear. The commentary provided a concrete interpretation of the diagram by recasting the definition into an observation about the physical relationship of the line and the plane. As Clavius described it, the image was intended to show that "*AB* stands upon plane *CD* equally, and does not incline more to one part than the other,"[46] which is just a physical description of what perpendicularity means. In order to make it easier for the reader to visualize this scenario, Clavius allowed only one line in the plane to pass through the point of intersection with the perpendicular line, focusing the reader on the relationship between those two lines. Indeed, in his commentary, Clavius used the continued line as an example of what it means for *AB* to be perpendicular to one particular line, *DH,* which is that angles *BAH* and *BAD* must be equal to one another,

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[46] Clavius, *Euclidis Posteriores,* 118*r*. "Hac enim ratione fiet, ut AB, aequaliter insistat plano CD, & non magis in unam partem, quam in aliam inclinet." As noted above, the definition for a line perpendicular to a plane tells the reader that a line is perpendicular to a plane when it is perpendicular to all of the lines it intersects in that plane. No mention is made of inclination towards one side or another of the plane.

meaning that line *AB* does not lean more towards *D* or *H*.[47] Since the same must be true of any line in the plane that intersects line *AB,* line *AB* cannot incline towards one part of the plane more than it does towards any other.

Clavius's second diagram (produced again in Figure 31) returned to the more abstract formulation of Euclid's definition to show that the impossibility of diagramming an infinite number of lines was not an issue if one wanted to check the perpendicularity of a line to a plane. The diagram does this by showing that for a line to be perpendicular to a plane it is necessary and sufficient for it to be perpendicular to two lines in that plane. His heuristic proof unites the concrete and the abstract by using an imaginary manipulation of diagrams to provide physical reasoning for the claim that two lines can represent the infinite set of the plane. It is based on an image of a plane, CD, a line in that plane, EF, and a second line, AB, that is perpendicular to the line in the plane (EF). This diagram does not use perspective, and it is not clear what direction *AB* should be imagined to point. The ambiguity of *AB*'s orientation was intentional because it allowed Clavius to instruct the reader to imagine rotating *AB* around *EF,* keeping point *B* at the same distance from *EF*. Through this rotation, *AB* would remain perpendicular to *EF*, but its relationship to the plane CD would change. Thus, through the imaginary physical manipulation of a diagram Clavius led the reader to consent that a line can be perpendicular to one line in the plane without

[47] In Book One, a right angle is defined as one of the two equal angles formed by two intersecting lines when the one line does not incline more to one side or the other of the second line. Clavius, *Euclidis Elementorum,* 5*v*. "Cum vero recta linea super rectam consistens lineam eos, qui sunt deinceps, angulos aequales inter se fecerit, rectus est uterque aequalium angulorum: Et quae insistit recta linea, perpendicularis vocatur, eius cui insistit."

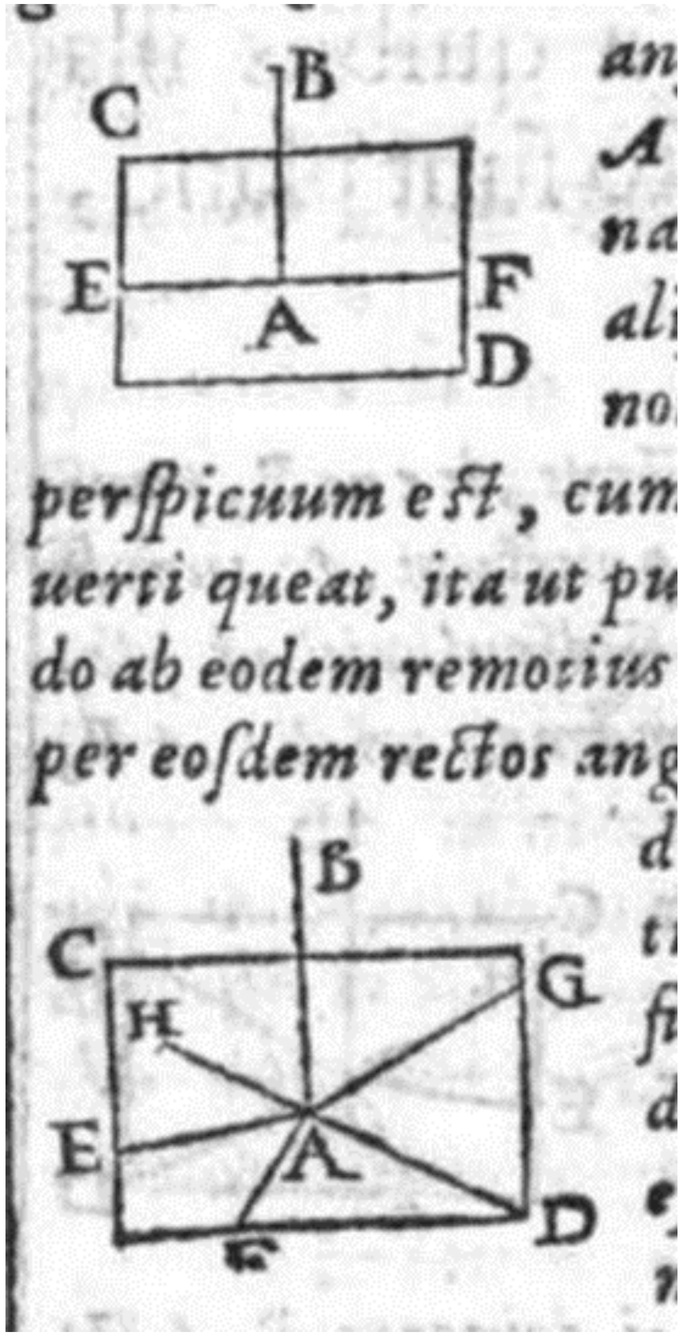

Figure 31: **Clavius's Second Diagram for the Definition of a Line Perpendicular to a Plane**

being perpendicular to the plane itself. Clavius then compared this diagram with his original diagram for the definition (which he had printed a second time right below the diagram just described) to create a challenge of a similar imaginary rotation of *AB* around either of two lines in the plane in an attempt to keep it perpendicular to both but not to the plane. Using the original diagram, Clavius chose the lines *AD* and *AG* from the plane and informed the reader that if *AB* formed a right angle with both of those lines, then it would make right angles with all of the others that it intersected.[48] In case the reader doubted that claim, Clavius also noted that the impossibility of a line being perpendicular to two lines in the plane and not perpendicular to the plane is formally proven in the fourth proposition of the book.

After the first eleven definitions, Euclid's text turns to identifying the various kinds of solid bodies and some of their features. Eleven kinds of solid figures are defined: pyramids, prisms, spheres, cones, cylinders, cubes, tetrahedrons, octahedrons, dodecahedrons, icosahedrons, and parallelepipeds. A few more definitions identify features of some of the shapes, such as the diameter of a sphere or the base of a cone. Because these are the definitions that deal with physical bodies and their properties, they are the place in which commentators were most completely confronted with the physicality of mathematics. In each of the three commentaries studied here, the treatment of the visualization of solid bodies more clearly reveals the author's vision

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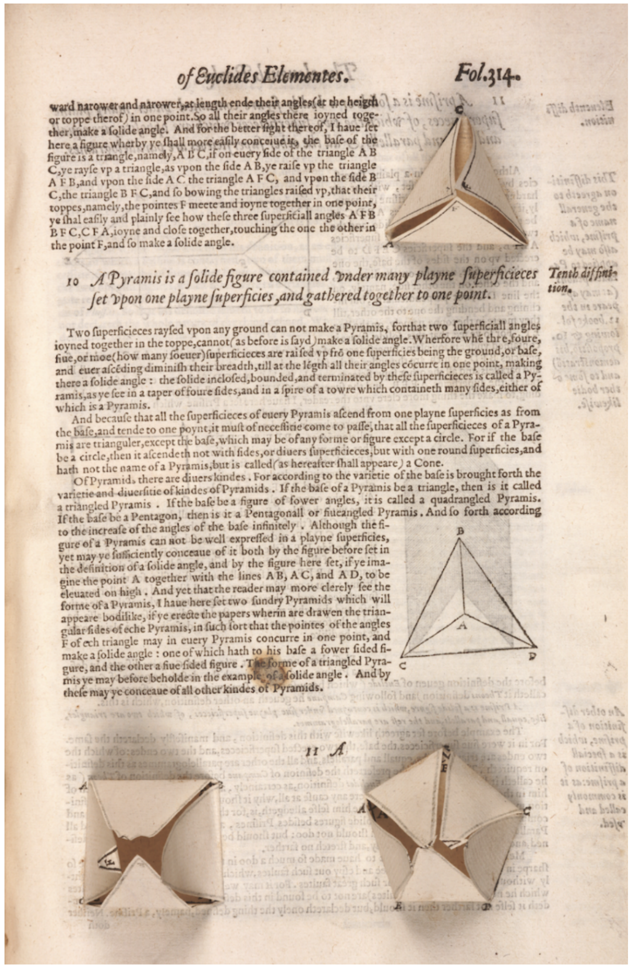
[48] Clavius, *Euclidis Posteriores,* 118*v*. "Quod si eadem *AB*, cum duabus angulum componentibus, quales sunt *AG*, *AD*, in priori figura, rectos constituat angulos, tunc demum cum omnibus aliis rectos angulos efficiet, ut diximus."

of mathematics and the role he ascribed to physical bodies in its study here than any other part of *The Elements*.

For Billingsley, mathematics was a study of concrete objects in which reason enabled the discovery of relationships between solid bodies. Thus, the definitions of the solid bodies marked the culmination of the Euclidean text. The tremendous value he attached to these definitions is manifest in the number of diagrams he included. All but one definition, that for the diameter of a sphere, had a diagram. Each shape received illustration by at least one, often two or three, diagrams using perspective or shading to indicate dimensionality. Two shapes, pyramids and prisms even received pop-up diagrams (Figure 32). Concerned that the shapes defined were "not by these figures [the two-dimensional drawings accompanying each definition] here set, so fully and lively expressed, that the studious beholder can throughly [sic] according to their definitions concyve them," Billingsley also included a set of templates which could be copied by the reader onto paper and folded into the various shapes so that his reader might "most plainly and manifestly see the formes and shapes of these bodies, even as their definition is shewn."[49] Eight of the eleven varieties of solids defined are represented in his templates.[50] See Figure 33 for some examples. For three shapes, a sphere, a cone, and a cylinder, it was not possible to provide a pop-up or a template for

[49] Billingsley, *Elements of Geometrie,* 340. (N.B. The folio number on what should be 320 is printed as 340. The next folio shows 341; the following is numbered 327. After that, the number is 323, which is what it should be. From there the folios count up as one expects.)
[50] Billingsley's templates are for a tetrahedron, a cube, an octahedron, a dodecahedron, an icosahedron, three varieties of pyramids, a prism, and a parallelepiped. The quotation I used above in which Billingsley laments the inaccuracies of the two-dimensional diagrams only directly references the last five definitions which were for the five regular solids. Still, since he included templates for pyramids, prisms, and parallelepipeds, it is reasonable to assume his lament about the shortcomings of two-dimensional representations would still apply.



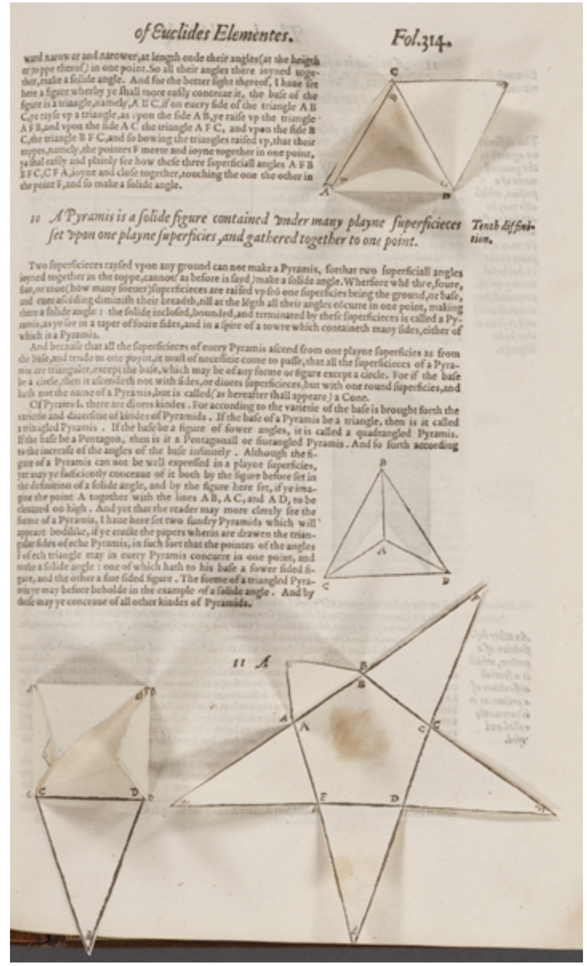
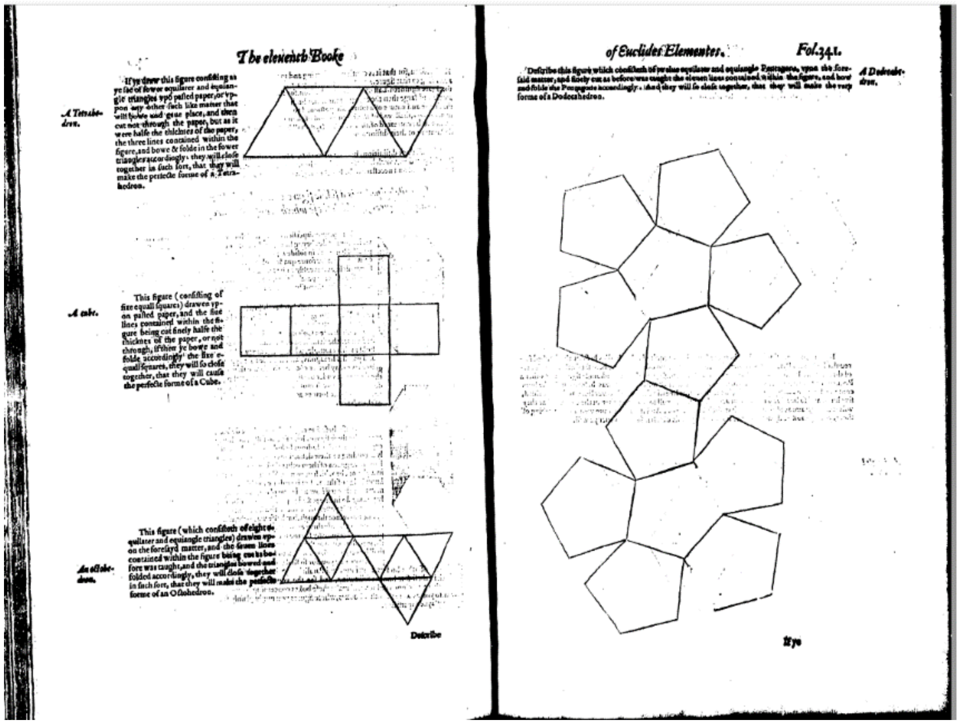


Figure 32: Billingsley's Pop-up Diagrams for the Definition of a Pyramid

Sources: Left: Huntington Library; Right: http://www.maa.org/press/periodicals/convergence/mathematical-treasures-billingsley-euclid, Physical copy held by Columbia University in the Plimpton Collection



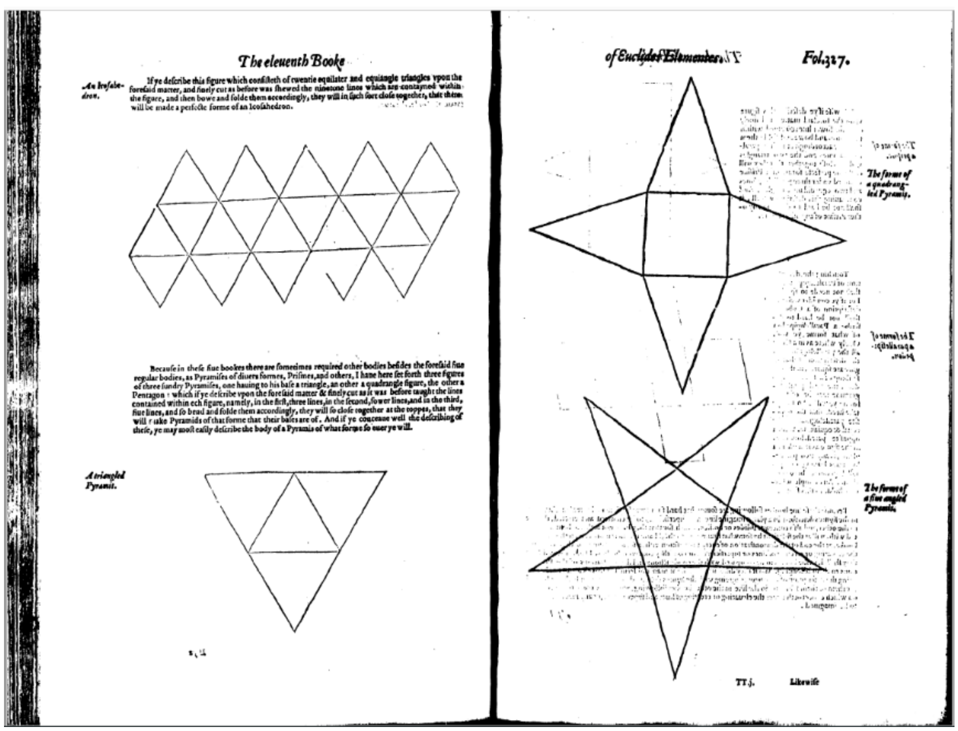


Figure 33: **Eight of Billingsley's Templates**

These cover the five regular solids and three kinds of pyramids.

the figure.[51] Still, for the first of those definitions, the sphere, Billingsley offered his reader a two-dimensional diagram to explain how the shapes could be generated. That diagram appears in his commentary on a sphere, and shows a semicircle mounted between two posts such that it could rotate about its diameter (Figure 34). It effectively concretizes the imagined rotation of a semicircle required by the definition.

For Commandino, mathematics was an abstract study in which logical reasoning about the forms of physical entities could aid the discovery of universal truths about the structure of the world. Thus, the definitions of the solid bodies were examples of how the earlier definitions' study of relationships between objects in three-dimensions could be put to use, but were not in and of themselves defining features of mathematical constructs. As such, they had little need for further development, and, consequently, Commandino left most of the solid body definitions without any diagram. He included diagrams only for cones and cylinders, and those were designed to illustrate concepts about the structure of the shapes, rather than simply illustrating the forms themselves. Commandino's diagram for a cylinder is the most focused on the physical. In it he provided a straightforward rendering of the definitions of a cylinder, its base, and its axis, which together assert that a cylinder is generated by a rotation of a rectangle around one of its sides. The image shows a rectangle at two points of its rotation as well as the circular paths of the two corners

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[51] Because they are defined by the rotation of a planar figure, providing the means to construct a three-dimensional model of any of these shapes with two-dimensional planes would have been exceedingly difficult. It is impossible to construct these shapes by folding planar figures as Billingsley's pop-ups and templates require. The reader may think of round paper decorations that have been cut along their axis so that they can fold flat. The paper between the two flat ends is made into a honeycomb weave to maintain its shape as either end is "rotated" around the axis.

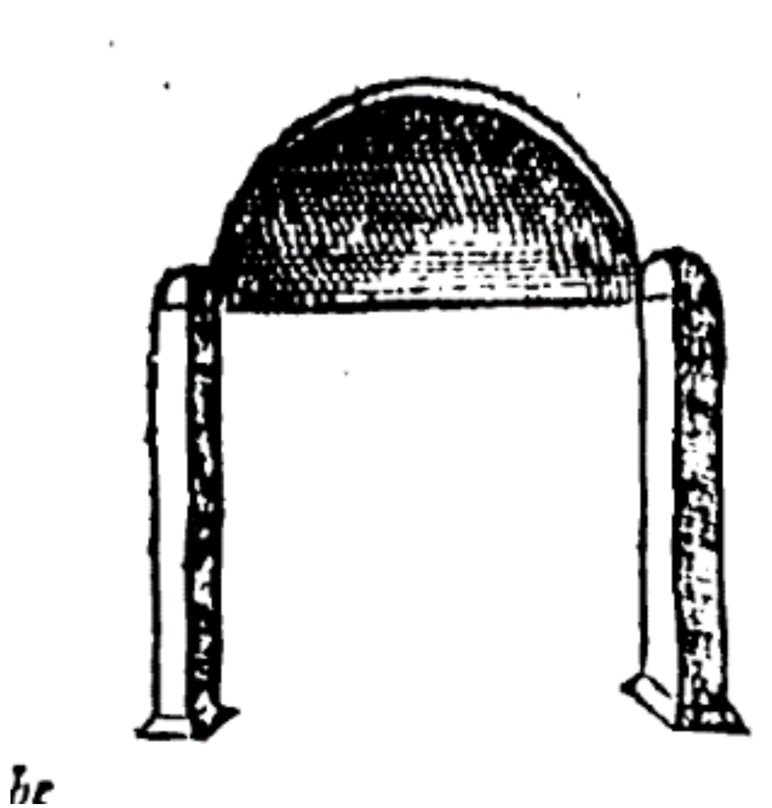

Figure 34: **Billingsley's Image for the Generation of a Sphere**

In his commentary Billingsley observes that similar images can be found in Sacrobosco's *Sphere*.

that define the cylinder.[52] However, unlike Billingsley's diagram for the generation of a sphere, there are no extraneous supports for the axis, leaving the rotation itself somewhat abstract. Since the cylinder is the last of the shapes generated by rotation to be defined, this diagram can also be seen as representative of the definitions for a cone and a sphere. The reader will recall that, in the first book of *The Elements,* Commandino established a practice of grouping illustrations for similar figures into diagrams accompanying the definition for the last figure. Here, he only included one illustration, but this diagram clearly shows the rotation process central to the definition for all three figures. A simple change in the planar shape being rotated would create a different figure.[53]

The diagrams Commandino included for cones serve as part of the explanations for the abstract concepts of orthogonal and scalene as they apply to cones (Figure 35). The first diagram is part of a demonstration that the angle at the apex, rather than the angle between the base and the axis (the line from the center of the circle to the vertex), is right angle for which a cone generated by the rotation of an

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[52] Commandino, *Euclidis Elementorum,* 192*r.* "Sit parallelogrammum rectangulum *ABCD,* & latere *AB* manente intelligatur latus *CD* convertum, quousque ad eum locum redeat, a quo capit moveri. Erit ita descripta fiugra, cuius axis est *AB* recta linea manens, & basis circuli ipsi a punctis *CD* circa contra *BA* descripti." N.B. The page number was misprinted as 129.

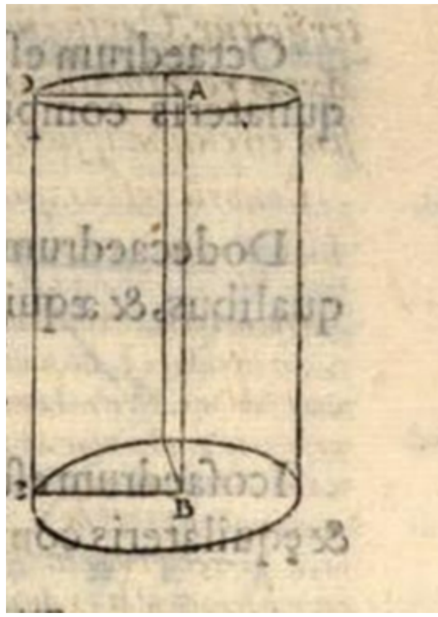
[53] The definitions for the axis of each shape are all completely analogous, as well. The axis is the line held fixed in the rotation – in the sphere it is the diameter, and in the cone it is one leg of a right triangle. The base of the cylinder and the base of the cone are analogous, too. Thus, the diagram for the cylinder can represent eight definitions (a sphere, axis of a sphere, a cone, axis of a cone, a base of a cone, a cylinder, axis of a cylinder, and base of a cylinder). While all of these analogies hold, I did not count Commandino's diagram of a cylinder for more than one definition in the table because he never explicitly refers his reader to it in other definitions. Also, while Billingsley's diagram for the rotation of a semi-circle to form a sphere could also be said to represent the rotations of the other three shapes, that representation is less clear because the sphere is the first of the shapes generated by rotation to be defined. The reader would have to refer back to the image as he worked through the later definitions. Billingsley provided no such references.

Figure 35: Commandino's Diagrams for the Definitions of a Cone and a Cylinder

Commandino's diagram for a cylinder (top) includes a line drawn at the back of the solid body that makes the rotation of the rectangle *ABCD* clear.

Commandino's image for the definition (bottom left) of a cone shows (from top to bottom) an acute, orthogonal, and obtuse cone. All three of these cones are right cones because the axis and the base are perpendicular to one another (i.e. the cones are generated by the rotation of a right triangle). However, only the middle cone, that generated by the rotation of an isosceles right triangle, has a right angle at the apex.

Commandino's image for his discussion of scalene cones (bottom right) shows two cones, one right (top) and one scalene (bottom). The image and discussion are both drawn from Apollonius's work. In the case of a scalene cone, the rotation that generates the cone is that of the line AD, not a triangle. Triangles ADC and BCD are distinct scalene triangles.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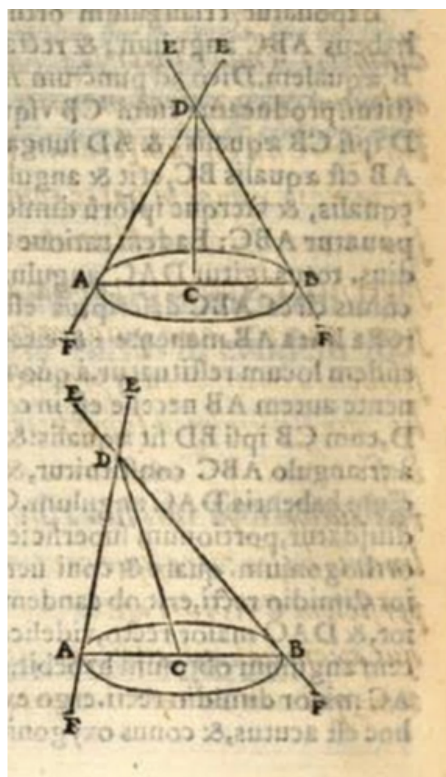
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isosceles right triangle about one of its legs is classified orthogonal.[54] This distinction illustrates that Euclid's definition leaves room for multiple kinds of structures. All Euclidean cones are right cones generated by the rotation of a right triangle about one of its legs, and, therefore, all cones have a right angle between the base and the axis, but only those generated by an isosceles right triangle will also have a right angle at the apex.[55] The second diagram accompanies an addition that expands the definition of cones to include scalene cones by generalizing the procedure of creating a cone. Instead of rotating a right triangle, scalene cones are created by rotating a line that connects a circle to a point outside of its plane. These cones are called scalene because the triangles made by the lines that define the surface of the cone, the axis of the cone, and a radius of the circle is a scalene triangle.

For Clavius, the definitions of solid bodies were neither the climax nor the dénouement of Euclid's work. Because Clavius treated mathematics as both a study of physical bodies and a path to universal truths, these definitions were much like the definitions in the first book in that they provided the physical entities necessary to either study, both as individual bodies and as universal categories. In one sense, Clavius's diagrams in this section are simply physical renditions of the solid bodies defined. The diagrams he provided, which accompany the ten definitions for which

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[54] Commandino, *Euclidis Elementorum,* 191*r*. "Ostendendum quomodo conus orthogonius sit, vel angulum rectum ad verticem habeat."

[55] The right angle at the apex arises from the rotation of the isosceles right triangle about one of its legs (i.e. one of the two shorter sides). Taking the original triangle and its counterpart at 180 degrees, a larger triangle can be seen to be contained by the diameter of the base of the cylinder and two lines on the surface of the cylinder from the apex to the base (i.e. the hypotenuse of the original triangle at two points of its rotation). Since the angle at the top of the original triangle was 45 degrees, and the larger triangle is formed by joining two of the original triangles along the leg running from the apex to the base of the cone, the angle at apex is the sum of two 45 degree angles, a right angle.

there are multiple possible physical cases, showed examples of those cases.[56] For example, pyramids and prisms can both be formed using different polygons. Clavius's diagrams show those shapes formed on triangles, rectangles, and pentagons (Figure 36). However, unlike the multiple cases for the pyramid and the prism, not all the cases Clavius depicted are actually covered by the definitions Euclid provided. Like Commandino, Clavius included a discussion and diagrams of scalene cones, even though such cones are not included in Euclid's definition. He also included a similar discussion with diagrams of scalene cylinders (Figure 37). Thus, Clavius's diagrams go beyond the Euclidean definitions to create more universal categories of these physical bodies. Even in the instances for which all the cases are encompassed by Euclid's definition, the inclusion of diagrams of the multiple cases served to show the reader the breadth of these definitions, rather than simply focusing on the physicality of the bodies.

Within the definitions of the solid bodies, the five regular solids (the cube, tetrahedron, octahedron, icosahedron, and dodecahedron) are the most revealing of all three authors' approaches to the role of physical bodies within mathematics. According to Billingsley, the five regular solids "are as it were the ende and perfection of all Geometry, for whose sake is written whatsoever is written in Geometry."[57] In order to ensure that the reader could fully grasp the physical forms of the bodies,

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[56] A sphere and the five regular solids did not receive diagrams because they do not have multiple physical representations.
[57] Billingsley, *Elements of Geometrie,* 319*v*-320*r*. (N.B. The folio number on what should be 320 is printed as 340. The next folio shows 341; the following is numbered 327. After that, the number is 323, which is what it should be. From there the folios count up as one expects.)

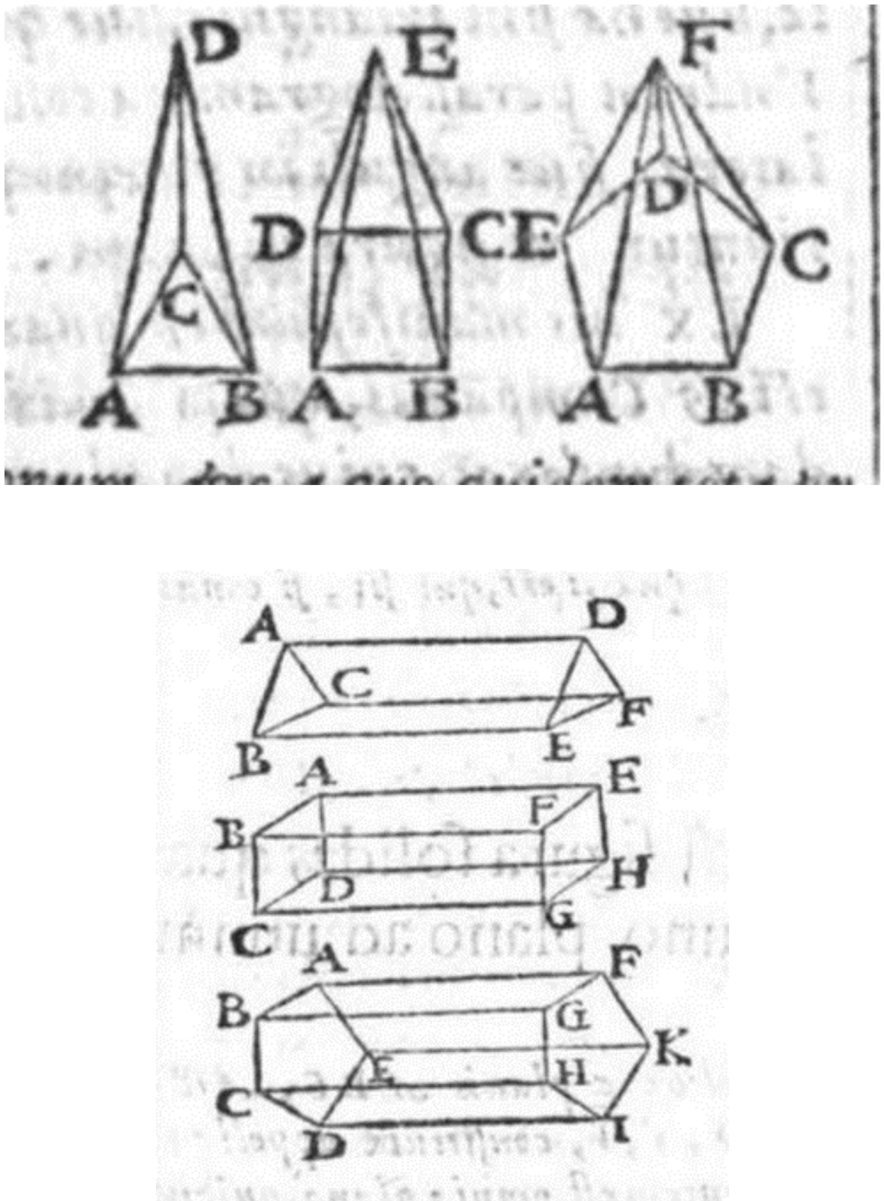

Figure 36: **Clavius's Diagrams for the Definitions of a Pyramid and a Prism**

For both shapes, Clavius shows examples generated on triangular, rectangular, and pentagonal bases.

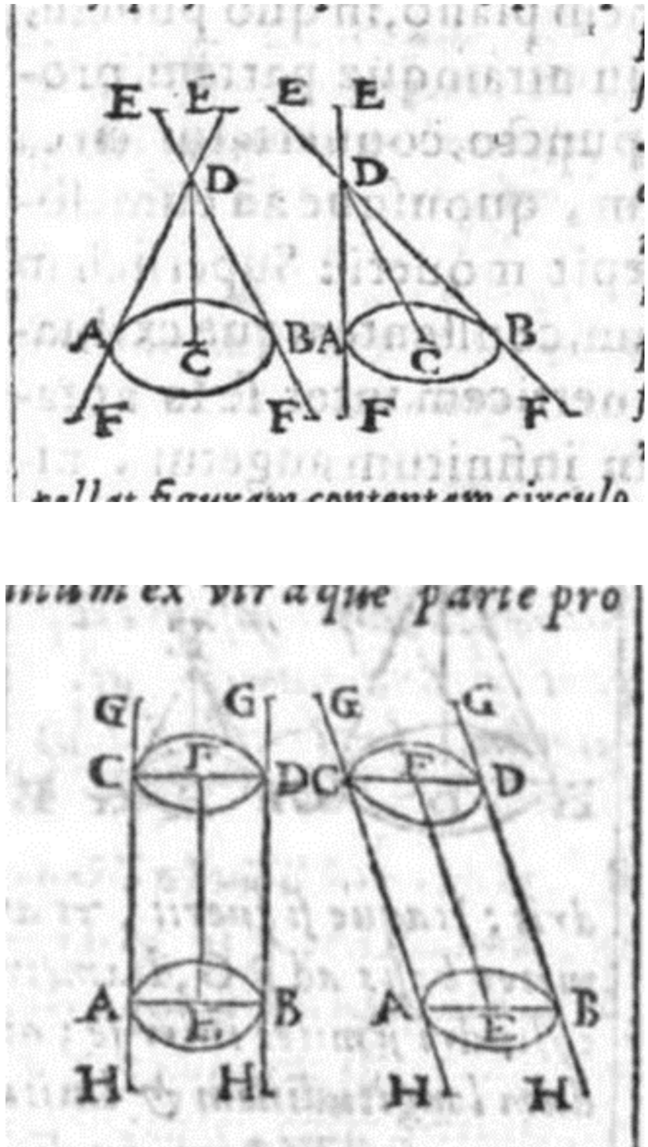

Figure 37: **Clavius's Diagrams for Right and Scalene Cones and Cylinders**

Note how he uses his labels to indicate what goes through a rotation to generate the shape. The diagrams for cones are nearly identical to those found in Commandino's text. Both credit the discussion of scalene cones to Apollonius. It is possible that the diagrams are also based on a Greek text.

Billingsley provided three images for each definition: a template from which the reader could create the physical body and two diagrams – one that showed all of the faces of a figure drawn in the plane and one that offered a perspectival artistic rendition of the figure. (See Figure 38). In his commentary he described how to see the first of the figures as a solid body (a feat that requires some imagination for the last three bodies). He contended that, even though, of the two diagrams he provided, the artistic rendering looked more like a body, the planar figure was necessary because without it, "ye can not conceave the draught of lines and sections in any one of the … sides which are sometimes in the descriptions of some of those Propositions required."[58] After the definition for the last of the regular solids, Billingsley included a brief discussion of these bodies' history and role in philosophy. Even here, his emphasis was on the physical. As he described the association of each of the solids with an element (or the heavens in the case of the dodecahedron – the sphere was not a regular solid), he explained how the physical attributes of each body merited its assignment to a particular element.

Commandino gave no diagrams and no commentary for the five regular solids. He did not even acknowledge that they were the only solids that could be composed of equal regular polygons. In this case, silence speaks volumes. Without a connection to some more universal truth, even the significance of these solids to ancient

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[58] Ibid., 319*r*. I have taken his description of the utility for the first diagram of the octahedron. I elided the word "eight" because he gives similar justifications for the planar diagram of each of the five solids.

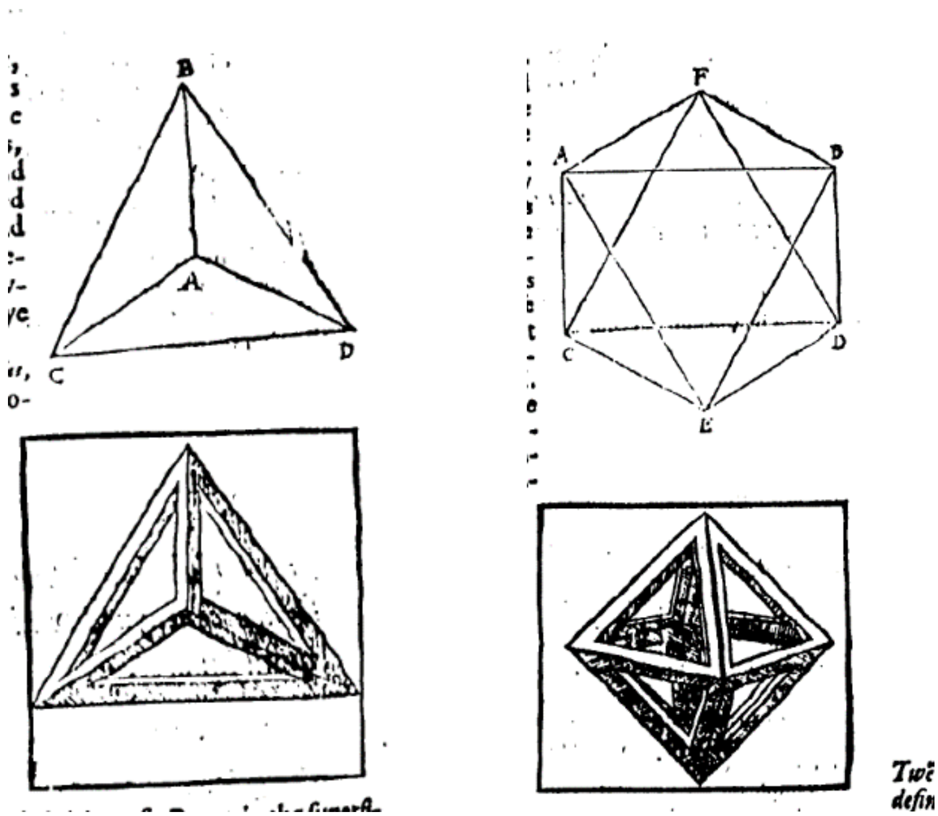

Figure 38: Billingsley's Diagrams for a Tetrahedron and an Octahedron

In the first diagram of each shape, all of the edges are drawn as complete lines. In the artistic rendering, some edges cannot be seen completely, but the forms of the bodies are visible. Billingsley noted that the line diagrams were how the figures were "commonly described" in a plane. The same linear schematics can be found in Candalla's commentary as diagrams for these solids.

mathematics and philosophy was not enough for Commandino to offer an analysis of the solid bodies, let alone to provide images.

For Clavius, it was the importance of the regular solids to philosophy as the only solids composed of equal regular polygons that earned these definitions their commentary. Following the last definition, he included a brief paragraph in which he observed that these solids, known as the Platonic solids, were identified by Plato in the *Timaeus* as the bodies that "the five forms of the world, which are called simple by philosophers, namely the heavens, fire, air, water, and earth," imitate.[59] While he did not go into the details of these imitations, he noted that he offered an extensive explanation of them in his commentary on Sacrobosco's *Sphere*. Nevertheless, by devoting his commentary on the five regular solids to their philosophical value as the forms of the elements of the world instead of their own physical properties, he showed his reader that mathematics was essential to the study of philosophy. However, by avoiding the specifics, Clavius kept the emphasis on the category of regular solids rather than on the individual bodies. In fact, Clavius did not even provide diagrams to accompany the definitions of the regular solids, thereby maintaining his emphasis on their philosophical status as Platonic forms. Nevertheless, in a nod to the physical reality of these solids, he promised further elaboration, noting that these definitions "will be more plainly and perfectly understood" in the thirteenth book where he provided "exceedingly simple practices according to which anyone could construct the

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[59] Clavius, *Euclidis Posteriores,* 128*v*. "A nonnullis corpora Platonica dicuntur, propterea quod Plato in Tymaeo quinque mundi corpora, quae simplicia a philosophis nuncupantur, nempe Caelum, Ignem, Aerem, Aquam, atque, Terram, quinque dictis corporibus assimilet, ut in Sphaera Ioannis a Sacrobosco latius explicavimus."

solids."[60] Clavius followed through on his promise by including templates for each of the five solids in scholia to the propositions in the thirteenth book which required their construction. Thus, Clavius ensured that the reader would be able to build the regular solids when the manipulation of their physical forms could be used to uncover their properties. See Figure 39 for an example. He credited the templates to Albert Dürer, who had provided them in his 1525 book *Underweysung der Messung mit dem Zirckel und Richtscheyt in Linien ebnen unnd gantzen Corporen.*[61] Still, Clavius reduced the importance of the physical bodies by making his templates quite small. Both Dürer and Billingsley included full-page or near full-page templates as stand-alone images, but Clavius's each took up less than a quarter of a page and were embedded in his commentary on the relevant propositions. Thus, the physical diagrams became little more than tools for the completion of the propositions.

The definitions of the eleventh book lay the foundations for the propositions in the last five (or six, if Candalla's book was included) books of *The Elements*. Like their counterparts in the plane geometry books, the diagrams in the propositions of the solid geometry books were restricted in form by the necessity of illustrating the demonstration as it was described in the texts. However, as was the case in the plane geometry books, small differences between the diagrams for the proposition and

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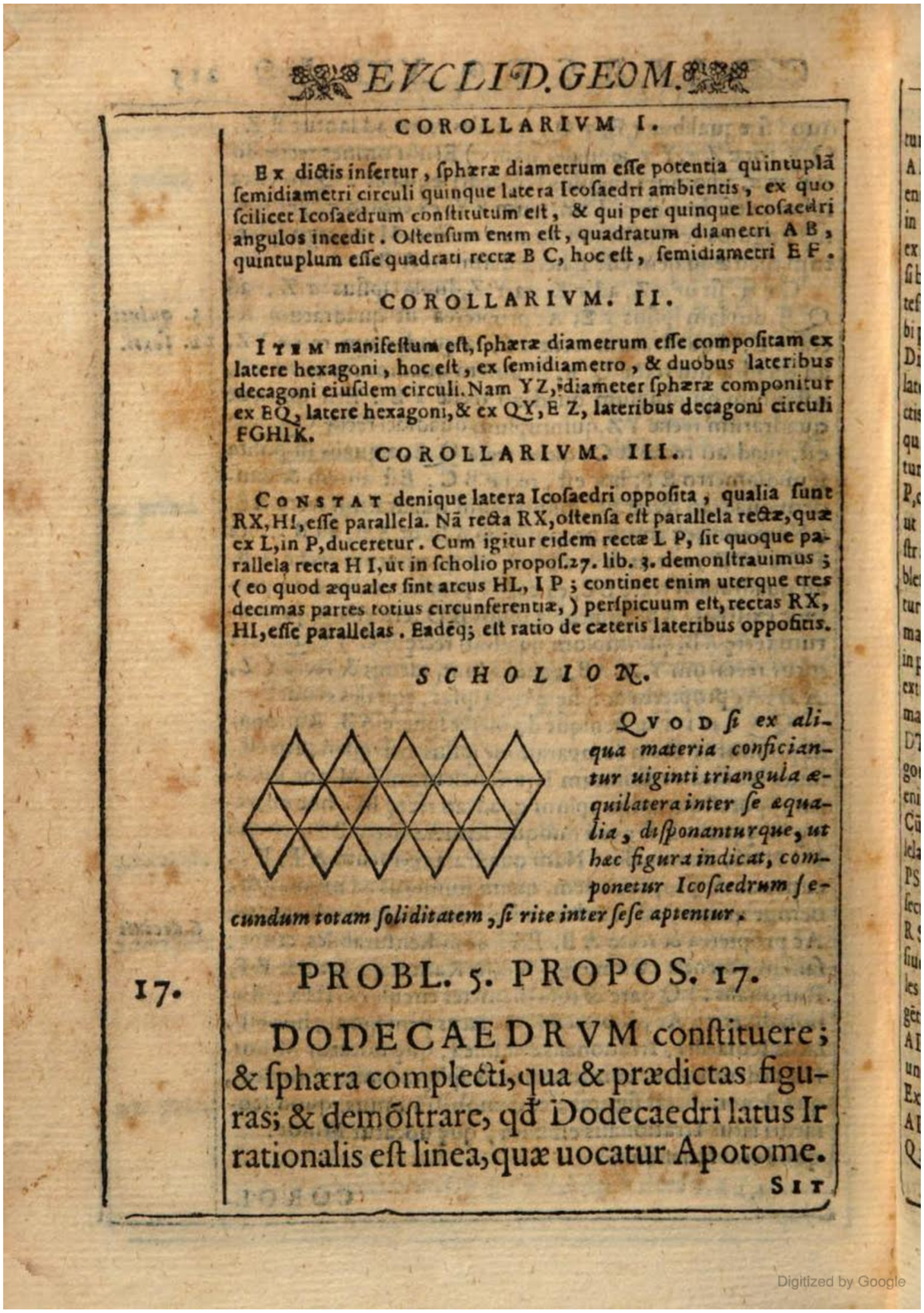
[60] Ibid., p. 128*v*. "Ubi planius perfectiusque definitiones horum corporum intelligentur. In plano enim difficillimum est, ea ita depingere, ut veram eorum effigiem, atque formam quis intueatur. Trademus tamen propriis in locis praxes admodum faciles, quibus ea quilibet secundum eorum soliditatem possit conficere."

[61] Ibid., 209*v* -219*v*. On the tetrahedron (p. 209*v*): "Hanc vero praxim, & sequentes quatuor, quibus reliqua quatuor solida regularia in materia quavis sensibili conficiuntur, desumpsimus ex Alberto Durero non ignobili scriptore." See Albrecht Dürer, *Underweysung der Messung mit dem Zirckel und Richtscheyt in Linien ebnen unnd gantzen Corporen,* (Nuremberg, 1525), NRiii*v* – NRv-*v*.



EVCLID. GEOM.

COROLLARIVM I.

Ex dictis infertur, sphæræ diametrum esse potentia quintuplā semidiametri circuli quinque latera Icosaedri ambientis, ex quo scilicet Icosaedrum constitutum est, & qui per quinque Icosaedri angulos incedit. Ostensum enim est, quadratum diametri A B, quintuplum esse quadrati rectæ B C, hoc est, semidiametri E F.

COROLLARIVM. II.

Item manifestum est, sphæræ diametrum esse compositam ex latere hexagoni, hoc est, ex semidiametro, & duobus lateribus decagoni eiusdem circuli. Nam Y Z, diameter sphæræ componitur ex EQ, latere hexagoni, & ex QY, E Z, lateribus decagoni circuli FGHIK.

COROLLARIVM. III.

Constat denique latera Icosaedri opposita, qualia sunt RX, HI, esse parallela. Nā recta RX, ostensa est parallela rectæ, quæ ex L, in P, duceretur. Cum igitur eidem rectæ L P, sit quoque parallela recta H I, ut in scholio propos. 27. lib. 3. demonstrauimus; (eo quod æquales sint arcus HL, I P; continet enim uterque tres decimas partes totius circunferentiæ,) perspicuum est, rectas RX, HI, esse parallelas. Eadēq; est ratio de cæteris lateribus oppositis.

SCHOLION.

Quod si ex aliqua materia conficiantur uiginti triangula æquilatera inter se æqualia, disponanturque, ut hæc figura indicat, componetur Icosaedrum secundum totam soliditatem, si rite inter sese aptentur.

17. PROBL. 5. PROPOS. 17.

DODECAEDRVM constituere; & sphæra complecti, qua & prædictas figuras; & demōstrare, qđ Dodecaedri latus Irrationalis est linea, quæ uocatur Apotome.

Sit

Figure 39: Clavius's Template for an Icosahedron

Compare this image to Billingsley's template shown in Figure 33 to see how much less of the page Clavius devoted to his templates.

additional diagrams in the commentary could illustrate the authors' varied views on mathematics. In one particularly notable example, the complexity of the proof makes the diagram the central focus of all three authors' demonstrations. However, an examination of the proposition shows that where Billingsley focused on providing an accurate physical representation of the problem and Commandino sought to reveal the universal truths used in the demonstration, Clavius helped his readers understand the physical form of the construction through universal principles by ensuring that every step of the demonstration received a formal, theoretical proof based on universal principles.

The proposition requires the reader to inscribe a polyhedron into a sphere such that it never touches a smaller sphere concentric with the first sphere (Figure 40). There are a few differences between the diagrams. The most noticeable of those is Commandino's choice not to represent any of the shapes completely, illustrating his disinterest in representing the physical situation described by the diagram. Even though all three authors only include a few faces of the polyhedron in the upper hemisphere of the spheres, Clavius and Billingsley still included great circles representing the complete spheres, but Commandino showed only the upper hemisphere of either sphere because that was sufficient to the demonstration. While there are a few other differences between the diagrams, the authors' approaches to mathematics are most evident in their use of diagrams in the commentary.

Billingsley's commentary ignores any abstract or universal claims in order to focus on the concrete representation of the proposition. It is devoted entirely to the

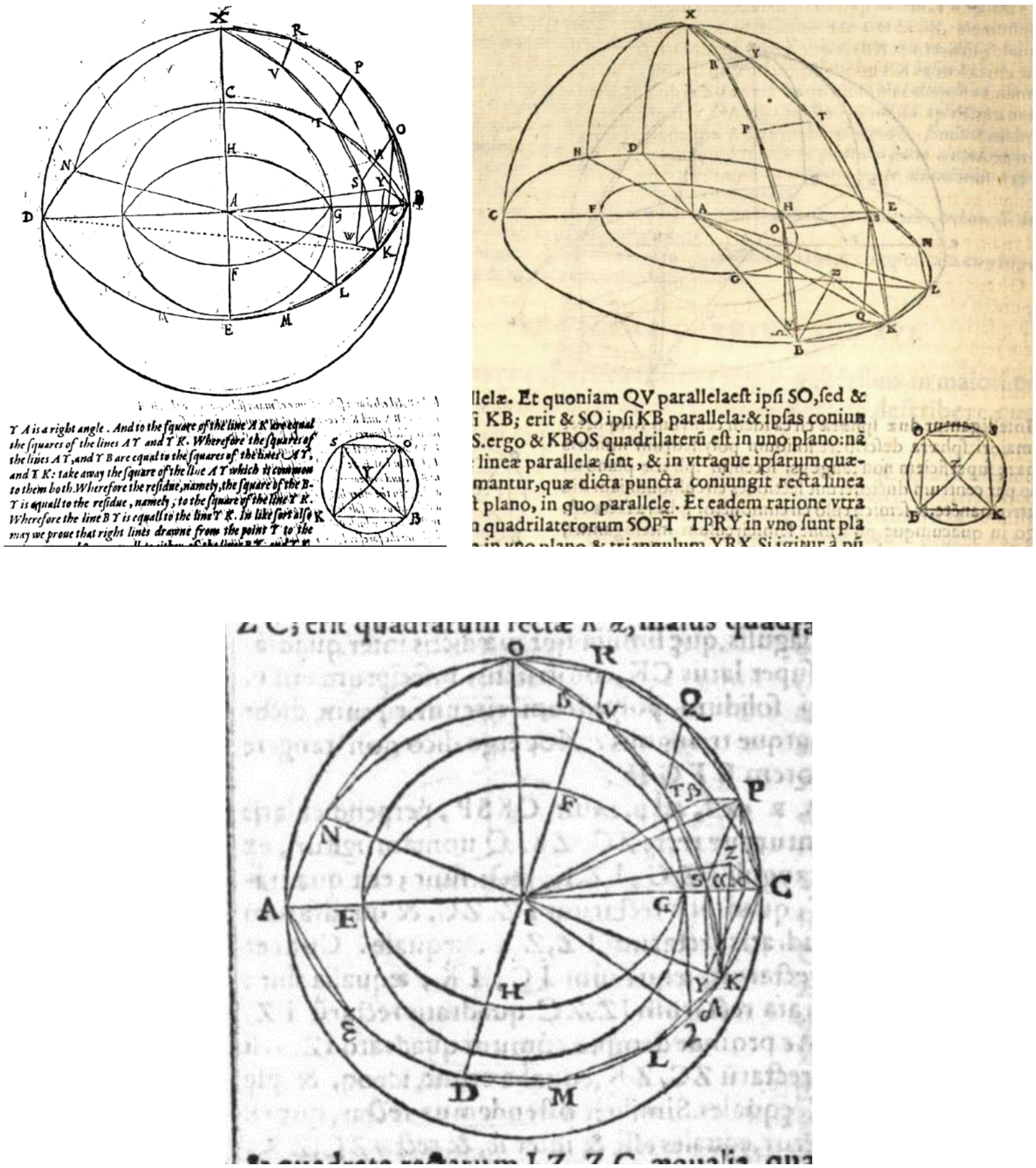
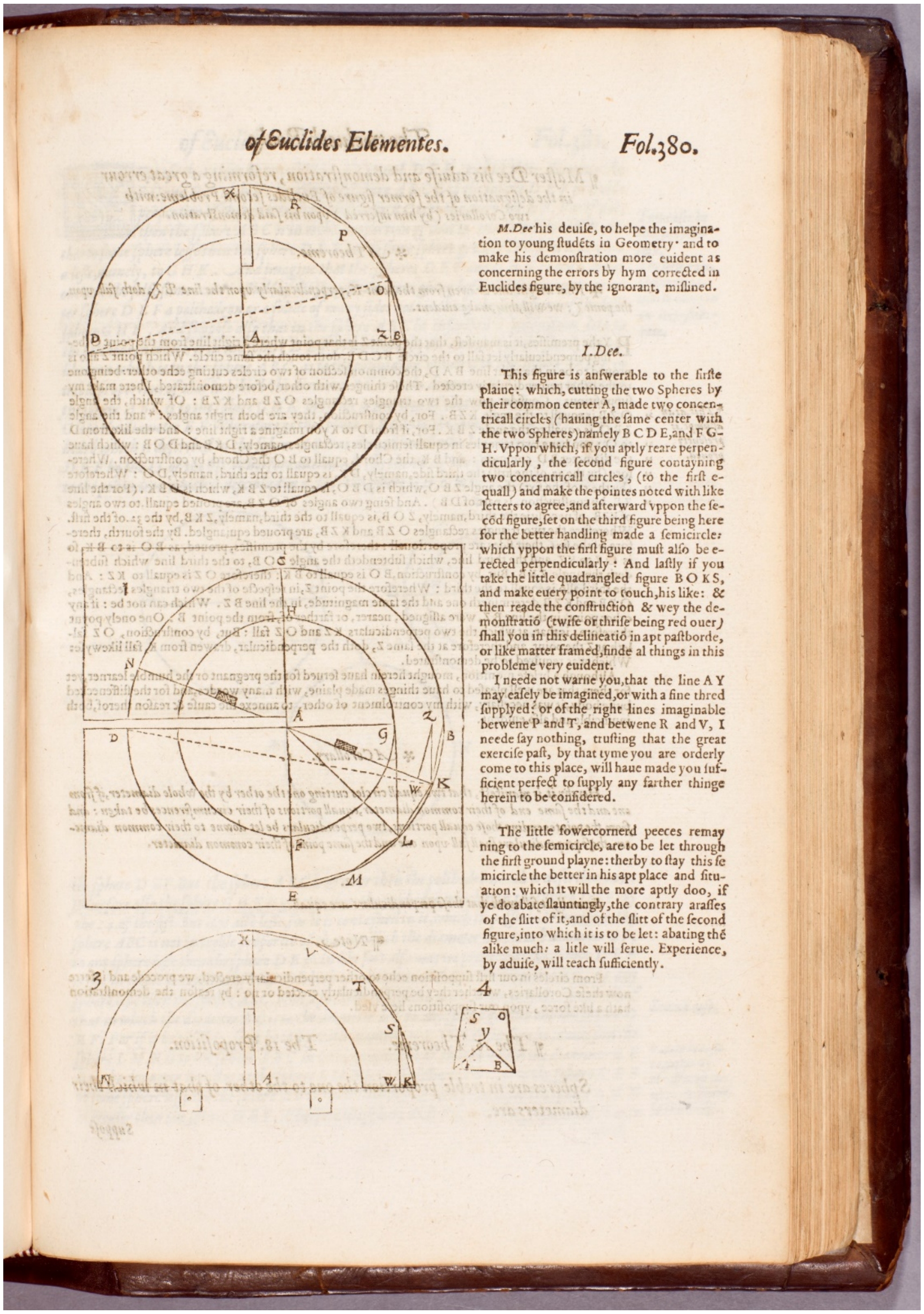


Figure 40: Diagrams for Book Twelve, Proposition 17

Billingsley's (top left), Commandino's (top right), and Clavius's (bottom) images for Book Twelve, Proposition 17. Clavius's diagram does not have the front on view of the face of the polyhedron inscribed in a circle because he simplified the section of the demonstration that relied on that diagram. Because it is such a complex figure that accompanies a lengthy demonstration, all three authors printed the diagram multiple times such that the reader could see the diagram as he read through the entire demonstration. For a translation of the demonstration, see Appendix C.

diagram as its own object. Indeed, most of the commentary is simply the presentation of a template from which the reader could create a three-dimensional version of the diagram. (Figure 41) The template comes complete with instructions on which lines to cut and fold and how to paste the various pieces together. The only other points Billingsley makes are corrections of two errors found in earlier authors' diagrams, even though neither error has any bearing on the completion of the proof. By identifying these errors, despite their irrelevance to the proof itself, Billingsley showed his dedication to providing accurate physical models. One is the identification of two pairs of lines (*GL* and *AG,* and *KZ* and *AB*) that are perpendicular to one another, but are not always drawn as such. The other is the identification of a point (point *Z*) that usually received two labels because it comes up in two separate parts of the proof. Billingsley's commentary included a demonstration that the points described in the two parts of the proof are necessarily one and the same in order to make it possible for him to label the point only once.

In contrast, Commandino, who made both of the errors Billingsley corrected, used his commentary only to clarify points made in the text of the proof. While most of the clarifications were simple references to past propositions, three could be expressed as general claims of their own and required diagrams to accompany their demonstrations. Instead of reproducing the relevant segments of the diagram from the proposition, Commandino created new diagrams, thereby showing the universality of these claims. For example, one of the claims in his commentary is that in the original diagram, lines *OV* and *SQ* are equal and lines *BV* is equal to *KQ*. In order to show


of Euclides Elementes. Fol.380.

M.Dee his deuiſe, to helpe the imagination to young ſtudẽts in Geometry· and to make his demonſtration more euident as concerning the errors by hym corrected in Euclides figure, by the ignorant, miſlined.

I.Dee.

This figure is anſwerable to the firſte plaine: which, cutting the two Spheres by their common center A, made two concentricall circles (hauing the ſame center with the two Spheres) namely B C D E, and F G H. Vppon which, if you aptly reare perpendicularly, the ſecond figure contayning two concentricall circles, (to the firſt equall) and make the pointes noted with like letters to agree, and afterward vppon the ſecõd figure, ſet on the third figure being here for the better handling made a ſemicircle: which vppon the firſt figure muſt alſo be erected perpendicularly: And laſtly if you take the little quadrangled figure B O K S, and make euery point to touch, his like: & then reade the conſtruction & wey the demonſtratiõ (twiſe or thriſe being red ouer) ſhall you in this delineatiõ in apt paſtborde, or like matter framed, finde al things in this probleme very euident.

I neede not warne you, that the line A Y may eaſely be imagined, or with a fine thred ſupplyed: or of the right lines imaginable betwene P and T, and betwene R and V, I neede ſay nothing, truſting that the great exerciſe paſt, by that tyme you are orderly come to this place, will haue made you ſufficient perfect to ſupply any farther thinge herein to be conſidered.

The little fowercornerd peeces remayning to the ſemicircle, are to be let through the firſt ground playne: therby to ſtay this ſemicircle the better in his apt place and ſituation: which it will the more aptly doo, if ye do abate ſlauntingly, the contrary araſſes of the ſlitt of it, and of the ſlitt of the ſecond figure, into which it is to be let: abating thẽ alike much: a litle will ſerue. Experience, by aduiſe, will teach ſufficiently.

Figure 41: Billingsley's Template for Book Twelve, Proposition 17

A template created by John Dee for Book Twelve, Proposition Seventeen in Billingsley's Euclid. The accompanying commentary provides instructions for its assembly into a three-dimensional form. (Source: Huntington Library.)

those equalities to be true, Commandino recast the proof as a planar demonstration for the relevant lines in two semicircles. His diagram (Figure 42) and proof show that lines *BG* and *EH* and lines *AG* and *DH* are equal if *AB* and *DE* are equal arcs on equal semicircles. The lines from the original diagram have the same relationships to one another as the lines in the commentary, but by providing a new diagram complete with new labels, Commandino generalized the named equality into a universal principle.

Clavius's commentary and accompanying diagrams united the physical diagram of the original proof with the universal truths of mathematics by formalizing three seemingly obvious claims that had appeared in the original demonstration without proof. While these formalizations eliminated the need for the reader to rely on physical intuition, Clavius maintained the links to the bodies represented in the original diagram by reproducing relevant portions of that diagram complete with its labels. The only changes he made were the eliminations of unnecessary lines to make the images easier to read. For example, Figure 43 shows the diagram accompanying the last of Clavius's additions, in which he sought to show the reader that line *βP* is shorter than *ZC* as part of the proof that the second face of the polyhedron Clavius drew, *PTSQ*, does not touch the surface of the smaller sphere. The reader could easily assume that *βP* is shorter than *ZC* because the polygon *PTSQ* had already been shown to be smaller than the polygon *CKSP*, and it seems obvious that the lines connecting the centers and corners of the polygons, *βP* and *ZC* respectively, would reflect that relationship. Thus, Clavius was able to eliminate physical intuition from the proof without generalizing his claims into their own universal truths.

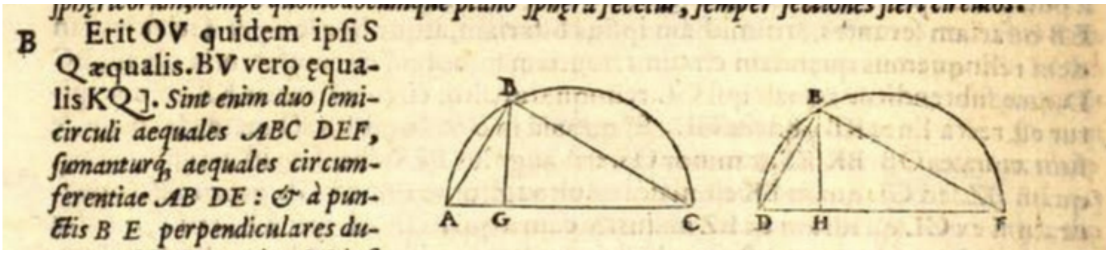


Figure 42: One of Commandino's Diagrams for his Commentary on Book Twelve, Proposition 17

This diagram accompanies the section of his commentary explaining why lines *OV* and *SQ* and line *BV* and *KQ* must be equal to one another. Any reference to the original diagram is gone.

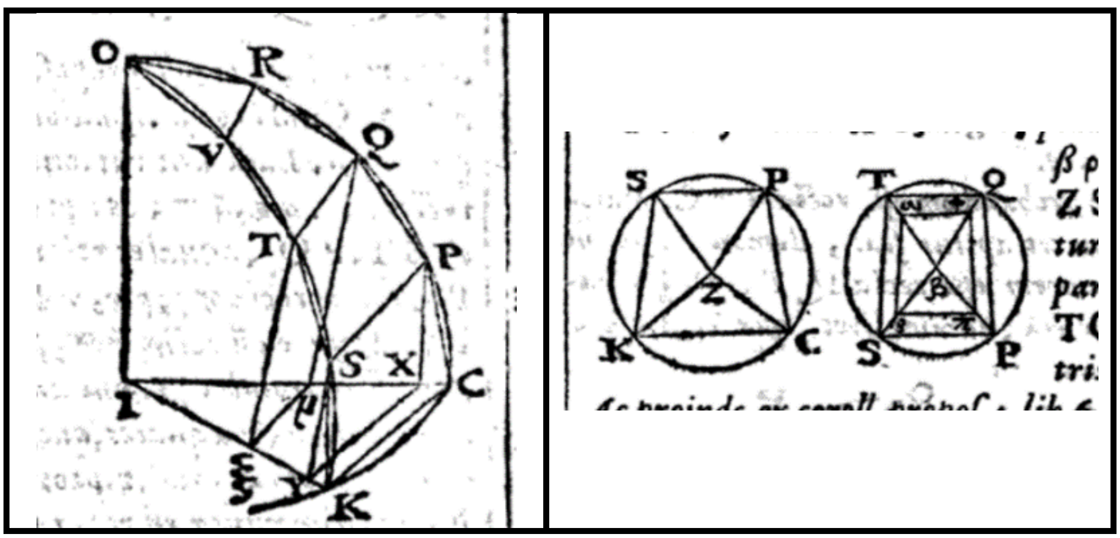


Figure 43: Clavius's Diagrams for Part of his Commentary on Book Twelve, Proposition 17

These two diagrams accompany Clavius's proof that *βP* is shorter than *ZC*. The diagram on the left is clearly a segment of the original image. The diagram on the right focuses on the two faces of the polyhedron separately from one another in order to make the necessary relationships visible, but the labels remain those found in the original diagram.

As this example has shown, the visual arguments created by diagrams in the definitions for the solid geometry book also appear in the authors' propositions. Billingsley embraced the challenges of representing three-dimensional objects, showing the high value he placed on the physical components of mathematics. Commandino was satisfied to offer a few images to aid his reader in recognizing the abstract principles that allowed for the creation of solid bodies, rather than providing images of the forms of the bodies themselves. Clavius sought to show both the forms of the bodies and the abstract principles embedded in them, allowing the physical bodies and the abstract principles to inform each other.

Conclusion

The challenges of representing three-dimensional objects on a two-dimensional page forced the authors to decide how much effort accurate physical representations were worth, but, as I have shown in this chapter, the diagrams in the planar geometry books were also able to provide arguments about the value of mathematics. Throughout the entire text of *The Elements,* commentators used images to present their own visions of the discipline of mathematics. Despite the necessary similarities of diagrams in *The Elements*, variations in the presentation of the images and their relationships to the prose reveal the authors' individual conceptions of the value of mathematics. Through his emphasis on his diagrams as particular instances of the objects described, Billingsley showed mathematics to be a physical discipline valuable for its ability to describe and model concrete objects. Commandino treated his images as representative of broader categories, using them to aid the reader in uncovering the certain and universal truths of mathematics. Clavius consistently balanced the roles of diagrams as physical instances and as representations of universal concepts, making his version of mathematics a bridge between the disciplines depicted by his contemporaries.

In *Painting and Experience* Baxandall argued that the style of art is a subject for study in social history because the social position and setting of the artists shaped their artistic choices. Likewise, the versions of mathematics revealed in each text's images show the influence of the social positions and corresponding goals of each author that were discussed in earlier chapters. Billingsley's focus on the physical aspects of mathematics can be explained by his express desire that his version of *The Elements* aid English inventors, a wish consistent with his own role as a merchant.[62] Commandino's interest in the universal principles of mathematics fits with his background as a humanist and his commitment to restoring the ancient dignity of mathematics based on its certainty. His position at the court at Urbino provided him with the opportunity to study mathematics as a branch of philosophy rather than a tool with which to manipulate the physical world. Clavius's diagramming practices are the result of his position as an educator. He used his images to help his reader grasp the universal truths found in geometry while providing a physical object that could be translated into practical knowledge. By offering different versions of mathematics to appeal to the varied Jesuit student body, Clavius's diagrams served as part of his

[62] Billingsley, ij*r;* Chapter 3 of this dissertation.

efforts to establish mathematics within the curriculum. In his view, students who would go on to become philosophers or mathematicians, especially those in his academy, could appreciate the diagrams as representatives of universal truths, and students who would pursue more mundane activities could understand them as examples of physical bodies. Where his contemporaries' each consistently emphasized either the physical instances of the diagrams or the universal nature of mathematics, Clavius used his visualizations for both purposes such that the physical and conceptual clearly informed each other. He thereby gave mathematics a place in the curriculum as the bridge from the study of the physical world – necessary to missionaries – to the study of universal truths – necessary to theologians. Future research into the readers of these texts could perhaps show that how students learned mathematics influenced their own later work.[63]

[63] Such a study would be an interesting development of Baxandall's argument that the mathematics people knew had a profound influence on the visual cultures reflected in the paintings of the fifteenth century. See Baxandall, *Painting and Experience,* 86-108.

Conclusion

In 1611 Clavius used the Prolegomena that he had written for his 1574 commentary on *The Elements* as the preface to his collected works. In it, he had outlined the bipartite vision of mathematics that defined his pedagogical project from the beginning of his writing career to the final compilation of his life's efforts in 1611. For Clavius, mathematics was both a study of perfect abstract entities through which "the mind's eyes" could come to understand the "great work of God and nature," and an "abundant fountain" of practical uses.[1] Both parts of mathematics appear throughout Clavius's extensive body of work, most of which was the development of a mathematics curriculum, complete with textbooks, for the Jesuit schools.

As is evident from the frontispiece to his *Opera Mathematica* (Figure 44), in Clavius's estimation, his was a divinely blessed project, and, thus, mathematics was a means through which Jesuit schools could promote the true faith.[2] In it, at top of the page, there are images of angels and saints looking down benevolently at Clavius's

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[1] Christopher Clavius, *Euclidis Elementorum Libri XV Accessit XVI de solidorum Regularium comparatione* (Rome: Vincentium Accoltum, 1574), b4*v*, "Hoc denique ingens Dei, & naturae opus, mundum, inquam, totum, mentis nostrae oculis munere, ac beneficio Geometriae subiectum conspicimus." And "Ex his etenim elementis, veluti fonte uberrimo, omnis latitudinum, longitudinum, altitudinum, profunditatum, omnis agrorum, monitum, insularum dimensio, atque divisio; omnis in caelo per instrumentae syderum observatio, omnis horologiorum sciotericorum composito, omnis machinarum vis, & ponderum ratio, omnis apparentiarum variorum, quails cernitur in speculis, in picturis, in aquis, & in aere varie illuminato, diversitas manat."

[2] Although Clavius's collected works were published in Mainz rather than Rome, he does still seem to have had some say in the design of the frontispiece. In a 1611 letter to Johann Gottfried von Aschhausen, who is named in thanks for assisting with the publication of the *Opera Mathematica,* Clavius requested that St. Henry (Henry II of the Holy Roman Empire in the eleventh century) be commemorated on the frontispiece of his work. The saints shown are Henry and his wife Cunigunde. Christoph Clavius a Johann Gottfried von Aschhausen [in Bamberg] Roma 1611, in *Chirstoph Clavius: Corrispondenza Vol. VI* ed. Ugo Baldini, P.D. Napolitani, (Pisa: Università di Pisa Dipartimento di Matematica, 1992), 183.

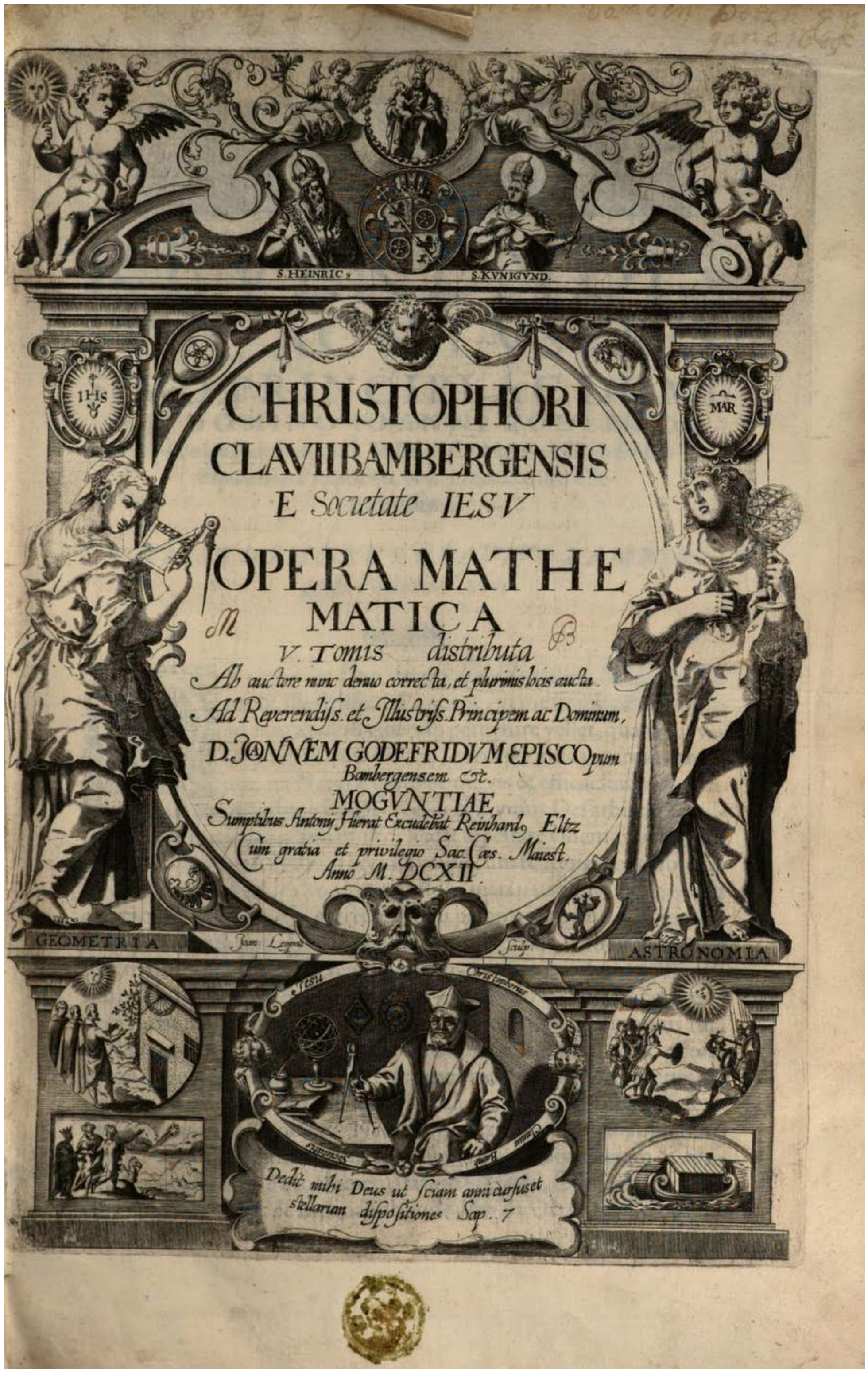

Figure 44: The Frontispiece to Clavius's *Opera Mathematica*

project. Along the sides of the title section are female figures representing astronomy and geometry. Each one holds the tools of her discipline. Below that, a portrait of Clavius is centered between four biblical scenes with ties to astronomy.[3] The allegory created on this frontispiece is an illustration of the key point in Clavius's arguments for the nobility of mathematics: through the study of geometry and astronomy he and his readers could illuminate the works of God.

As discussed in Chapter One, Clavius made the arguments for the nobility of mathematics explicit both in the prefaces to his texts and in his treatises on the elevation of the status of mathematics within the Jesuit schools. In these arguments he defended his discipline as a source of universal and certain truths that could bridge the studies of physics and metaphysics and insisted that without knowledge of mathematics the Jesuits would be embarrassed by other scholars, a fate which would thwart the Order's efforts to gain influence as educators to the elite.[4] Students could learn how to discover mathematical truths in Clavius's commentaries and textbooks on geometry, including his commentaries on *The Elements* and Theodosius's *Sphere*. These could then be applied to the mixed study of astronomy, which led to

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[3] The scenes, clockwise from the upper left, are Hezekiah receiving the sign of the shadow on the sundial moving backwards ten degrees (2 Kings: 20:8-11), the sun standing still for Joshua during a battle (Joshua 10:13), Noah's ark with the rainbow (Genesis 9:14-15), and the magi looking at the star of Bethlehem (Matthew 2:2), and. While the rainbow may seem out of place with the other images since it was not astronomical in nature, all four images show signs from God that could be studied mathematically. Optics, like astronomy, was a branch of mixed mathematics. For a discussion of the mathematical treatment of the rainbow in the sixteenth century see Carl B. Boyer, *The Rainbow: From Myth to Mathematics* (Princeton: Princeton University Press, 1987 [Sagamore Press, 1957]), 151-177.
[4] Christopher Clavius, "Modus quo disciplinae mathematicae in scholis Societatis possent promoveri (1582)," in ed. Ladislaus Lukacs, *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu* (Rome: Institutum Historicum Societatis Iesu, 1992), 116. "Pari ratione oporteret praeceptores philosophiae callere disciplinas mathematicas, saltem mediocriter, ne in similes scopulos magna famae, quam Societas in litteris habet, iactua et dedecore incurrerent."

mathematical descriptions of the structure of the universe that Clavius included in his commentary on Sacrobosco's *Sphere*.[5]

However, as discussed in Chapter Two, the arguments for the nobility of mathematics did not persuade everyone, and the Jesuits ultimately gave mathematics a place amongst the higher faculties in their curriculum because they perceived the practical value of the discipline to mundane tasks as a means to gain patronage from ruling classes. In order to secure such patronage and to create skilled missionaries who could carry out necessary tasks from timekeeping to irrigation, Clavius provided textbooks on various topics in practical mathematics, implicitly promising that students of the Order would be able to apply their mathematical knowledge to whatever task was most necessary. Concern for the practical use of mathematics even appeared in his texts on abstract topics, including his commentary on *The Elements*. Indeed, Clavius had that commentary printed in two volumes, so that the itinerant missionaries could carry it with them as a handbook.[6]

Clavius's combination of the nobility of mathematics as a contemplative study and the utility of mathematics as a practical means to describe and manipulate the physical world was not simply rhetoric. The idea that mathematics was the language

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[5] Clavius, *Euclidis Elementorum,* b4*v*, "Ex his, inquam, elementis machinae totius huius mundanae est inventum medium, atque centrum, inventi cardines, circa quos perpetuo convertitur, orbis denique totius explorata figura acquantitas."

[6] Ibid., a5*v,* "Nam cum Euclides, propter singularem utilitatem, instar enchiridii, manibus semper debeat circumgestari, neque unquam deponi ab his, qui fructum aliquem serium ex hoc suavi Matheseos studio capere volunt, in eoque progredi, id vero in hunc diem exemplaribus omnibus maiore forma impressis, necdum factum videamus; hoc nostra editio certe, si nihil aliud, attulerit commodi, atque emolumenti. Sunt enim hi nostri commentarii in universum Euclidem conscripti commodiore nunc forma, quam vulgo caeteri, (id quod magnopere a nobis, qui nos audierunt, efflagitabant,) volumineque editi, ut facile iam queant, nulloque negotio, e loco in locum, cum restulerit, ferri atque portari."

of the universe, which Galileo famously asserted in *The Assayer,* underpins all of Clavius's work and can be seen in his union of pure mathematics with its myriad potential applications in the bodies of his textbooks, including the first edition of his commentary on *The Elements* (1574). That text has particular significance because it was the first book presented in the mathematics portion of the Jesuit curriculum. Thus, the vision of mathematics that Clavius outlined in its preface, the same vision that guided his entire pedagogical project, was the means through which the discipline was defined for Jesuit students. In the last three chapters of this dissertation, I have examined the 1574 commentary on *The Elements* as a means to understand how Clavius made the versatility of his discipline apparent in a textbook.

Through a comparison of Clavius's commentary to two other significant sixteenth-century commentaries on *The Elements,* those of Federico Commandino and Sir Henry Billingsley (1572 and 1570, respectively), I have shown that Clavius's commentary, while part of a widespread sixteenth-century effort to revitalize the study of mathematics, was so rich that it transcended its genre and became a resource book for students and teachers in which the current state and scope of the discipline of mathematics was illustrated. James Lattis described Clavius's Euclid as "a paraphrase and a commentary," as opposed to a translation of *The Elements.*[7] While accurate, this characterization understates the significance of Clavius's work, which approaches a modern textbook because the paraphrasing and commentary were used systematically

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[7] James Lattis, *Between Copernicus and Galileo: Christoph Clavius and the Collapse of Ptolemaic Cosmology* (Chicago: University of Chicago Press, 1994),, 16.

to describe and teach his particular vision of mathematics. Indeed, Ugo Baldini has credited Clavius with taking the first steps towards the development of the "modern scientific textbook."[8] Instead of providing readers with a translation of a classical text with clarifying glosses, as Commandino did, or an anthology of demonstrations for Euclid's propositions, as Billingsley did, Clavius presented a textbook with detailed analyses of classical, medieval, and renaissance contributions to the Euclidean corpus. Through these analyses he created a clear picture of geometry as the foundation of a mathematical description of the structure of the universe with the ability to effectively guide manipulations of the physical world.

Clavius's approach to mathematics in *The Elements* was driven by what he believed the Jesuit Order required in its mathematics education. The result was a combination of nobility and utility of mathematics within its foundational text not seen as sharply in other versions of the text. Although Commandino translated a great variety of treatises on mathematics, including some of Archimedes' practical works, his commentary on Euclid set the tone for treating mathematics primarily as an abstract study of universal truths. Even when he translated his commentary on Euclid into Italian, ostensibly for its applicability to various crafts, he did not suppress his disparaging opinion of the use of mathematics as a path to material profit as vulgar.[9]

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[8] Ugo Baldini, "The Academy of Mathematics of the Collegio Romano from 1553 to 1612" in *Jesuit Science and the Republic of Letters* ed. Mordechai Feingold, (Cambridge MA: The MIT Press, 2003), 67.

[9] Federico Commandino, *De gli Elementi d'Euclide Libri Quindici con gli Scholii Antichi,* (Urbino: Domenico Frisolino, 1575), **3*r*-*v*. "Hora, perche la maggior parte de gli huomini, & massimamente à questi tempi per l'utile solo aprono gli occhi à gli studi dell'arti nobilissime, & solo con questo disegno danno opera alle liberali discipline, vediamo di gratia se e vero, che le matematiche non vagliono punto, ne arrechino, aiuto alcuno all' uso del viver humano; come il cieco & vergognoso desiderio del guadagno fece già dire falsamente a certi: i quali hanno fatto, che gli studiosi di questa facoltà siano da

His vision of mathematics was one of a noble discipline: a science, rather than an art. Indeed, historians have found that the Urbino school, of which Commandino was the head, primarily worked on recovery of ancient mathematical works and the invention of instruments that could improve mathematical precision, advancing mathematics as a noble discipline.[10] For example, Commandino's most famous student, Guidobaldo del Monte, author of *Mechanicorum Liber,* seems to have remained uninterested in the actual operation of the machines whose principles he described.[11] In contrast, while Billingsley and John Dee acknowledged that mathematics was an abstract source of universal truths in the prefatory material they provided for the first English version of *The Elements*, the content of the text itself, written by Billingsley, emphasized mathematics' physicality, thereby preparing readers for the practical study of mathematics that the English merchant hoped to inspire. For Billingsley, mathematics was of interest as an art. Even John Dee, who was well aware of the Aristotelian distinction between demonstrative sciences, which could explain causes, and descriptive studies, which could only describe facts, wrote the "Mathematicall Preface" for Billinglsey's text primarily as a description of the mathematical arts.[12]

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ignoranti, & da quelli, che hanno altro studio alle mani publicamente beaffti, come genti, che in cosa uana & di niun momento perdano il tempo, & la fatica." This comment appears in the preface to the text, as it had in the Latin version. In the letter to Duke Francesco Maria II of Urbino, written by after Commandino's by his son-in-law, the younger author claimed that the translation was done to enable non-Latinate Italians to benefit from the utility of *The Elements*.

[10] Alexander Marr, *Between Raphael and Galileo: Mutio Oddi and the Mathematical Culture of Late Renaissance Italy*, (Chicago: University of Chicago Press, 2011), 221-224.

[11] M. Henninger-Voss, "Working Machines and Noble Mechanics: Guidobaldo del Monte and the Translation of Knowledge." *Isis,* Vol. 91, No. 2 (June, 2000), pp. 233-259.

[12] John Dee, "Mathematicall Preface," second page "Wherefore, seying I find great occasion (for the causes alleged, and farder, in respect of my *Art Mathematike generall*) to use a certaine forewarnying Preface, whose content shalbe, that mighty, most pleasunt, and frutefull *Mathematicalll Tree,* with his chief armes and second (grifted) braunches: Both, what every one is, and also, what commodity, in

Clavius's efforts to prepare Jesuit students both for future scholarly endeavors in philosophy and theology and for practical mathematical tasks for which they might be called upon, united the two visions of mathematics found in his contemporaries' works by emphasizing pure mathematics as the foundation of both mathematical sciences and mathematical arts.[13] His commentary on *The Elements* thereby paved the way for the development of a realist-mathematical science, which Floris Cohen has identified as the first revolutionary transformation of the Scientific Revolution.[14] Such a science required pure mathematics as a driving force.

In fact, Cohen treated Clavius's work as a necessary precursor to the Scientific Revolution, though he is careful to note that the Jesuit's promotion of mixed mathematics did not fundamentally break with an Aristotelian natural-philosophical analysis.[15] What this dissertation has shown is that Clavius's contributions to the early stages of the Scientific Revolution can only be understood as the result of his pedagogical goals and their influence on his presentation of pure mathematics as a versatile means to engage with the world. His combination of the contemplative and the practical visions of his discipline within his commentary on Euclid was driven by his efforts to position mathematics within the Jesuit's philosophy curriculum. It is

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genereall, is to be looked for, aswell of griff as stocke."; For a discussion of the Aristotelian separation of knowledge based on the ability to explain causes see Robert Westman, *The Copernican Question,* 32-33.

[13] Clavius was familiar with the request of superiors to apply mathematics to practical problems. In the early 1580s his efforts were focused on the development of a new calendar at the request of Gregory XIII. The calendar he created, the Gregorian calendar, is still in use today.

[14] The emergence of a realist-mathematical science is the first of the six revolutionary transformations outlined by Cohen. H. Floris Cohen, *How Modern Science Came Into the World: Four Civilizations, One 17th-Century Breakthrough,* (Amsterdam: Amsterdam Unisversity Press, 2010), xvi and 159-220.

[15] Ibid*.,* 146-147.

through his pedagogy that Clavius clearly belongs in a discussion of the renaissance of mathematics in the sixteenth century and is necessarily part of that "indispensable prelude to the scientific revolution" in which mathematical humanists, such as Federico Commandino, successfully argued to raise the status of their discipline for its contemplative value.[16] Like Commandino, Clavius was devoted to the restoration of ancient mathematics and believed that his discipline could lead to discoveries about the fundamental nature of Creation. Through looking up to the heavens, mathematicians could elevate their discipline to the status of natural philosophy. Yet, Clavius would also not be out of place in a discussion of mathematical practitioners, such as Henry Billingsley, who hoped for concrete benefits from the study of mixed branches of mathematics. Like Billingsley, Clavius was eager to explore the benefits of mathematics that came from looking down towards the earth. As it was presented in Clavius's commentary on *The Elements,* mathematics was both a science and an art, and pure mathematics, especially geometry, was the source of both knowledge and skill.

By providing a close comparison of three significant commentaries on *The Elements,* this dissertation provides insight into the ways in which Euclid, and thus the subsequent study of mathematics, was understood in the sixteenth century. Each of the three commentaries studied here contains a unique understanding of mathematics as a science, an art, or as a combination of these two approaches to the world. This

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[16] Paul Rose, *The Italian Renaissance of Mathematics: Studies on Humanists and Mathematicians from Petrarch to Galileo* (Geneva: Librairie Droz, 1975), 2.

dissertation thus offers a necessary starting point to further examinations of how sixteenth-century mathematicians learned their subject. Antonella Romano has already called attention to the need for such work, and contributed some of her own. However, she has focused on the branches of mathematics, usually some form of mixed mathematics, taught after Euclid.[17] A study of readers of various versions of Euclid and the influence of early mathematics education on later mathematical practice is much to be desired as *The Elements* was almost always the first mathematical text from which a student would learn. Annotated copies of *The Elements* would provide a valuable resource to understand how such elementary texts were read.[18] Another approach would be to compare works written by various scholars in the seventeenth century to determine how the available versions of *The Elements* were used and incorporated into advanced mathematical scholarship. Within the Jesuit context, a study of readers could be done as part of an examination of educational practice and the classroom content of Jesuit schools, where Clavius's texts were part of the curriculum. Otto Cattenius's lectures in Mainz (1610/1611) could offer a valuable starting point for such research. A preliminary reading of these

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[17] Antonella Romano, "Teaching Mathematics in Jesuit Schools: Programs, Course Content, and Classroom Practices," in *The Jesuits II: Cultures, Sciences, and the Arts: 1540-1773,* ed. John O'Malley, Gauvin Alexander Bailey, Steven J Harris, T. Frank Kennedy. (Toronto: University of Toronto Press, 2006), 355-370.

[18] Renée Raphael has studied the readers of Galileo's *Two New Sciences.* Her approach could be an example for studies on readers of Euclid. She used marginal annotations as well as course materials from Pisa and Jesuit colleges to understand how Galileo's contemporaries read his work. Similar methods could prove fruitful for various editions of *The Elements.* Renée Raphael, *Reading Galileo: Scribal Technologies and the Two New Sciences,* (Baltimore: Johns Hopkins University Press, 2017).

lectures suggests that Cattenius relied heavily on Clavius's commentaries on *The Elements* and Sacrobosco's *Sphere*.[19]

Although the commentaries on *The Elements* and Sacrobosco's *Sphere* were Clavius's largest works, he wrote over a dozen textbooks on mathematical topics to accompany his ideal curriculum. Lattis has done an excellent study of Clavius's commentary on Sacrobosco's *Sphere* in which he traced the content of the text through all of the editions published during Clavius's life, but most of Clavius's other textbooks remain little examined. This dissertation only examined the first edition of Clavius's commentary on *The Elements.* It remains necessary to compare the first edition to the later versions. Such comparisons could shed light on the evolution of Clavius's methods for presenting his vision of mathematics within his commentary and, possibly, discover areas he found to be difficult for his students. In order to create a complete picture of Clavius's union of theory and practice in mathematics, it would be especially valuable to know how Clavius presented practical arithmetic in his *Epitome arithmeticae practicae* and practical geometry in his *Geometria practica* and how those textbooks were used by their readers both within and outside of Jesuit schools. Comparative studies between Clavius's work and similar books by other authors would further clarify his position in the intellectual discourse of sixteenth-

[19] Otto Cattenius, "Edition" in Albert Krayer, Mathematik im Studienplan der Jesuiten: Die Vorlesung von Otto Cattenius an Der Universität Mainz (1610/1611), (Stuttgart: Franz Steiner Verlag, 1991), 181-360. The beginning portion of the lectures defining mathematics and introducing Euclid's *Elements* follows the same outline as Clavius's prolog to *The Elements* (181-184). The diagram included for the first proposition (p. 188) follows Clavius's labelling scheme. For the Pythagorean Theorem (pp. 197-198), Cattenius differed from Clavius in that he only fully developed the first half of the proof, but he did outline the second half more completely than Commandino or Billingsley did in their texts.

century mathematicians. Such a study could also be extended into an examination of the later editions of his works. Some of his texts, including his commentary on Euclid, were republished long after his death.[20] Some were translated into various vernaculars. For example, Clavius's commentary on Theodosius's *Sphere* was published in English as late as 1721, and his practical arithmetic appeared in Italian in 1738. Those late editions could provide insight into how readers continued to use Clavius's texts after his death. This dissertation has laid the groundwork for further studies into readers in this Jesuit tradition of elementary mathematics texts.

Studies of Clavius's texts and their readers could shed light on the early development of the mathematical ideas that characterized the Scientific Revolution. Because of his position at the Collegio Romano, Clavius played a significant role in the education of those who lived at the start of the Scientific Revolution. His treatment of mathematics likely influenced early pioneers of the Scientific Revolution. One of his students, Gregory of St. Vincent (1584-1687) has been credited with developing mathematical techniques necessary to analytic geometry and, later, calculus.[21] Such luminaries as Descartes and Galileo can be found among Clavius's readers and correspondents.[22] And, the fact that some of his texts were translated and

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[20] For example, the commentary on Euclid was published in Frankfurt by Ionas Rosa in 1654. It is a reprint of Clavius's 1607 edition.

[21] Margaret Baron, *The Origins of Infinitesimal Calculus*, (Oxford: Pergamon Press, 1969), 134.

[22] Descartes studied at the Jesuit college of La Flèche in the second decade of the seventeenth century. Galileo corresponded with Clavius and met him in Rome in the late 1580s, when Clavius was working on his 1589 edition of *The Elements*. Wallace has posited that Clavius was the source through which Galileo obtained lecture notes from the Jesuit philosophy professors, and it stands to reason that the Jesuit mathematician would have shared his own work with Galileo as well. See William Wallace, "Galileo's Jesuit Connections and Their Influence on His Science," ed. by Mordechai Feingold, (Cambridge, MA: The MIT Press, 2003), 103-104. The list of members of Clavius's academy in Rome constructed by Ugo Baldini includes John Hay, James Bosgrave, Matteo Ricci, Christoph Grienberger,

republished in the eighteenth century suggests that his works remained valuable to students throughout the Scientific Revolution. Much remains to be done in order to understand what exactly gave Clavius's textbooks their longevity.

Clavius's contemporaries would not have been surprised by the argument that his significance to the development of mathematics and science lay in his teaching. In fact, they knew him as the "Euclid of our times." The one portrait made of Clavius in his lifetime was created in 1606 by the Italian engraver Francesco Villamena, and it shows Clavius as a Jesuit teacher.[23] In it, the priest is seen seated at his desk surrounded by the various tools of his trade (Figure 45).[24] On the wall behind him hang an astrolabe and a quadrant. On his desk lie an armillary sphere, a straightedge, a pen, a penknife, a book and a sheet of paper with geometrical figure on it. In front of Clavius are stacks of books, on the pages of one of which the viewer can make out what appears to be Clavius's pair of diagrams for the Pythagorean Theorem. Clavius himself is holding a compass and seems to be deep in thought. While the three astronomical tools suggest Clavius's role as an astronomer and his own interest in that field, the image primarily depicts a teacher. Indeed, the astrolabe and the quadrant are subjects of Clavius's practical astronomy textbooks, and the armillary sphere was a teaching tool. The double diagram for the Pythagorean Theorem suggests that the

Giuseppe Biancani, Otto van Malecote and Paul Guldin, among others. See Baldini, "The Academy of Mathematics," 72-74.

[23] The portrait the appears on the frontispiece of Clavius's *Opera Mathematica* is clearly based on Villamena's 1606 engraving.

[24] Villamena was known for portraits in unromanticized settings that depicted tools of the subjects' trade. See Lattis, *Between Copernicus and Galileo,* 24-25.

Figure 45: Francesco Villamena's Portrait of Christopher Clavius (1606)

N.B. If for some reason the writing on the pages is not visible, please consult the Metropolitan Museum of Art's copy online.

Source: The Metropolitan Museum of Art,
http://www.metmuseum.org/art/collection/search/342372

open book is meant to be Clavius's commentary on Euclid's *Elements*. It is easy to imagine the man in the picture contemplating how best to teach mathematics.

And yet, in Villamena's portrait, the true significance of Clavius's efforts to combine mathematical sciences and mathematical arts is not visible. While the image captures that Clavius dedicated most of his life to developing the mathematics curriculum of the Jesuit schools, and accurately depicts the commentary on Euclid's *Elements* and various works on astronomy as the most significant results of the priest's efforts, it did not show that Clavius in any way advanced mathematics education. A curriculum composed of Euclid and astronomy could hardly distinguish Clavius from other mathematics teachers, before or during the sixteenth century. In medieval universities, geometry and astronomy were taught as part of the quadrivium, and both Euclid and Sacrobosco's *Sphere*, Clavius's choice for astronomy instruction, were commonly used texts, and several scholars wrote commentaries on both of those works.[25] And while the Jesuit curriculum required only Euclid and one other branch of mathematics (for which Clavius suggested astronomy), it was a far cry from Clavius's ideal program of mathematics study, and even though Euclid and Sacrobosco were not new, Clavius's commentaries were so substantial that they are more properly considered as independent texts based on the older works than as versions of those works. A close examination of Clavius's pedagogical project shows that, through a consistent combination of mathematics as a theoretical study and mathematics as a practical study, the Jesuit mathematics professor fulfilled the

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[25] Lynn Thorndike, *The Sphere of Sacrobosco and its Commentators,* 42-43.

promise of Neoplatonist rhetoric about mathematics' intermediate role between physics and metaphysics.

Appendix A
Clavius's Three Curricula

Translated from Christopher Clavius, "Ordo servandus in addiscendis disciplinis mathematicis (1581)." In *Monumenta Paedagogica Societatis Iesu Vol. VII,: Collectanea de Ratione Studiorum Societatis Iesu,* edited by Ladislaus Lukacs. Rome: Institutum Historicum Societatis Iesu, 1992.

The order to be kept in learning the mathematical disciplines (In the year 1581)

1. The first four books of Euclid.

    After our interpretation, with the superfluous scholia omitted.

2. The precepts of practical arithmetic are necessary to learned men. Those are addition, subtraction, multiplication and division of numbers as integers and as fractions. And also a very brief treatment of proportion, proportionality, and progressions, and one of the golden rule of proportion, which is called the Rule of Three, and the extraction of roots.

    Of these things we will write a brief compendium. In the interim the practical arithmetic of Gemma Frisius can be read, or the more useful precepts of this here enumerated can be selected from the first book of Michael Stiefelius's Arithmetic, which treats a great abundance of them with brevity.

3. The Sphere as briefly as possible or any other preferable introduction to astronomy. To which can be added the rules for looking at ecclesiastical computation which are necessary to learned men.

On this also we are putting forth a compendium. In the interim however, our commentary on the Sphere of Sacrobosco is sufficient, with the operations of sines, the treatment of isoperimetry, etc. omitted since they are added to this below.

4. The fifth and sixth books of Euclid.

From our interpretation, as above.

5. The use of the geometrical quadrant and the astronomical quadrant, and if it seems good, also of other instruments, which are managed around measurements, which is things like Jacob's staff, a torquetum, etc.

Of these we will compose a short work. In the interim, truly the use of these things can be selected from Orontius, or from any other who is pleasing, as from Gemma Frisius in the universal astrolabe, or Peurbach etc.

6. The next four books of Euclid, clearly the seventh, eighth, ninth and tenth. Or preferably in their place, the first three books of Jordanus's Arithmetic, or that of Gaspar Lax, where there is more contained, than in the arithmetic written by Euclid; to which could be added the Arithmetic of Maurolico.

On account of speculative music, from all of these I would choose the arithmetic of Jordanus. Moreover, if the meanness of time bears down, all of these things can be omitted, or put off to the end with the art of algebra. Especially if we only want to investigate that which are looking at astronomy and geography and those that can be reduced to them, as are the description of horology, perspective, etc.

7. The art of algebra, along with those things which are required for its use; which is the algorithms of numbers proportionality of denominators, roots, binomials etc. Whereby, indeed the algorithoms of roots and binomials cannot be understood and demonstrated before the tenth book of Euclid.

This we will treat. In the interim, however, the Algebra of Stiefelus, Johannes Seubel, or Peletier can be taken up. Others of these arts, because they are not necessary to all things, can be put off, as was said of the preceding four books of Euclid.

8. The last of books of Euclid; which are of course the eleventh, the twelfth, the thirteenth, and the fifteenth, together with the sixteenth which we have added from Candalla.

Form our edition, as prior. However, on account of shortness of time, books eleven and twelve can be read, and the rest can be put off to a more convenient time for the reason produced in number six. However, it does not seem good to omit the tenth proposition of the thirteenth book and the converse of the ninth proposition of the same book and the scholion of the tenth proposition. For these are necessary to understanding the study of the science of sines. Moreover, we have divided all the of the books of Euclid into the four above mentioned classes, not because not all of them can be learned at once from the beginning, but lest so many geometrical demonstrations cause difficulty for the studious, if they should be learned all at once and continually.

9. The treatment of sines, together with the use of tables of sines, and also the use of sines without tables, but only for straight lines.

We will elaborate on this, accepting as well the table of Peter Apianus or Johannes Regiomontanus, which are extended to the single minute of a degree.

10. The elements of the Sphere of Theodosius.

From the tradition of Maurolico.

11. A compendium of spherical triangles. For Menelaus, Johannes Regiomontanus and Maurolico have written very broadly on these. This pertains some to the elements of conics of Apollonius, which kind of study are the first fourteen propositions, which do the most to make understanding those things which are said of the construction of the astrolabe and of the shadows of the gnomon in horology. Indeed the rest seems to be as good as it is necessary for learned men; granted that they give the most pleasant and acute speculations.

This we will compose and provide for, as it will be printed in one volume with the elements of the sphere of Theodoisus. And this compendium will be together with the early propositions of Apollonius.

12. The structure of astrolabes demonstrated, together with its use. Leaving out however, the last part of the altimetry scale, since that is already explained in the use of the geometric quadrant etc. This ought to be related together with the use of the material sphere or globe.

> We will treat this, in a certain compendium encouraged by Maurolico and will will put in the use of the globe or material sphere.

13. Description of solar horology of all kinds. Where even the reason for describing horology can be explained from the *Analemmate* of Ptolemy, if time allows.

> This we will briefly sum up, together with a description of horology from Ptolemy's diagrams of sundials, in a compendium describing horology.

14. Geography.

> We will compose this. In the interim, however Gemma Frisius can be read, *De orbis divisione* et alii.

15. Precepts for measuring the areas of all figures, both of planes and of solids. To which can be added a treatment of the isoperimetry of figures, if it was earlier surpassed the little book of Archimedes on the dimensions of circles, and the divisions of surfaces from Machometo or Federico Commandino.

> These we will also treat. In the interim, however, Orontius can be read. Isoperimetry of figures is contained in our commentary on the Sphere. A truly impressive treatment of the division of surfaces is the work of Federico of Pisa [i.e. Commandino].

16. Perspective together with burning mirrors.

This we will write. Orontius has published a treatment of burning mirrors. They even say that Archimedes or Ptolemy published on this, but I have not seen it.

17. Various phenomena and problems of astronomy for the comprehension of the whole doctrine of the prime mover. To which can be added the little work of Peter Nunez on twilights.

We will also briefly investigate twilights. Truly we will report on the problem in another way than is contained in the little work of Peter Nunez.

18. A treatment of the motion of the planets and eight spheres, together with the use of the Alphonsine tables, or others. Also, if was earlier surpassed, a short treatment of algorithms of fractions of astronomy.

This we will publish. It will answer, moreover, to the treatment of the remaining eleven books of the Almagest of Ptolemy. We will also write the models briefly and clearly in in tables. But perhaps it will be a greater influence if we write commentaries on the Epitome of Johannes Regiomontanus.

19. The speculative music of Lefeve d'Etaples, which precedes the Arithmetic of Jordanus, if it was not previously taught.

This is related by Lefevre d'Etaples.

20. A few works of Archimedes, together with the discovery the mean proportions of two lines, and the doubling of the cube, and also, last, the squaring of the circle.

We will illustrate some commentaries on these.

21. Questions of mechanics from Hero, Pappus, and Aristotle, etc.

Perhaps we will write some compendium on these.

22. They will also add a few a few propositions from Serenus on cylindrical sections, in which the second proposition ought to be demonstrated, likewise the same for cylindrical portions evidently having elliptical bases.

A second order, shorter than those, to be followed, which does not provide for a most perfect understanding of mathematical things

1. The first four books of Euclid

   From our account.

2. The precepts of practical arithmetic of addition, subtraction, etc, of both integer numbers and fractions, together with a short treatment of proportions, proportionality, and progressions; which should immediately be followed by the golden rule of proportions and the extraction of roots.

   Ours or Michael Stifelius.

3. Most briefly, the sphere and ecclesiastical computation.

   Johannes de Sacrobosco with our commentary.

4. Books five and six of Euclid.

   From our account.

5. The use of the geometrical quadrant and the astronomical quadrant.

   Ours or from Orontius, Gemma Frisius, etc.

6. Books 11 and 12 of Euclid together with the tenth proposition of the thirteenth book with its scholion, and also the converse of the ninth proposition of the same book. For these all are necessary to understanding the science of sines.

   From our account.

7. The treatment of sines, as above.

From our account.

8. Theodosius's elements of the sphere.

9. A compendium on spherical triangles together with the first fourteen propositions of Apollonius on the elements of conics.

10. The structure and use of astrolabes, leaving out the last part of the measurement, since that was related in the use of the geometrical quadrant.

Ours.

11. A demonstrative description of horology.

Ours.

12. Geography.

Ours or Gemma Frisius's *De orbis divisione*.

13. The precepts of measuring figures either plane or solid. Together with our treatment of isoperimetric figures and the division of surfaces from Federico Commandino.

Ours or Orontius.

14. Perspective, together with Orontius's burning mirrors.

Ours or that which is common.

15. Phenomena and problems of astronomy regarding the prime mover necessary to learned men, together with computations of twilights.

Ours.

16. A treatment of the motions of the planets and eight spheres, together with the use of the Alphonsine tables, etc. Still, these tables should precede the algorithm of fractions of astronomy.

Ours or the Epitome of Johannes Regiomontanus on Ptolemy's *Almagest*.

17. The dimension of the circle from Archimedes together with the discovery of the place of mean proportions between two lines, and the doubling of the cube, and also, last, the squaring of the circle.

From our account.

18. The speculative arithmetic of Jordanus and music from Lefevre d'Etaples.

From the account of Lefevre d'Etaples.

19. The rule of algebra, together with the practice of those things which are required for that, without the demonstrations which depend on the tenth book of Euclid. Truly, those which are deduced from the second book can be brought in.

Ours, or Peletier, or Johannes Seubel.

A third most concise order adapted to a course of mathematics which should be summed up in two years.

The First Year

1. The first four books of Euclid.

    These two can be read from the beginning of studies just to the end of January.[1]

2. Practical arithmetic as prior.

3. The sphere and most briefly ecclesiastical computation.

    This can be summed up just to Easter.

4. Books five and six of Euclid.

5. The use of the geometrical quadrant and the astronomical quadrant.

    These two are treated up to Pentecost.

6. Perspective.

7. A compendium of horology without demonstrations.

    These two are read up to the end of the year.

[1] It seems that the "duo" refers to the first four books of Euclid and practical arithmetic.

The Second Year

8. Books eleven and twleve of Euclid together with the tenth proposition of the thirteenth book and its scholion, and also the converse of the ninth proposition of the same book.

This can be done from the renewal of studies to the Nativity of the Lord.

9. A treatment of sines together with their uses for various phenomena and problems for studying the prime mover, without demonstrations. There can be a few demonstrations from the sixth of Euclid about straight lines, in order that the quantity of daylight and hours may be found from the altitude of the sun and vice versa, etc.

10. Geography

These two are read just to Lent.

11. The structure and use of the astrolabe, leaving off the last part, as prior. They should precede, however, Apollonius's first fifteen propositions of conic elements.

12. The theory of the planets bringing in a few demonstrations, together, most briefly, with the use of the Alphonsine tables.

These two can be quickly done up to the feast of John the Baptist.

13. The dimension of the circle from Archimedes, and its quadrature, which approaches the next truth, together with the discovery of the mean proportions of two lines from Eratosthenes, Diocles, and Nicomedes, and the doubling of the cube.

14. The standard of algebra, together with the practice of those things which are required for it, adding only those demonstrations which depend on the second book of Euclid.

15. The precepts for the measuring of figures as above.

These three are executed just to the end of the year.

Students are to be encouraged after the eleventh and twelfth books of Euclid to read over Theodosius's elements of the sphere and something from speculative arithmetic by themselves.

However, I recommend the second order, rather than this one, if it can be summed up in two years. Moreover, it can, if the students are capable and desire to be taught. The first however, is the most complete of all.

Appendix B
The Postulates and Axioms, untranslated

The postulates and axioms found in *The Elements* are listed in full in Table 1 in Chapter 3 and Table 4 in Chapter 4. In order to allow the reader to see the small differences between Commandino's and Clavius's Latin versions of these postulates and axioms, I have reproduced those tables here with those two authors' enunciations in their original Latin. I have also added brief lists of the equivalences between the authors' enunciations.

Table 7: The Postulates and Axioms of Book One (Latin)

Commandino's Postulates	**Billingsley's Postulates**	**Clavius's Postulates**
1.Postuletur a quovis puncto ad quodvis punctum rectam lineam ducere.	1. From any point to any point, to draw a right line.	1. Postuletur, ut a quovis puncto in quodvis punctum, rectam lineam ducere concedatur.
2.Rectam lineam terminatam in continuum, & directum producere.	2. To produce a right line finite, straight forth continually.	2. Et rectam lienam terminatam in continuum recta producere.
3.Quovis centro, & interuallo circulum describere.	3. Upon any centre and at any distance, to describe a circle.	3. Item quovis centro, & interuallo circulum describere.
4. Omnes angulos rectos inter se aequales esse.	4. All right angles are equall the one to the other.	4. Item quacunque magnitudine data, sumi posse aliam magnitudinem vel maiorem, vel minorem.
5.Et si in duas lineas recta linea incidens interiores & ex eadem parte angulos duobus rectis minores fecerit, rectas lineas illas in infinitum productas, inter se convenire ex ea parte in qua sunt anguli duobus rectis minores.	5. When a right line falling upon two right lines, doth make on one and the selfe same syde, the two inwarde angles less then two right angles, then shall the two right lines beying produced in length concurre on that part, in which are the two angles lesse then two right angles.	
	6. That two right lines include not a superficies.	

Table 7: The Postulates and Axioms of Book One (Latin), continued

Commandino's Axioms	**Billingsley's Axioms**	**Clavius's Axioms**
1. Quae eidem aeqaulia, et inter se sunt aequalia.	1. Thinges equall to one and the selfe same thyng: are equal also the one to the other.	1. Quae eidem aequalia, & inter se sunt aequalia.
2.Et si aequalibus aequalia adijciantur tota sunt aequalia.	2. And if ye adde equall thinges to equall things: the whole shalbe equall.	2. Et si aequalibus aequalia adiecta sint, tota sunt aequalia.
3. Et si ab aequalibus aequlia auferantur, reliqua sunt aequlia.	3. And if from equall thinges, ye take away equall thinges: the things remayning shall be equall.	3. Et si ab aequalibus abalata sint, quae reliquuntur, sunt aequalia.
4. Et si inaequalibus aequalia adijciantur, tota sunt inaequlia.	4. And if from unequall thinges ye take away equall thinges: the thynges which remayne shall be unequall.	4. Et si inaequalibus aequalia adiecta sint, tota sunt inaequalia.
5. Et si ab inaequalibus aequalia auferantur, reliqua sunt inaequalia.	5. And if to unequall thinges ye adde equall thinges: the whole shall be unequal.	5. Et si ab inaequalibus aequalia ablta, reliqua sunt inaequalia.
6. Et quae eiusdem dupla, inter se sunt aequalia.	6. Thinges which are double to one and the selfe same thing: are equall the one to the other.	6. Et quae eiusdem duplicia sunt, inter se sunt aequalia.
7. Et quae eiusdem dimidia inter se sunt aequalia.	7. Thinges which are the halfe of one and the selfe same thing are equal the one to the other.	7. Et quae eiusdem sunt dimidia, inter se aequalia sunt.
8. Et quae sibi ipsis congruint, inter se sunt aequlia.	8. Thinges which agree together; are equall the one to the other.	8. Et quae sibi mutuo congruunt, ea inter se sunt aequalia.
9. Totum est sua parte maius.	9.Every whole is greater then his part.	9. Et totum sua parte maius est.
10. Duae rectae lineae spacium non comprehendunt.		10. Item, omnes angule recti sunt inter se aequales.

Table 7: The Postulates and Axioms of Book One (Latin), continued

		Clavius's Axioms, Cont. 11. Et si in duas rectas lineas altera recta incidens, internos ad easdemque partes angulos duobus rectis minores faciat, duae illae rectae lineae in infinitum productae sibi mutuo incident ad eas partes, ubi sunt anguli duobus rectis minores. 12. Duae rectae lineae spatium non comprehendunt. 13. Duae lineae rectae non habent unum & idem segmentum commune. 14. Si aequalibus inaequalia adijciantur, erit totorum excessus, adiunctorum excessui aequalis. 15. Si inaequlibus aequalia adiungantur, erit totorum excessus, excessus eorum, quae a principio errant, aequalis. 16. Si ab aequalibus inaequalia demantur, erit residuorum excessus, excessui ablatorum aequalis. 17. Si ab inaequalibus aequalia demantur, erit residuorum excessus, exessui totorum aequalis. 18. Omne totum aequale est omnibus suis partibus simul sumptis. 19. Si totum totius est duplum, & ablatum ablati; erit & reliquum reliqui duplum.

Equivalencies between postulates and axioms in Book One:

- Postulates 1, 2, and 3 are the same in all three texts.
- Commandino and Billingsley also share postulates 4 and 5. These are, respectively, the tenth and eleventh axioms in Clavius's text.
- Clavius's fourth postulate is unique to his text.
- The first nine axioms in all three texts are equivalent.
- Commandino's tenth axiom can be found in Billingsley's sixth postulate and Clavius's twelfth axiom.
- The remaining axioms found in Clavius included are unique to his text.

Table 8: The Postulates and Axioms of Book Seven (Latin)

	Billingsley	**Clavius**	**Commandino**
Postulate 1		Postuletur, cuilibet numero quotlibet posse sumi aequlaes ,vel multiplices.	Cuilibet numero quotlibet sumi posse aequales, vel multiplices.
Postulate 2		Quolibet numero sumi posse maiorem.	Quolibet numero sumi posse maiorem.
Postulate 3			Numerus infinite augetur, sed non infinite diminuitur.
Axiom 1	The lesse part is that which hath the greater denomination: and the greater part is that, which hath the lesse denomination.	Qui numeri aequalium numerorum, vel eiusdem aeque multiplices sunt, inter se sunt aequales.	Quicumque eiusdem, vel aequalium aequemultiplices fuerint, & ipsi inter se sunt aequales.
Axiom 2	Whatsoever numbers are equemultiplices to one & the selfe same number, or to equall numbers, are also equall the one to the other.	Quorum idem numerus aeque multiplex est, vel aeque multiplices sunt aequales, inter se aequales sunt.	Quorum idem numerus aeque multiplex fuerit, vel quorum aeque multiplices fuerint aequales, & ipsi inter se aequales sunt.
Axiom 3	Those numbers to whome one and the selfe same number is equimultiplex, or whose euqemultiplices are equall: are also equall the on to the other.	Qui numeri aequalium numerorum, vel eiusdem, eadem pars, vel eaedem partes fuerint, aequales inter se sunt.	Quicumque eiusdem numeri, vel aequalium eadem pars, vel eadem partes fuerint, & ipsi inter se sunt aequales.
Axiom 4	If a number measure the whole, and a part taken away: it shall also measure the residue.	Quorum idem numerus, vel aequales eadem pars, vel eaedem partes fuerint, aequales inter se sunt.	Quorum idem, vel aequales numeri eadem pars, vel aedem partes fuerint, inter se sunt aequales.
Axiom 5	If a number measure any number: it also measureth every number that the sayd number measureth.	Unitas omnem numerum per unitates quae in ipso sunt hoc est, per ipsummet numerum metitur.	Omnis numeri pars est unitas ab eo denominata, binarii enim numeri unitas pars est ab ipso binario denominata, quae dimidia dicitur, ternarii vero unitas est pars, quae a ternario denominata tertia dicitur, quaternarii quarta, & ita in aliis.
Axiom 6	If a number measure two numbers, it shall also measure any number composed of them.	Omnis numerus se ipsum metitur per unitatem.	Unitas omnem numerum metitur per unitates, quae in ipso sunt.

Table 8: The Postulates and Axioms of Book Seven (Latin), continued

	Billingsley	Clavius	Commandino
Axiom 7 -	If in numbers there be proportions how manysoever equall or the selfe same to one proportion: they shall also be equall or the selfe same the one to the other.	Si numerus numerum multiplicans, aliquem produxerit, metietur multiplicans productum per multipicatum, multiplcatus autem eundem per multiplicantem.	Omnis numerus se ipsum metitur.
Axiom 8		Si numerus numerum metiatur, & ille, per quem metitur, eundem metietur per eas, quae in metiente sunt, unitates, hoc est, per ipsum numerum metientem.	Si numerus metiatur numerum, & ille, per quem metitur, eundem metietur per eas, quae sunt in metiente unitates.
Axiom 9		Si numerus numerum metiens, multiplicet eum per quem metitur, vel ab eo multiplicetur, illum quem metitur, producet.	Quicumque numerus alium metitur, multiplicans eum, vel multiplicatus ab eo, per quem metitur, illum ipsum producit.
Axiom 10		Numerus quotcunque numeros metiens, compositum quoque ex ipsis metitur.	Si numerus numerum alium multiplicans aliquem produxerit, multiplicans quidem productum metitur per unitates, quae sunt in multiplicato; multiplicatus vero metitur eundem per unitates, quae sunt in multiplicante.
Axiom 11		Numerus quemcunque numerum metiens, metitur quoque omnem numerum, quem ille metitur.	Quicumque numerus metitur duos, vel plures, metietur quoque eum, qui ex illis componitur.
Axiom 12		Numerus metiens totum & ablatum, metitur & reliquum.	Quicumque numerus metitur aliquem, metietur quoque eum, quem mille ipse metitur.
Axiom 13			Quicumque numerus metitur totum & ablatum etiam reliquum metietur.

Equivalencies between postulates and axioms in Book Seven:

- Clavius and Commandino share the first two postulates. The third is unique to Commandino's text.
- Clavius and Commandino's first four axioms are equivalent to each other. Of those, the first axiom in Clavius's and Commandino's texts are the same as the second in Billingsley's.
- Likewise, the second axiom in the Latin texts is Billingsley's third axiom.
- Billingsley does not include the third and fourth axioms from the Latin texts.
- Commandino's fifth axiom is related to Billingsley's first. Both are explanations of denomination, but they are not quite the same in that Commandino focuses on creating the denominations by dividing unity, and Billingsley merely establishes the relative sizes between denominative numbers (i.e. His axiom spells out that larger denominators create smaller numbers, e.g. a third is less than a half because three is greater than two.) Clavius included no such axiom.
- Clavius's fifth axiom is equivalent to Commandino's sixth. Their respective sixth and seventh axioms likewise cover the same principle, although only Clavius stated that it is by unity that each number measures itself. Billingsley included no such axioms.

- Billingsley does not include equivalents for the next three axioms in the Latin texts either. Of those axioms, Clavius's seventh is equivalent to Commandino's tenth. The eighth and ninth axioms in each text are equivalent.
- Clavius's tenth, eleventh, and twelfth axiom are equivalent to Commandino's eleventh, twelfth, and thirteenth, respectively. And they are equivalent to Billingsley's sixth, fifth and seventh, respectively.

Appendix C
Clavius's Demonstrations for the Propositions Discussed

Book One, Problem 1/Proposition 1

On a given bounded straight line to create an equilateral triangle.

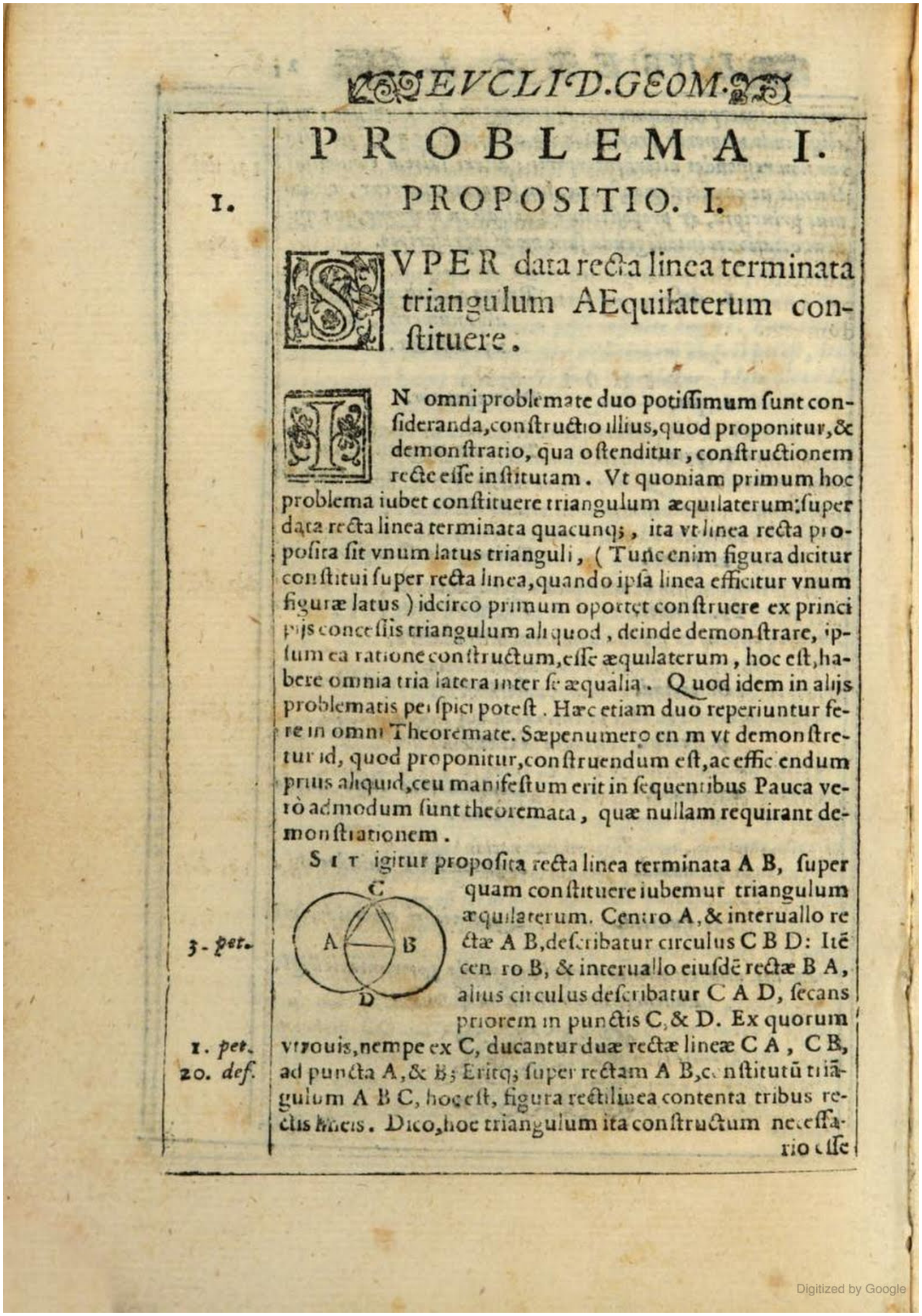
Let there be, then, a straight line bounded by AB on which we are instructed to create an equilateral triangle. On center A and with an interval of the straight line AB, let there be drawn circle CBD. Likewise, on center B and with an interval of the same line BA, let there be described another circle, CAD (Postulate 3), cutting the first in points C and D. From either of these points, for example from C, let there be drawn two straight lines, CA and CB, to points A and B (Postulate 1).

Super data recta linea terminata triangulum Aequilaterum constituere.

Sit igitur proposita recta linea terminata AB, super quam constituere iubemur triangulum aequilaterum. Centro A, & intervallo rectae AB, describatur circulus CBD: Item centro B, & intervallo eiusdem rectae BA, alius circulus describatur CAD (3. pet.), secans priorem in punctis C, & D. Ex quorum utrovis, nempe ex C, ducantur duae rectae lineae CA, CB, ad puncta A, & B (1. pet.);

And there will be constructed on the straight line AB the triangle ABC, that is a straight-lined figure contained in three straight lines (Definition 20). I say that this triangle so constructed is necessarily equilateral. Since the straight lines AB and AC are drawn from the center A to the circumference of circle CBD, the straight line AC will be equal to the straight line AB. In turn, because the straight lines BC and BA are drawn from the center B to the circumference of circle CAD, straight line BC will be equal to straight line BA (Definition 15). So, therefore, AC, like BC is equal to the straight line AB (Axiom 1). Therefore, AC and BC are equal to each other, and, thus, triangle ABC will be equilateral. Therefore, on the given bounded straight line, etc. Which was required to be done.

Eritque super rectam AB, constitutum triangulum ABC, hoc est, figura rectilinear contenta tribus rectis lineis (20. def.). Dico, hoc triangulum ita constructum necessario esse aequilaterum. Quoniam rectae AB, AC, ducuntur ex centro A, ad circumferentiam circuli CBD, erit reca AC, rectae AB, aequalis: Rursus quia rectae BC, BA, ducuntur ex centro B, ad circunferentiam circuli CAD, erit recta BC, rectae BA, aequalis (15. def.). Tam igitur AC, quam BC, aequalis est rectae AB. Quare & AC, BC, inter se aequales erunt (1. pron.), atque idcirco triangulum ABC, erit aequilaterum. Super data ergo recta linea terminate, etc. Quod faciendum erat.



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PROBLEMA I.

1. PROPOSITIO. I.

SVPER data recta linea terminata triangulum AEquilaterum constituere.

IN omni problemate duo potissimum sunt consideranda, constructio illius, quod proponitur, & demonstratio, qua ostenditur, constructionem recte esse institutam. Vt quoniam primum hoc problema iubet constituere triangulum æquilaterum: super data recta linea terminata quacunq; , ita vt linea recta proposita sit vnum latus trianguli, (Tunc enim figura dicitur constitui super recta linea, quando ipsa linea efficitur vnum figuræ latus) idcirco primum oportet construere ex principijs concessis triangulum aliquod , deinde demonstrare, ipsum ea ratione constructum, esse æquilaterum , hoc est, habere omnia tria latera inter se æqualia . Quod idem in alijs problematis perspici potest . Hæc etiam duo reperiuntur fere in omni Theoremate. Sæpenumero enim vt demonstretur id, quod proponitur, construendum est, ac efficiendum prius aliquid, ceu manifestum erit in sequentibus Pauca verò admodum sunt theoremata , quæ nullam requirant demonstrationem .

SIT igitur proposita recta linea terminata A B, super quam constituere iubemur triangulum æquilaterum. Centro A, & interuallo rectæ A B, describatur circulus C B D: Itē centro B, & interuallo eiusdē rectæ B A, alius circulus describatur C A D, secans priorem in punctis C, & D. Ex quorum vtrouis, nempe ex C, ducantur duæ rectæ lineæ C A , C B, ad puncta A, & B; Eritq; super rectam A B, cōnstitutū triāgulum A B C, hoc est, figura rectilinea contenta tribus rectis lineis. Dico, hoc triangulum ita constructum necessario esse

3. pet.

1. pet. 20. def.

Figure 46: Book One, Problem 1/Proposition 1

rio esse æquilaterum. Quoniam rectæ A B, A C, ducuntur ex centro A, ad circunferentiam circuli C B D, erit recta A C, rectæ A B, æqualis: Rursus quia rectæ B C, B A, ducuntur ex centro B, ad circunferentiam circuli C A D, erit recta B C, rectæ B A, æqualis. Tam igitur A C, quam B C, æqualis est rectæ A B. Quare & A C, B C, inter se æquales erunt, atq; idcirco triangulum A B C, erit æquilaterum. Super data ergo recta linea terminata, &c. Quod faciendum erat. 15. def. 1. pron.

SCHOLION.

Vt autẽ videas, plures demõstrationes in vna propositione contineri, placuit primã hanc propositionẽ resoluere in prima sua principia, initio facto ab vltimo syllogismo demõstratiuo. Si quis igitur probare velit, triangulũ A B C, constructũ methodo prædicta, esse æquilaterum, vtetur hoc syllogismo demonstrante.

Omne triangulum habens tria latera æqualia, est æquilaterum. 23. def.

Triangulum A B C, tria habet æqualia latera.

Triangulum igitur A B C, est æquilaterum.

Minorem confirmabit hoc alio syllogismo.

Quæ eidem æqualia sunt, inter se quoq; sunt æqualia. 1. pron.

Duo latera A C, B C, æqualia sunt eidem lateri A B.

Igitur & duo latera A C, B C, inter se æqualia sunt. Ac propterea omnia tria latera A B, B C, A C, æqualia existunt.

Minorem verò huius syllogismi hac ratione colliget.

Lineæ rectæ a centro ductæ ad circunferentiam circuli, inter se sunt æquales. 15. def.

Lineæ A B, A C, sunt ductæ a centro A, ad circunferentiam C B D.

Sunt igitur lineæ A B, A C, æquales inter se.

Eademq; ratione erunt lineæ A B, B C, æquales; cũm ducantur a centro B, ad circunferentiam C A D. Quamobrem minor præcedentis syllogismi tota confirmata erit.

Non aliter resolui poterunt omnes aliæ propositiones non solum Euclidis, verum etiam cæterorum Mathematicorum.

Negligunt

Figure 46: Book One, Problem 1/Proposition 1 (continued)

Book One, Problem 2/Proposition 2

Note in my discussion of this proof (Chapter 5) I used Billingsley's diagram as the foundation for my discussion because he included a diagram for all four cases. Clavius's labels are a little different and the order of the proof is also changed. His changes ensure that the desired line, in his text, AG, is the last line drawn.

On a given point, to draw a straight line equal to a given straight line.

Let there be a given point A and a given straight line BC, equal to which another straight line is required on be drawn on point A. With either extreme of line BC, for example C, made the center, describe a circle BC on the interval of straight line BC (Postulate 3.). And from A to the center C draw the straight line AC (Postulate 1), (unless the point A was on the straight line BC: indeed, then from the drawn line AC is selected, as the second figure indicates.).

Ad datum punctum, datae rectae lineae aequalem rectam lineam ponere.

Sit punctum datum A, & data recta linea BC; cui aliam rectam aequalem ponere oportet ad punctum A. Facto alterutro extremo lineae BC, nempe C, centro describatur circulus BE, intervallo rectae BC (3 pet.). Et ex A, ad centrum C, recta ducatur AC (1 pet.); nisi punctum A, intra rectam BC, fuerit: Tunc enim pro linea ducta sumetur AC, ut secunda figura indicat.)

On the straight line AC, the equilateral triangle ACD is constructed, turned either above or below, as is pleasing (I.1). The two sides of the triangle which were just created, DA and DC, are extended in the directions of AC. DC, which is opposite the given point A is extended to the circumference in E. DA, which is opposite the center C, is extended however far to point F (Postulate 2). Then with center D and an interval of the straight line DE, crossing over the center C, describe another circle EG, cutting straight line DF in G (Postulate 3). I say that straight line AG, which is set on the given point A, is equal to the given line BC. Since DE, DG are led from the center D to the circumference EG, they are equal to each other (Definition 15).

Super recta vero AC, construatur triangulum aequilaterum ACD (1 primi.), sursum, aut deorsum versus, ut libuerit; cuius duo latera modo consituta DA, DC, versus rectam AC, extendantur; DC, quidem opposita puncto dato A, usque ad circunferentiam in E; DA, vero opposita centro C, quantumlibet in F (2 pet.). Deinde e centro D, intervallo vero rectae DE, per C, centrum transeuntis, alter circulus describatur EG, secans rectam DF in G (3 pet.). Dico rectam AG, quae posita est ad punctum datum A, aequalem esse datae rectae BC. Quoniam DE, DG, ductae sunt ex centro D, ad circunferentiam EG, ipsae inter se aequales erunt (15 def.).

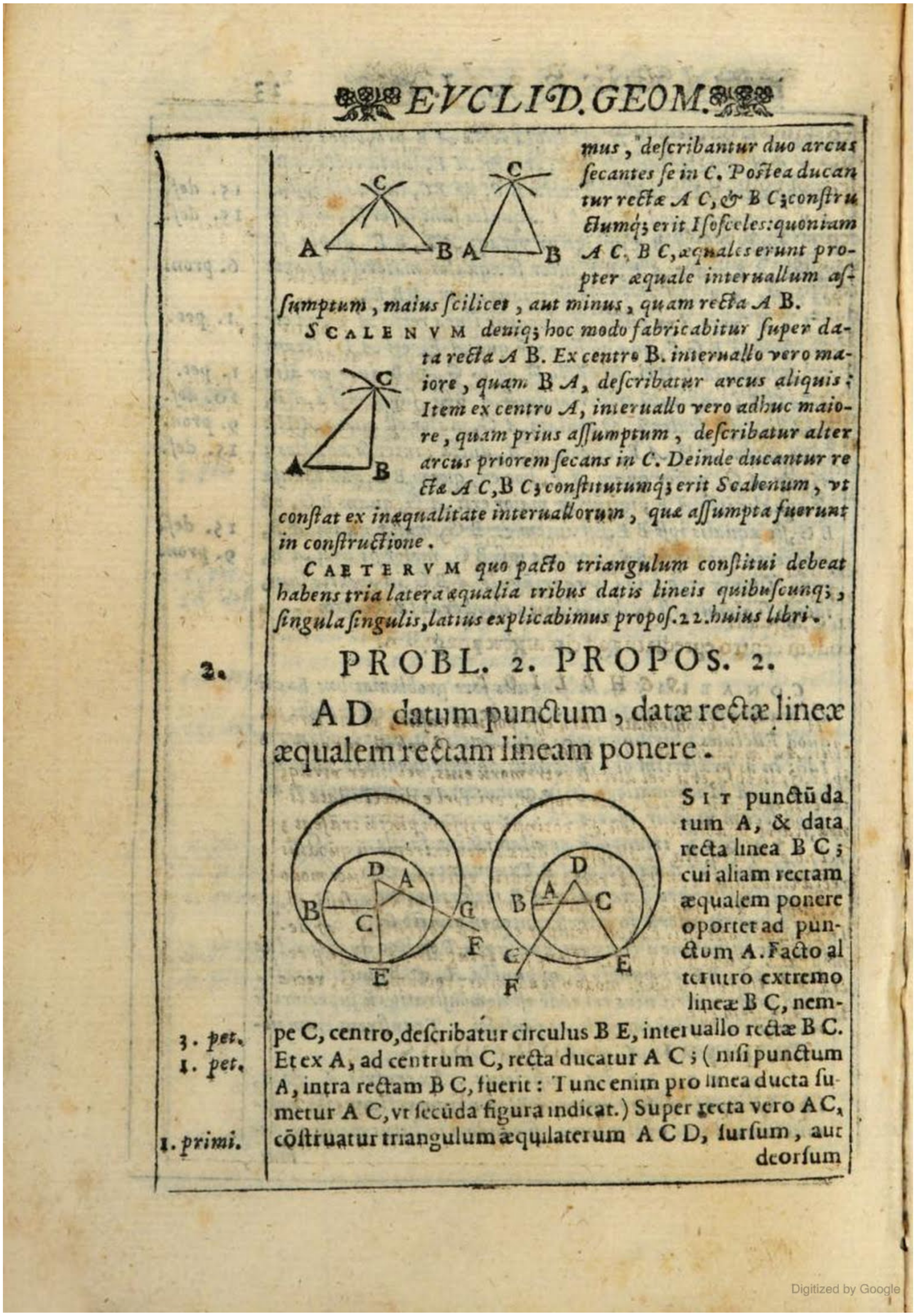
Therefore, taking away DA and DC, as equal sides of the equilateral triangle ACD, there will remain AG equal to the straight line CE (Axiom 3). But, the same CE is equal to straight line BC (since both straight lines CB and CE fall from the center C to the circumference BE (Definition 15).) Therefore, since the straight lines AG and BC are each shown to be equal to CE, they are equal to each other (Axiom 1). Thus, on a given point, etc. which was required to be done.

Ablatis igitur DA, DC, aequalibus lateribus trianguli aequilateri ACD, remanebit AG, aequalis rectae CE (3 pron.). Sed eidem CE, aequalis est recta BC. (cum ambae rectae CB, CE, cadant e centro C, ad circunferentiam BE (15 def.).) Igitur rectae AG, & BC, quandoquidem utraque aequalis est ostensa rectae CE (1 pron.), inter se aequales erunt. Ad datum igitur punctum, etc. Quod erat faciendum.

But if the given point was on the extreme of the given line, if it is C, it is easy to sum up the problem. If then on center C and interval CB, a circle is described (Postulate 3) to whose circumference any straight line whatever, like CE, is drawn

Quod, si punctum datum fuerit in extremo date lineae, quale est C, facile absoluetur problema. Si enim centro C, et intervallo CB, describatur circulus (3 pet.), ad cuius circunferentiam recta ducatur utcumque CE (1 pet.), (Postulate 1), this is a line placed on the given point C equal to the given line BC, since both BC and CE from the same center are led out to the circumference BE (Definition 15).

erit haec posita ad punctum datum C, aequalis datae rectae BC, cum utraque & BC, & CE ex eodem centro egrediatur ad circunferentiam BE (15 def.).



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mus, describantur duo arcus secantes se in C. Postea ducantur rectæ A C, & B C; constructumq; erit Isosceles: quoniam A C, B C, æquales erunt propter æquale interuallum assumptum, maius scilicet, aut minus, quam recta A B.

SCALENVM *deniq; hoc modo fabricabitur super data recta A B. Ex centro B, interuallo vero maiore, quam B A, describatur arcus aliquis; Item ex centro A, interuallo vero adhuc maiore, quam prius assumptum, describatur alter arcus priorem secans in C. Deinde ducantur rectæ A C, B C; constitutumq; erit Scalenum, vt constat ex inæqualitate interuallorum, quæ assumpta fuerunt in constructione.*

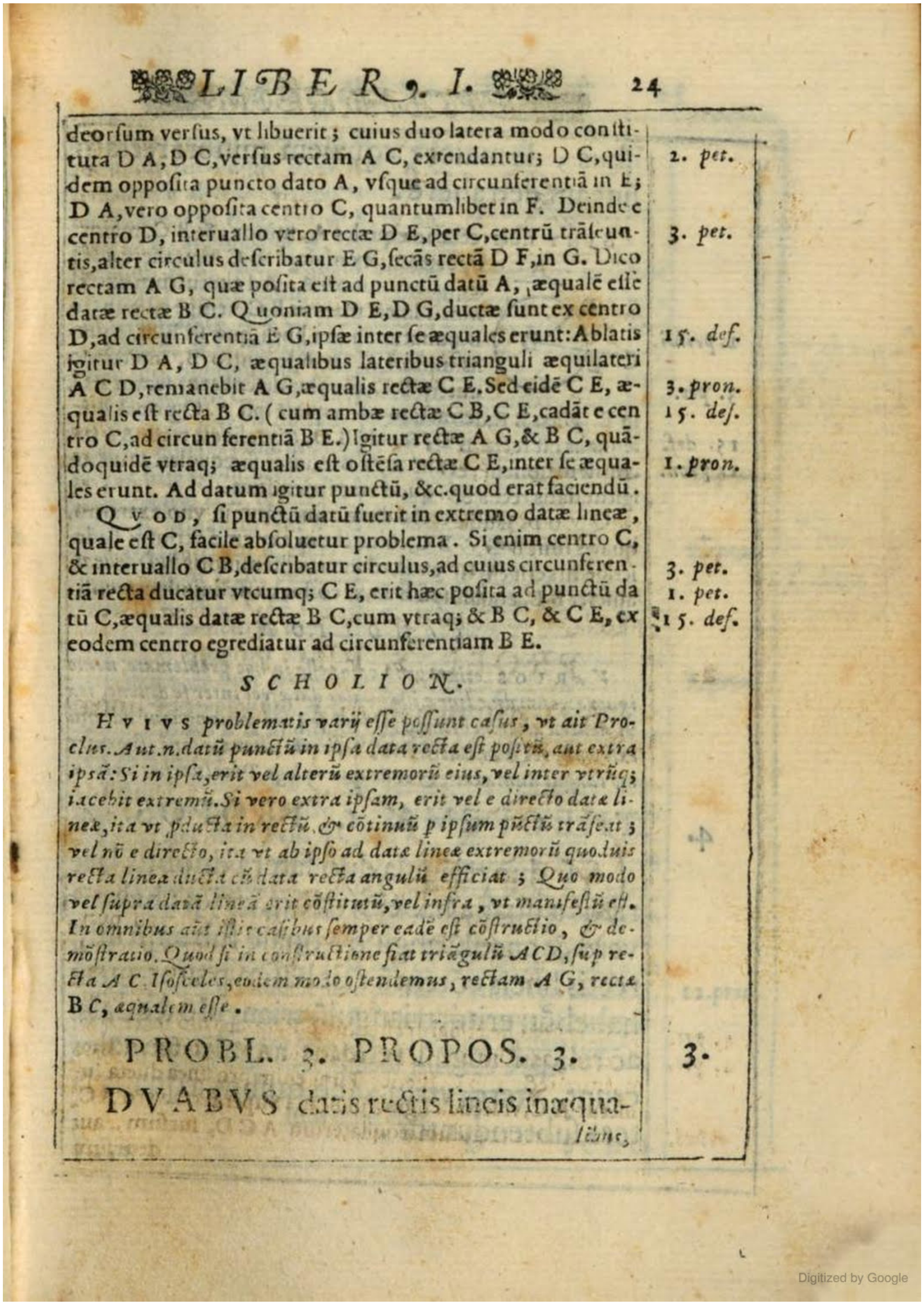
CAETERVM *quo pacto triangulum constitui debeat habens tria latera æqualia tribus datis lineis quibuscunq;, singula singulis, latius explicabimus propos. 22. huius libri.*

2. PROBL. 2. PROPOS. 2.

AD datum punctum, datæ rectæ lineæ æqualem rectam lineam ponere.

Sit punctũ datum A, & data recta linea B C; cui aliam rectam æqualem ponere oportet ad punctum A. Facto altertutro extremo lineæ B C, nempe C, centro, describatur circulus B E, interuallo rectæ B C.
3. *pet.* Et ex A, ad centrum C, recta ducatur A C; (nisi punctum
1. *pet.* A, intra rectam B C, fuerit: Tunc enim pro linea ducta sumetur A C, vt secũda figura indicat.) Super recta vero A C,
1. *primi.* cõstruatur triangulum æquilaterum A C D, sursum, aut deorsum

Figure 47: Book One Problem 2/Proposition 2



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deorſum verſus, vt libuerit; cuius duo latera modo conſtituta D A, D C, verſus rectam A C, extendantur; D C, quidem oppoſita puncto dato A, vſque ad circunferentiã in E; D A, vero oppoſita centro C, quantumlibet in F. Deinde e centro D, interuallo vero rectæ D E, per C, centrũ trãſeuntis, alter circulus deſcribatur E G, ſecãs rectã D F, in G. Dico rectam A G, quæ poſita eſt ad punctũ datũ A, æqualẽ eſſe datæ rectæ B C. Quoniam D E, D G, ductæ ſunt ex centro D, ad circunferentiã E G, ipſæ inter ſe æquales erunt: Ablatis igitur D A, D C, æqualibus lateribus trianguli æquilateri A C D, remanebit A G, æqualis rectæ C E. Sed eidẽ C E, æqualis eſt recta B C. (cum ambæ rectæ C B, C E, cadãt e centro C, ad circun ferentiã B E.) Igitur rectæ A G, & B C, quãdoquidẽ vtraq; æqualis eſt oſtẽſa rectæ C E, inter ſe æquales erunt. Ad datum igitur punctũ, &c. quod erat faciendũ.

2. *pet.* 3. *pet.* 15. *def.* 3. *pron.* 15. *def.* 1. *pron.*

QVOD, ſi punctũ datũ fuerit in extremo datæ lineæ, quale eſt C, facile abſoluetur problema. Si enim centro C, & interuallo C B, deſcribatur circulus, ad cuius circunferentiã recta ducatur vtcumq; C E, erit hæc poſita ad punctũ datũ C, æqualis datæ rectæ B C, cum vtraq; & B C, & C E, ex eodem centro egrediatur ad circunferentiam B E.

3. *pet.* 1. *pet.* 15. *def.*

SCHOLION.

HVIVS problematis varij eſſe poſſunt caſus, vt ait Proclus. Aut. n. datũ punctũ in ipſa data recta eſt poſitũ, aut extra ipsã: Si in ipſa, erit vel alterũ extremorũ eius, vel inter vtrũq; iacebit extremũ. Si vero extra ipſam, erit vel e directo datæ lineæ, ita vt pducta in rectũ, & cõtinuũ p ipſum pũctũ trãſeat; vel nõ e directo, ita vt ab ipſo ad datæ lineæ extremorũ quoduis recta linea ducta cũ data recta angulũ efficiat; Quo modo vel ſupra datã lineã erit cõſtitutũ, vel infra, vt manifeſtũ eſt. In omnibus aũt iſtis caſibus ſemper eadẽ eſt cõſtructio, & demõſtratio. Quod ſi in conſtructione fiat triãgulũ A C D, ſup recta A C Iſoſceles, eodem modo oſtendemus, rectam A G, rectæ B C, æqualem eſſe.

PROBL. 3. PROPOS. 3. 3.

DVABVS datis rectis lineis inæqualibus,

Figure 47: Book One Problem 2/Proposition 2 (continued)

Book One, Theorem 33/Proposition 47: The Pythagorean Theorem

In a right triangle, the square which is described on the side that subtends the right angle, is equal to those described on the sides which contain the right angle.

In triangle ABC, let angle BAC be right, and let the squares ABFG, ACHI, and BCDE, be described on AB, AC, and BC (I.46). I say that the square BCDE described on side BC, which opposes the right angle, is equal to the two squares ABFG and ACHI which are described on the other two sides, whether those two sides are equal or unequal. Then draw straight line AK parallel to BE or CD, cutting BC in I (I.11). And let AD, AE, CF, and BH be joined by straight lines.

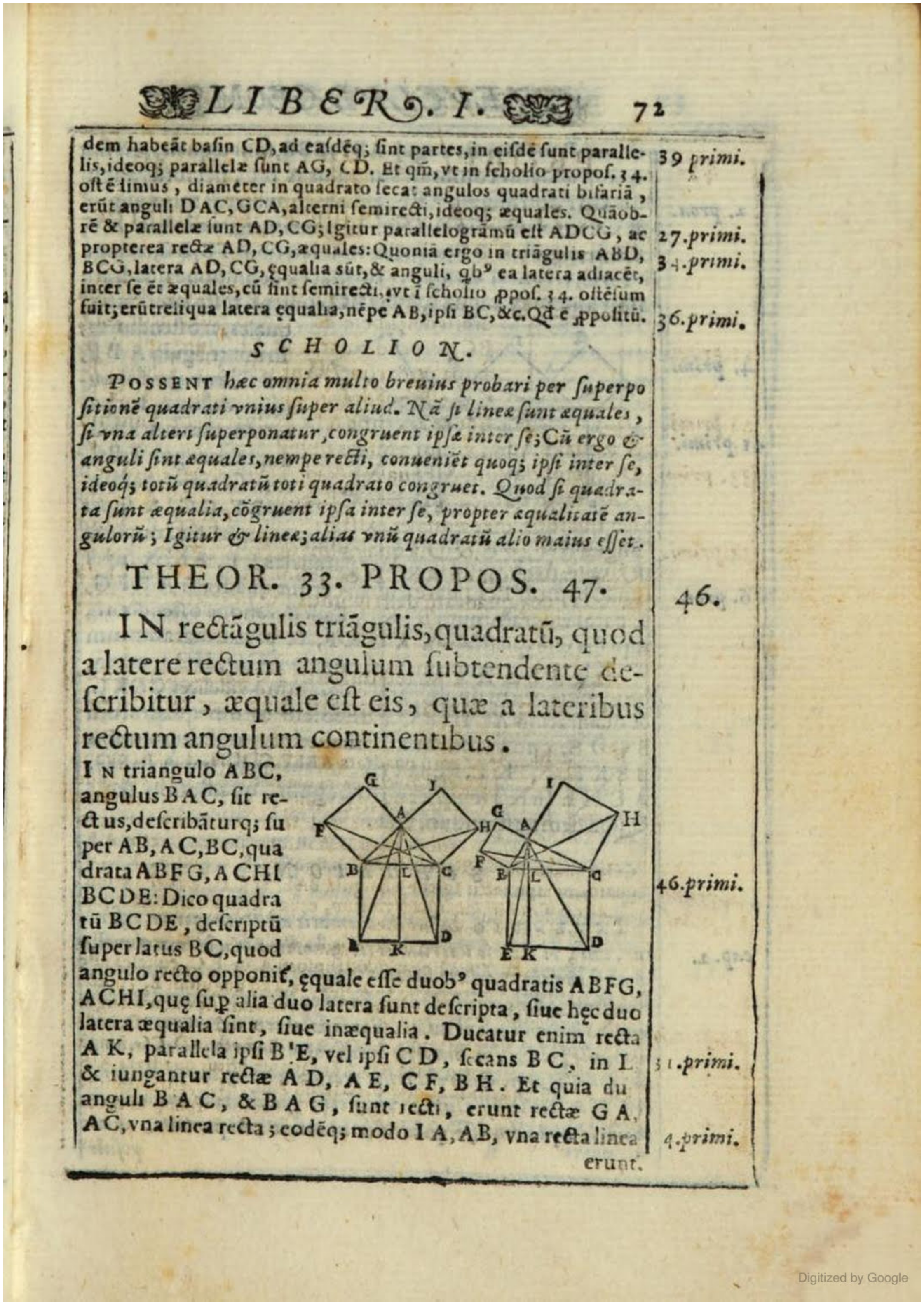
In rectangulis triangulis, quadratum, quod a latere rectum angulum subtendente describitur, aequale est eis, quae a lateribus rectum angulum contientibus.

In triangulo ABC, angulus BAC, sit rectus, describanturque super AB, AC, BC, quadrata ABFG, ACHI, BCDE (46 primi): Dico quadratum BCDE, descriptum super latus BC, quo angulo recto opponit, aequale esse duobus quadratis ABFG, ACHI, quae super alia duo latera sunt descripta, sive haec duo latera aequalia sint, sive inaequalia. Ducatur enim recta AK, parallela ipsi BE, vel ipsi CD, secans BC in L (11 primi). & iungantur rectae AD, AE, CF, BH.

And since the two angles BAC and BAG are right, the straight lines GA and AC are one straight line. And in the same way, IA and AB are one straight line (I.4). In turn because angles ABF and CBE are equal, because they are right. If the common angle ABC is added to them, the total angle CBF is equal to the total angle ABE; and similarly the total angle BCH is equal to the total angle ACD (Axiom 2). Therefore the sides AB and BE, of triangle ABE, are equal to the respective sides of FBC, as comes about from the definition of a square. Moreover, the angles ABE and FBC are contained by the equal sides, so, as we have shown, the triangles ABE and FBC are equal (I.4). Furthermore, the square, or parallelogram, ABFG is double to the triangle FBC because

Et quia du angulis BAC & BAG, sunt recti, erunt rectae GA, AC, una linea recta; eodemque modo IA, AB, una recta linea erunt (4 primi). Rursus quia anguli ABF, CBE, sunt aequales, cum sint recti, si addatur communis angulus ABC, fiet totus angulus CBF, toti angulo ABE, aequalis; similterque totus angulus BCH, toti angulo ACD (2 pron.). Quoniam igitur latera AB, BE, trianguli ABE, aequalia sunt lateribus FB, BC, triangul FBC, utrumque utrique, ut confiat ex definitione quadrati. Sunt autem & anguli ABE, FBC, contenti hisce lateribus aequales, ut ostendimus (4 primi); Eurnt triangula ABE, FBC aequalia. Est autem quadratum, seu parallelogramum ABFG duplum trianguli FBC, they are between parallels BF and CG and built on the same base BF (I.41). And parallelogram BEKL is double to triangle ABE, because they are between parallel lines BE and AK and on the same base, BE. Therefore the square ABFG and the parallelogram BEKL are equal (Axiom 6). By the same reason it can be shown that square ACHI and parallelogram CDKL are equal to each other. They are so because triangle ACD and HCB are equal. By that reason, one may see that the parallelogram CDKL and the square ACHI are double those triangles. Therefore, the total square BCDE, which is composed of the two parallelograms BEKL, CDKL, is equal to the two squares ABFG and ACHI. Thus, in a right triangle, the square etc. Which was to be demonstrated.

cum sint inter parallelas BF, CG & super eandem basin BF(41 primi) ; Et parallelogrammum BEKL, duplum trianguli ABE, quod sint inter parallelas BE, AK, & super eandem basin BE. Quare aequalia erunt quadratum ABFG, & parallelogrammum BEKL (6 pron.), Eadem ratione ostendetur, aequalia esse quadratum ACHI, & parallelogrammum CDKL. Erunt enim rursus triangula ACD, HCB, aequalia, ideoque eorum dupla, parallelogrammum videlicet CDKL, & quadratum ACHI. Quamobrem totum quadratum BCDE, quod componitur ex duobus parallelogrammis BEKL, CDKL, aequale est duobus quadratis ABFG, ACHI. In rectangulis ero triangulis, quadratum &c. Quod demonstrandum erat.



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dem habeāt basin CD, ad easdēq; sint partes, in eisdē sunt parallelis, ideoq; parallelæ sunt AG, CD. Et qm̄, vt in scholio propos. 34. ostēdimus, diameter in quadrato secat angulos quadrati bifariā, erūt anguli DAC, GCA, alterni semirecti, ideoq; æquales. Quāobrē & parallelæ sunt AD, CG; Igitur parallelogrāmū est ADCG, ac propterea rectæ AD, CG, æquales: Quoniā ergo in triāgulis ABD, BCG, latera AD, CG, ęqualia sūt, & anguli, qb⁹ ea latera adiacēt, inter se ēt æquales, cū sint semirecti, vt ī scholio ꝓpos. 34. ostēsum fuit; erūt reliqua latera ęqualia, nēpe AB, ipsi BC, &c. Qđ ē ꝓpositū.

39 primi. 27. primi. 34. primi. 36. primi.

SCHOLION.

POSSENT hæc omnia multo breuius probari per superpositionē quadrati vnius super aliud. Nā si lineæ sunt æquales, si vna alteri superponatur, congruent ipsæ inter se; Cū ergo & anguli sint æquales, nempe recti, conueniēt quoq; ipsi inter se, ideoq; totū quadratū toti quadrato congruet. Quod si quadrata sunt æqualia, cōgruent ipsa inter se, propter æqualitatē angulorū; Igitur & lineæ; alias vnū quadratū alio maius esset.

THEOR. 33. PROPOS. 47. — 46.

IN rectāgulis triāgulis, quadratū, quod a latere rectum angulum subtendente describitur, æquale est eis, quæ a lateribus rectum angulum continentibus.

IN triangulo ABC, angulus BAC, sit rectus, describāturq; super AB, AC, BC, quadrata ABFG, ACHI, BCDE: Dico quadratū BCDE, descriptū super latus BC, quod angulo recto opponit, ęquale esse duob⁹ quadratis ABFG, ACHI, quę sup alia duo latera sunt descripta, siue hęc duo latera æqualia sint, siue inæqualia. Ducatur enim recta AK, parallela ipsi BE, vel ipsi CD, secans BC, in L & iungantur rectæ AD, AE, CF, BH. Et quia du anguli BAC, & BAG, sunt recti, erunt rectæ GA, AC, vna linea recta; eodēq; modo IA, AB, vna recta linea erunt.

46. primi. 31. primi. 4. primi.

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Figure 48: Book One Theorem 33/Propostion 47: The Pythagorean Theorem

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erunt. Rursus quia anguli ABF, CBE, sunt æquales, cũ sint recti, si addatur communis angulus A B C, fiet totus angu-
2. pron. lus CBF, toti angulo ABE, ęqualis; similiterq; totus angul⁹ B C H, toti angulo A C D. Quoniam igitur latera A B, B E, trianguli A B E, æqualia sunt lateribus FB, BC, triangu li FBC, vtrumq; vtriq;, vt constat ex definitione quadrati; Sunt autem & anguli ABE, FBC, contenti hisce lateribus æquales, vt ostendimus;
4. primi. Erunt triãgula A B E, F B C, æqualia. Est autẽ quadratum, seu parallelogrãmũ A B F G,
41. primi. duplũ trianguli F BC, cum sint inter parallelas B F, C G, & super eandem basin B F; Et parallelogrammum BEKL, duplum trianguli ABE, quod sint inter parallelas B E, A K, & super
6. pron. eandẽ basin B E. Quare æqualia erunt quadratũ A B F G, & parallelogrammum BEKL; Eadem ratione ostendetur, æqualia esse quadratum A C H I, & parallelogrammum CDKL. Erunt enim rursus triangula ACD, HCB, æqualia, ideoq; eorum dupla, parallelogrammũ videlicet CDKL, & quadratum ACHI. Quamobrem totum quadratum B C D E, quod componitur ex duobus parallelogrammis B E K L, C D K L, æquale est duobus quadratis A B F G, A C H I. In rectangulis ergo triangulis, quadratum &c. Quod demonstrandum erat.

S C H O L I O N.

I N V E N T I O *huius theorematis ad Pythagoram referetur, qui, vt scribit Vitruuius lib. 9. hostias Musis immolauit,*
cap. 2. *quod se in tam præclaro inuento adiuuerint. Sunt qui putent, eum immolasse centum boues; si tamen Proclo credendum est, vnum tantummodo obtulit. Fortasse autem Pythagoras, vt nonnulli volunt, ex numeris occasionem sumpsit, vt theorema hoc inuestigaret. Cum enim hos tres numeros 3. 4. 5. diligenter esset contemplatus, vidissetq; quadratum numerum maioris æqualem esse quadratis numeris reliquorum, com-*

posuit

Figure 48: Book One Theorem 33/Propostion 47 (continued)

Book Two, Theorem 1/Proposition 1

If there are two straight lines, either of which is cut into however so many segments: The rectangle included by the two straight lines is equal to those rectangles which are included by the uncut line and every one of the segments.

Si fuerint duae rectae lineae, seceturque ipsarum altera in quotcunque segmenta: Rectangulum coprehensum sub illis duabus rectis lineis, aequale est eis, quae sub insecta, & quolibet segmentorum comprehenduntur, rectangulis.

Let there be two straight lines, A and BC, of which BC is cut into however so many pieces BD, DE, EC: I say that the rectangle contained under A and BC is equal to all of the rectangles which are contained under the uncut line A and all of the segments, in this example, the rectangles contained under A and BD, A and DE, and A and EC.

Sint duae rectae A, & BC quarum BC, sectetur quomodocunque in quotlibet segmenta BD, DE, EC: Dico rectangulum sub A, & BC, comprehensum aequale esse omnibus rectangulis simul sumptis quae sub linea indivisa A, & quolibet segment comprenduntur, nempe rectangulo sub A, & BD; Item sub A, & DE; Item sub A, & EC comprehnso.

For the rectangle BF is contained under lines A and BC, that is since the straight line GB is equal to the straight line A; for let two perpendiculars, BG and CF, equal to the straight line A be drawn to FG (I.23). Then from D and E let straight lines DH, EI be drawn parallel to the same BG or CF. Then DH and EI because they are parallel to BG are parallel to each other. In turn, the same lines, because by construction they made parallelograms BH and BI, are equal to the line BG, and thus to the line A (I.34). Therefore, seeing that the straight line BG is equal to the straight line A, the rectangle BH is contained under the uncut line A and the segment BD.

Rectangulum enim BF, comprehendatur sub A, & BC, hoc est, recta GB, aequalis sit rectae A; Quod quidem fiet, si erigantur ad BC, duae perpendiculares BG, CF, aequales rectae A, ducatrque recta FG (23 primi). Deinde ex D, & E, ducantur rectae DH, EI, parallelae ipsi BG, vel CF. Itaque DH, EI, cum parallelae sint ipsi BG inter se quoque parallelae erunt: Rursus eaedem, cum ex constructione parallelogramma sint BH, BI, aequales erunt rectae BG, ac propterea rectae A (34 primi). Quoniam igitur recta BG, aequalis est rectae A, erit rectangulum BH, comprehensum sub insecta linea A, & segmento BD;

By the same reason the rectangle DI is contained under A and the segment DE. Likewise, the rectangle EF under A and the segment EC. Thus, because the rectangles BH, DI, EF, are equal to the total rectangle BF, it is obvious that the rectangle contained under A and BC is equal to all of the rectangles which are contained under A and the segments BD, DE, and EC. Therefore, if there are two lines, either one of which is cut etc. Which was to be shown.

Eadem ratione erit rectangulum DI, comprehensum sub A, & segmento DE; Item rectangulum EF sub A, & segmento EC. Quare cum rectangula BH, DI, EF, aequalia sint toti rectangulo BF; perspicuum est rectangulum comprehensum sub A et BC, aequale esse rectangulis omnibus, quae sub A, et segmentis BD, DE, EC comprehenduntur. Si ergo fuerint duae rectae lineae, seceturque ipsarum altera, &c. Quod erat ostendendum.

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bus complementis B G, GD, *Gnomon appellabitur.*

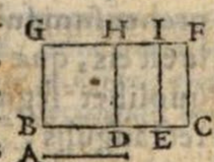
THEOR. 1. PROPOS. 1.

1.

SI fuerint duæ rectæ lineæ, seceturque ipsarum altera in quotcunque segmenta: Rectangulum comprehensum sub illis duabus rectis lineis, æquale est eis, quæ sub insecta, & quolibet segmentorum comprehenduntur, rectangulis.

SINT duæ rectæ A, & B C quarum BC, secetur quomodocunque in quotlibet segmenta B D, D E, E C: Dico rectangulum sub A, & B C, comprehensum æquale esse omnibus rectangulis simul sumptis; quæ sub linea indiuisa A, & quolibet segmento comprehenduntur, nempe rectangulo sub A, & B D; Item sub A, & D E; Item sub A, & E C, comprehenso.

Rectangulum enim B F, comprehendatur sub A, & B C, hoc est, recta G B, æqualis sit rectæ A; Quod quidem fiet, si erigantur ad B C, duæ perpendiculares BG, CF, æquales rectæ A, ducaturq; recta F G. Deinde ex D, & E, ducantur

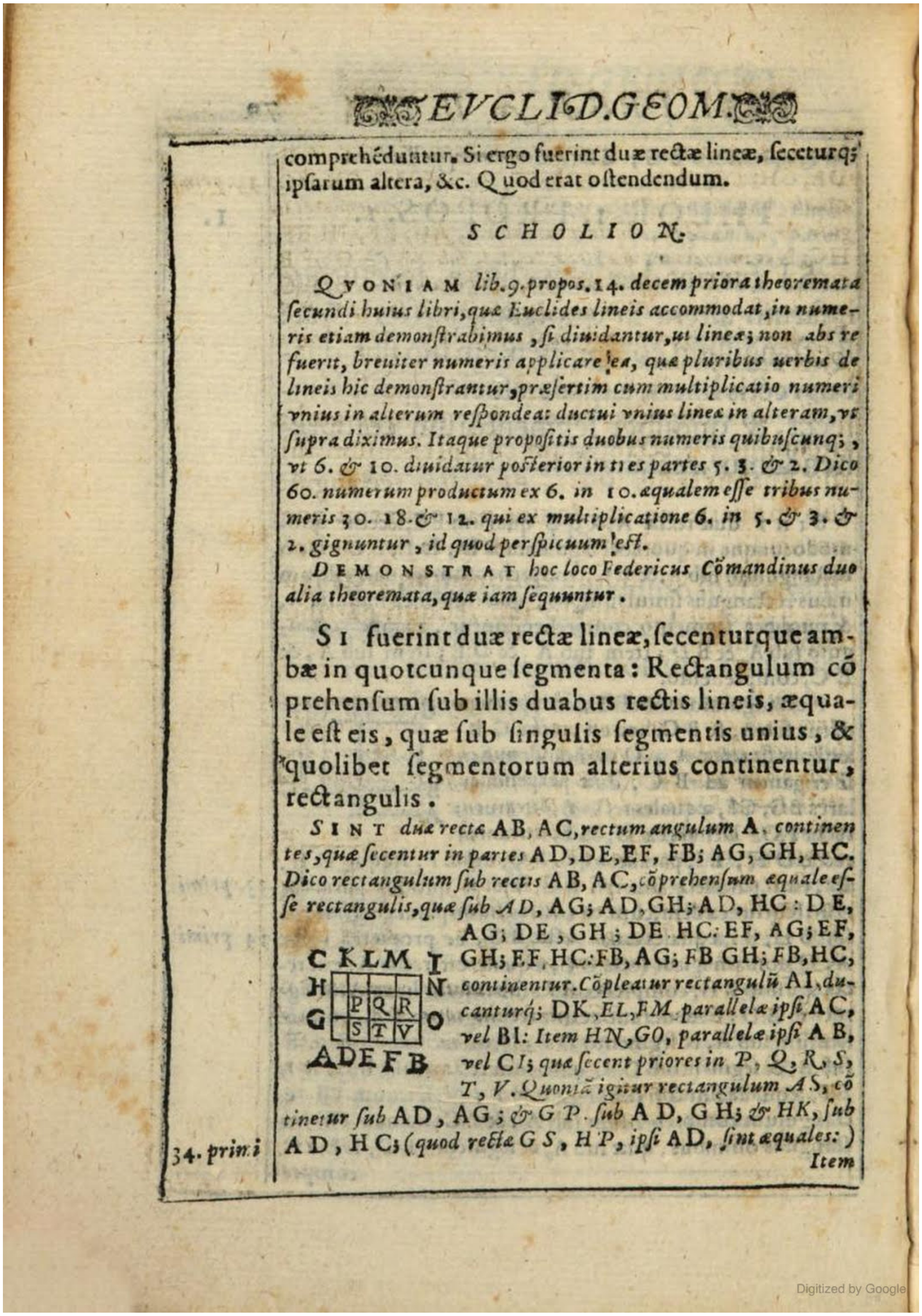


rectæ D H, E I, parallelæ ipsi B G, uel C F. Itaque D H, E I, cum parallelæ sint ipsi B G, inter se quoque parallelæ erūt: Rursus eædem, cum ex cōstructione parallelogramma sint B H, B I, æquales erunt rectæ B G, ac propterea rectæ A. Quoniam igitur recta B G, æqualis est rectæ A, erit rectangulum B H, comprehensum sub insecta linea A, & segmento B D; Eadem ratione erit rectangulum D I, comprehensum sub A, & segmento D E; Item rectangulum E F sub A, & segmento E C. Quare cum rectangula B H, D I, E F, æqualia sint toti rectangulo B F; perspicuum est rectangulum comprehensum sub A, & B C, æquale esse rectangulis omnibus, quæ sub A, et segmentis B D, D E, E C, compre-

33. primi

34 primi

Figure 49: Book Two Theorem 1/Proposition 1



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comprehẽduntur. Si ergo fuerint duæ rectæ lineæ, secetur q; ipsarum altera, &c. Quod erat ostendendum.

SCHOLION.

QVONIAM lib.9.propos.14. decem priora theoremata secundi huius libri, quæ Euclides lineis accommodat, in numeris etiam demonstrabimus, si diuidantur, ut lineæ; non abs re fuerit, breuiter numeris applicare ea, quæ pluribus uerbis de lineis hic demonstrantur, præsertim cum multiplicatio numeri vnius in alterum respondeat ductui vnius lineæ in alteram, vt supra diximus. Itaque propositis duobus numeris quibuscunq; , vt 6. & 10. diuidatur posterior in tres partes 5. 3. & 2. Dico 60. numerum productum ex 6. in 10. æqualem esse tribus numeris 30. 18. & 12. qui ex multiplicatione 6. in 5. & 3. & 2. gignuntur, id quod perspicuum est.

DEMONSTRAT hoc loco Federicus Cõmandinus duo alia theoremata, quæ iam sequuntur.

SI fuerint duæ rectæ lineæ, secenturque ambæ in quotcunque segmenta: Rectangulum cõprehensum sub illis duabus rectis lineis, æquale est eis, quæ sub singulis segmentis unius, & quolibet segmentorum alterius continentur, rectangulis.

SINT duæ rectæ AB, AC, rectum angulum A. continentes, quæ secentur in partes AD, DE, EF, FB; AG, GH, HC. Dico rectangulum sub rectis AB, AC, cõprehensum æquale esse rectangulis, quæ sub AD, AG; AD, GH; AD, HC: DE, AG; DE, GH; DE HC: EF, AG; EF, GH; EF, HC: FB, AG; FB GH; FB, HC, continentur. Cõpleatur rectangulũ AI, ducanturq; DK, EL, FM. parallelæ ipsi AC, vel BI: Item HN, GO, parallelæ ipsi AB, vel CI; quæ secent priores in P, Q, R, S, T, V. Quoniã igitur rectangulum AS, cõtinetur sub AD, AG; & GP. sub AD, GH; & HK, sub

34. primi AD, HC; (quod rectæ GS, HP, ipsi AD, sint æquales:)

Item

Figure 49: Book Two Theorem 1/Proposition 1 (continued)

Barlaam's Version of Book Two, Proposition 1

(Found in Book Nine)

If there are two numbers, and one or the other is cut into however so many parts, the plane number contained by those two numbers is equal to the numbers which are contained under the undivided number and however many parts of the divided number.

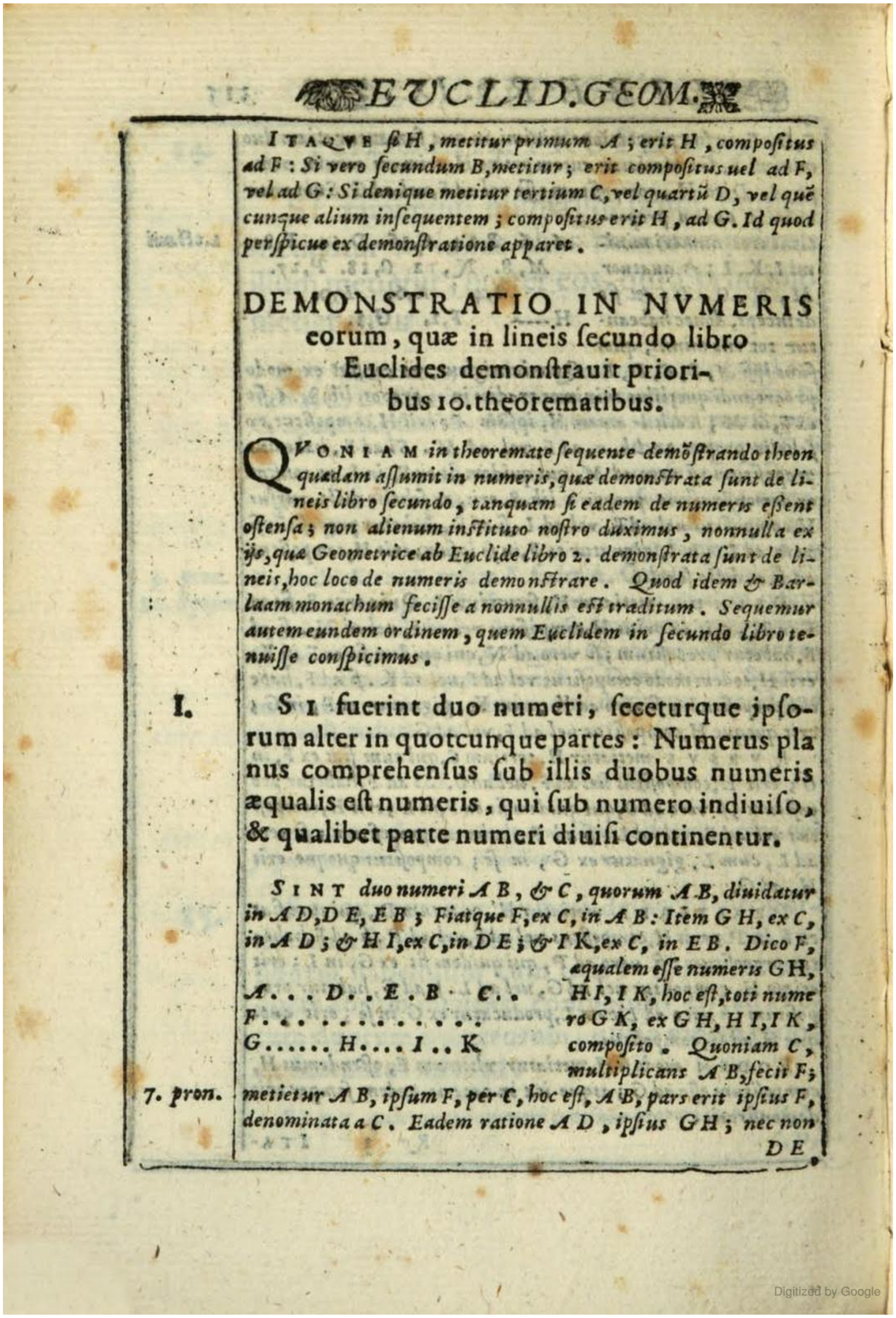
Si fuerint duo numeri, seceturque ipsorum alter in quotcunque partes: Numerus planus comprehensus sub illis duobus numeris aequalis est numeris, qui sub numero indiviso, & qualibet parte numeri divisi continentur.

Let there be two numbers, AB and C, of which AB is divided into AD, DE, and EB. And make F from C in AB. Likewise make GH from C in AD, and HI from C in DE and IK from C in EB. I say that F is equal to the numbers GH, HI, IK, that is the total number GK composed from GH, HI, IK.

Sint duo numeri AB, & C, quorum AB, dividatur in AD, DE, EB; Fiatque F, ex C, in AB: Item GH, ex C in AD; & HI, ex C, in DE; & IK, ex C, in EB. Dico F aequalem esse numerus GH, HI, IK, hoc est, toti numero GK, ex GH, HI, IK, composito.

Since C multiplied by AB makes F, that same F is measured by AB by C. That is, AB is a factor of F by the denomination of C. By the same reason AD, is the likewise a factor of GH by the denomination of C, and DE likewise of HI, and EB likewise of IK, truly like AB is of F (Axiom 7 – Book 7). And hence,from this, as we demonstrated in the poporition 5 of book 7, the whole of AB is in the same way a factor of the whole of GK, as AD is of GH. And AB is a factor of GK by the same number which AB is of F. And, thus, F and GK are equal to one another (Axiom 4 – Book 7). Which was proposed.

Quoniam C, multipilcans AB, fecit F; metietur AB ipsum F, per C, hoc est, AB, pars erit ipsius F, denominata a C. Eadem ratione AD, ipsius GH; nec non DE ipsius HIl & EB ipsius IK, pars erit a C, denominata, nempe eadem quae AB, ipsius F (pron. 7). Quia vero, per ea quae ad propos. 5. lib: 7. demonstravimus, totus AB, totius GK, eadem pars est, quae Ad, ipsius GH, erit quoque AB totus totius GK< pars eadem, quae AB ipsius F; Ac proinde inter se aequales erunt F, & GK (4 pron.). Quod est propositum.



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ITAQVE si H, metitur primum A; erit H, compositus ad F: Si vero secundum B, metitur; erit compositus uel ad F, vel ad G: Si denique metitur tertium C, vel quartũ D, vel quẽcunque alium insequentem; compositus erit H, ad G. Id quod perspicue ex demonstratione apparet.

DEMONSTRATIO IN NVMERIS corum, quæ in lineis secundo libro Euclides demonstrauit prioribus 10. theorematibus.

QVONIAM in theoremate sequente demõstrando theon quædam assumit in numeris, quæ demonstrata sunt de lineis libro secundo, tanquam si eadem de numeris essent ostensa; non alienum instituto nostro duximus, nonnulla ex ijs, quæ Geometrice ab Euclide libro 2. demonstrata sunt de lineis, hoc loco de numeris demonstrare. Quod idem & Barlaam monachum fecisse a nonnullis est traditum. Sequemur autem eundem ordinem, quem Euclidem in secundo libro tenuisse conspicimus.

I. SI fuerint duo numeri, seceturque ipsorum alter in quotcunque partes: Numerus planus comprehensus sub illis duobus numeris æqualis est numeris, qui sub numero indiuiso, & qualibet parte numeri diuisi continentur.

SINT duo numeri A B, & C, quorum A B, diuidatur in A D, D E, E B; Fiatque F, ex C, in A B: Item G H, ex C, in A D; & H I, ex C, in D E; & I K, ex C, in E B. Dico F, æqualem esse numeris G H, H I, I K, hoc est, toti numero G K, ex G H, H I, I K, composito. Quoniam C, multiplicans A B, fecit F;

A . . . D . . E . B · C . .
F . . . .
G . . . . . . H . . . . I . . K

7. pron. metietur A B, ipsum F, per C, hoc est, A B, pars erit ipsius F, denominata a C. Eadem ratione A D, ipsius G H; nec non

D E

Figure 50: Barlaam's Version of Book Two Proposition 1 (Found in Book Nine)

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D E, ipſius H I; & E B, ipſius I K, pars erit a C, denominata, nempe eadem quæ A B, ipſius F. Quia uero, per ea quæ ad propos: 5. lib: 7. demonſtrauimus, totus A B, totius G K, eadem pars eſt, quæ A D, ipſius G H; erit quoque A B, totus totius G K, pars eadem, quæ A B, ipſius F; Ac proinde inter ſe æquales erunt F, & G K. Quod eſt propoſitum. 4. pron.

II.

SI numerus in duas partes diuidatur: Numeri plani ſub toto, & ſingulis partibus comprehenſi æquales ſunt numero quadrato, qui a toto efficitur.

NVMERVS A B, diuidatur in A C, C B. Dico numeros, qui fiunt ex A B, toto in partes A C, C B, ſimul æquales eſſe numero quadrato, qui ex toto A B, efficitur. Sumpto enim numero D, qui æqualis ſit ipſi A B; erit per theor. 1. numerus factus ex D, hoc eſt, ex A B, in A B, nimirũ quadratus ipſius A B, æqualis numeris, qui fiunt ex D, hoc eſt, ex A B, in A C, & in C B. quod eſt propoſitum.

D . : . . . .

A . . . . C . . B

IDEM demonſtrabitur, ſi A B, in plures partes, quam in duas ſecetur, vt ex appoſita ſecunda figura apparere poteſt. Eadem enim ratione erit numerus factus ex E, hoc eſt, ex A B, in A B, nempe quadratus ipſius A B, æqualis numeris, qui gignuntur ex E, hoc eſt, ex A B, in ſingulas partes A C, C D, D B.

E . . . .

A . . . . C . . . D . . B

III.

SI numerus in duas partes diuidatur: Numerus planus ſub toto, & una parte comprehenſus æqualis eſt & illi, qui ſub partibus continetur, & quadrato, qui a prædicta parte efficitur.

SIT numerus A B, diuiſus in A C, C B. Dico numerum, qui fit ex toto A B, in partem A C, æqualem eſſe & ei, qui ſub partibus A C, C B, continetur, & quadrato dictæ partis A C.

Rr 4 Sumpto

Figure 50: Barlaam's Version of Book Two Proposition 1 (continued)

Book Nine, Proposition 15

If there are three numbers in continuous proportion, and they are the smallest numbers which have the same ratio between them, the number composed of any two, will be prime to the remaining number.

Si tres numeri deinceps proportionales, fuerint minimi omnium eandem cum ipsis rationem habentium; Duo quilibet compositi, ad reliquum primi erunt.

Let there be three numbers - A, B, C – which are the smallest numbers that have the same proportion between them. I say that the combination of any two numbers is prime to the remaining number, indeed A and B together are prime to C, and B and C together to A, and A and C together to B. For having taken D and E be the smallest number in the same proportion, from the scholion to proposition 35 in book 7

Sint tres numeri A, B, C, minimi omnium eandem cum illis proportionem habentium. Dico quoslibet duos compositos, ad reliquum primos esse nimirum AB, simul ad C, & BC, simul ad A, et AC, simul ad B. Sumptis enim dubus D, E, in eadem cum illis proportione minimis, ex scholio propos. 35. lib. 7.

it is manifest from the demonstration of proposition two in book 8, D multiplied by itself produces A, D multiplied with E produces B, and E multiplied with itself produces C. And since D and E are the smallest numbers in that proportion they are prime to each other (VII.24), and D and E together is prime to either of them (VII.30). Wherefore, because the number composed of D and E is prime to D and E themselves, together with the number made of D itself is prime to the number made from E itself (VII.26),. Moreover, that which is made from D and E, together with the one from D is – by the third theorem of the preceding scholion – the number A, made from D into self, and the number B made from D into E.

manifestum est ex demonstration propos. 2 lib. 8. D, seipusm multiplicantem facere A; multiplcantem vero ipsum E, facere B; atque E, se ipsum multiplicantem facere C. Quia igitur D, E, minimi in sua proportione inter se primi sunt (24 septimi), erit & uterque D, E, simul ad quemlibet illorum primus (30 septimi). Itaque cum tam compositus ex D, E, quam ipse D, ad E, primus sit; erit quoque numerus factus ex D, E (26 septimi), tanquam uno in D; ad eundem E, primus: Qui autem fit ex D, E, tanquam uno in D, aequalis est per 3 theorema scholii praecedentis, & numero A, facto ex D, in se, & numero B facto ex D, in E.

Therefore, A and B composed together are prime to E; and thus, A and B taken together are prime to C which is made from E into itself (VII.27).

Next, since, as above, the number composed from D and E is prime to either of D or E (VII.30) (because that which is composed of D and E, like that which is composed of E itself, is prime to D) the number made from D and E, is likewise to the number made from D and E together with the number made from E itself is prime to D (VII.26). Moreover, the number made from D and E together with the number made from E is equal, by the third theorem of the preceding scholion, the number C made from E into itself and the number B made from D into E.

Igitur & A, B, compositi primi sunt ad E; Ac proinde & ad C, qui factus est ex E in se, primi sunt A, B, simul compositi.

Deinde quia, ut prius, uterque D, E, simul primus est ad quemlibet ipsorum D, E; efficitur (30 septimi) (cum tam compositus ex D, E, quam ipse E, primus sit ad D,) numerum factum ex D, E, tanquam uno in E, primum esse ad D (26 septimi): Qui autem fit ex D, E; tanquam uno in E, aequalis est per 3. theorema scholii antecedentis, & numero C, facto ex E, in se, & numero B, facto ex D, in E.

Therefore, B and C composed together, are prime to D, and thus B and C composed together are prime to A, which is made from D into itself (VII.27).

Finally, because, as before, the number composed from D and E is prime to either of D or E (VII.30); and indeed from the opposite which is that either of D or E is prime to the number composed of D and E (VII.26), the number made from D into E will be prime to the number composed of D and E(VII.27). And, then likewise, that which is made from D into E will be prime to that which is made from D and E, as one number, into itself.

Igitur & B, C, simul compositi, ad D, primi sunt; atque adeo ad ipsum A, qui factus est ex D, in se primi sunt B, & C, simul compositi (27 septimi).

Postremo quia, ut prius, uterque D, & E, simul ad quemlibet ipsorum primus est (30 septimi); atque adeo e contrario, quilibet ipsorum D, E, primus est ad compositum ex D, E (26 septimi); erit quoque qui fit ex D, in E, ad compositum ex D, E, primus (27 septimi); Ac proinde & idem qui fit ex D, in E, ad eum qui fit ex D, E, tanquam uno, in se, primus erit.

Moreover, that number which is made from D and E, as one number, into itself is equal, by the fourth theorem of the preceding scholion, to those which are made from D and E into themselves and that which is made from D into E twice. Therefore, that which is made from D into E is prime to those which are made from D and E into themselves and D into E twice. Since when two numbers taken together, truly composed of those, one of which is made from D and E into themselves and from that which is made from D into E and the other of which is made from D into E, are prime to either of those numbers, as to that which is made from D into E, as has been shown.

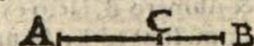
Qui autem fit ex D, E, tanquam uno, in se, aequalis est per 4. theorema antecedentis scholii, eis qui fiunt ex D, & E in se ipsos, una cum eo qui ex D, in E, bis: Igitur & factus ex D, in E, primus erit ad eos, qui fiunt ex D, & E, in se ipsos & ex D in E bis. Quoniam ergo duo numeri simul, nempe compositus ex iis, qui fiunt ex D, & E in se ipsos, & ex eo, qui fit ex D, in E, atque is qui fit ex D in E primi sunt ad aliquem ipsorum ut ad eum, qui fit ex D, in E, ut ostensum est;

Therefore there are those two numbers, evidently those which are composed of D and E into themselves and that composed of D into E, each of which is prime to that which is composed of D into E (VII. 30). Again, there are two numbers together, that is numbers composed from those which are made from D and E into themselves and D into E, which are each prime to each other and to that which is made from D into E, as has been shown. Therefore, evidently the two numbers composed from those made from D and E into themselves and that made from D into E, are prime to each other. Because D and E into themselves make A and C, and likewise D into E makes B, A and C taken together are prime to B. By which, if three numbers in continual proportion, etc. Which was to be shown.

Erunt etiam duo illi, nimirum compositus ex iis, qui fiunt ex D, & E, in se ipsos, & ex eo, qui fit ex D, in E; atque is, qui fit ex D, in E, inter se primi (30 septimi). Rursus quia duo numeri simul, videlicet compositus ex iis, qui fiunt ex D, & E, in se ipsos, atque is qui fit ex D, in E, ad aliquem ipsorum, ut ad eum, qui fit ex D in E, primi sunt, ut ostensum est; erunt etiam duo illi nimirum compositus ex iis qui fiunt ex D, & E, in se ipsos atque is, qui fit ex D, in E, inter se primi. Cum igitur ex D & E in se ipsos fiant A & C. Item ex D, in E, fiat B; erunt A, & C, simul compositi, primi ad B. Quam ob rem, sit res numeri deiceps poropotionales, &c. Quod erat ostendendum.

ad detractum C D, ita C B, ex A B, residuus, hoc est, C D, illi æqualis, ad A D, ex A C, residuum. Quia igitur est vt A C, ad C D, ita C D, ad A D; & est A C, maior quam C D; erit quoque C D, maior quam A D. Ablato ergo D E, qui ipsi A D, sit æqualis; erit etiam vt A C, ad C D, ita C D, ad D E. Itaque quia est, ut totus A C, ad totũ C D, ita C D, ex toto A C, detractus ad D E, ex toto C D, detractum; erit quoque vt totus A C, ad totum C D, vel ut detractus C D, ad detractum D E, ita A D, ex A C, residuus, hoc est D E, illi æqualis, ad E C, ex C D, residuum. Quare cum sit vt C D, ad D E, ita D E, ad A C; sit autem C D, maior, quam D E; erit etiam D E, maior, quam E C; Ac proinde ex E D, auferri poterit numerus ipsi E C, æqualis; & sic deinceps, nec unquam finis erit huius detractionis. Quod est absurdum; cum numerus non possit diuidi infinite. Non ergo numerus A B, diuidetur ita, ut planus numerus ex toto in vnam partium factus, æqualis sit quadrato reliquæ partis. Quod est propositum. 11. *septimi*

ALITER. *Quoniam numerus A B, in C, ita diuisus est, vt is qui fit ex A B, in C B, æqualis sit quadrato reliquæ partis A C; erit numerus, qui quater fit ex A B, in C B, quadruplus quadrati ipsius A C; Ac proinde numerus, qui fit quater ex A B,*



in C B, vna cum quadrato ipsius A C, quincuplus erit quadrati partis A C. Est autem numerus contentus quater sub A B, C B, vna cum quadrato ipsius A C, quadratus; quippe cum æqualis sit quadrato numero, qui fit ex numero composito ex A B, & C B, per 8. theorema. Igitur duo numeri quadrati (nimirum is qui quater continetur sub A B, C B, una cum quadrato ex A C; & quadratus ex A C,) proportionem habent, quã 5. ad 1. vel 25. ad 5. Quod est absurdum, ut constat ex coroll. propos. 24. lib. 8. Non ergo numerus A B, diuiditur ita in C, vt is qui producitur ex A B, in C B, æqualis sit quadrato ipsius A C. Quod est propositum.

THEOR. 15. PROPOS. 15. 16.

SI tres numeri deinceps proportionales, fuerint minimi omnium eandem cum ipsis

Figure 51: Book Nine, Proposition 15

ipsis rationem habentium; Duo quilibet compositi, ad reliquum primi erunt.

SINT tres numeri A, B, C, minimi omnium eandem cum illis proportionem habentium. Dico quoslibet duos compositos, ad reliquum primos esse, nimirum A B, simul ad C; & B, C, simul ad A; & A, C, simul ad B. Sumptis enim duobus D, E, in eadem cum illis proportione minimis, ex scholio propos. 35. lib. 7. manifestum est ex demonstratione propos. 2. lib. 8. D, seipsum multiplicantem facere A; multiplicantem vero ipsum E, facere B; atque E, se ipsum multiplicantem facere C. Quia igitur D, E, minimi in sua pro-

A, 9 B, 12. C, 16. D, 3. E, 4.

24. septimi portione inter se primi sunt; erit & uterque D, E, simul ad
30. septimi quemlibet illorum primus. Itaque cum tam compositus ex D, E, quam ipse D, ad E, primus sit; erit quoque numerus
26. septimi factus ex D, E, tanquam uno in D; ad eundem E, primus: Qui autem fit ex D, E, tanquam uno in D, æqualis est per 3. theorema scholij præcedentis, & numero A, facto ex D, in se, & numero B, facto ex D, in E. Igitur & A, B, composi-
27. septimi ti, primi sunt ad E; Ac proinde & ad C, qui factus est ex E, in se, primi sunt A, B, simul compositi.

DEINDE quia, ut prius, uterque D, E, simul pri-
30. septimi mus est ad quemlibet ipsorum D, E; efficitur (cum tam compositus ex D, E, quam ipse E, primus sit ad D,) numerum factum
26. septimi ex D, E, tanquam uno in E, primum esse ad D: Qui autem fit ex D, E, tanquam uno in E, æqualis est per 3. theorema scholij antecedentis, & numero C, facto ex E, in se, & numero B, facto ex D, in E. Igitur & B, C, simul compositi, ad D, primi sunt; atque adeo ad ipsum A,
27. septimi qui factus est ex D, in se, primi sunt B, & C, simul compositi.

B, 12. C, 16. A, 9. D, 3. E, 4.

POSTREMO quia, ut prius, uterque D, & E, simul
30. septimi ad quemlibet ipsorum primus est; atque adeo e contrario, quilibet ipsorum D, E, primus est ad compositum ex D, E;
26. septimi erit quoque qui fit ex D, in E, ad compositum ex D, E, pri-
27. septimi mus; Ac proinde & idem qui fit ex D, in E, ad eum qui fit ex D, E,

Figure 51: Book Nine, Proposition 15 (continued)

ex D, E, tanquã uno, in se, primus erit. Qui autem fit ex D, E, tanquam uno, in se, æqualis est, per 4. theorema antecedentis scholij, eis qui fiunt ex D, & E, in se ipsos, una cum eo qui ex D, in E, bis: Igitur & factus ex D, in E, primus erit ad eos, qui fiunt ex D, & E, in se ipsos, & ex D, in E, bis.

A, 9.		C, 16.
	B, 12.	
D, 3.		E, 4.

Quoniam ergo duo numeri simul, nempe compositus ex ijs, qui fiunt ex D, & E, in se ipsos, & ex eo, qui fit ex D, in E, atque is qui fit ex D, in E, primi sunt ad aliquem ipsorũ, ut ad eum, qui fit ex D, in E, ut ostensum est; Erunt etiam duo illi, nimirum compositus ex ijs, qui fiunt ex D, & E, in se ipsos, & ex eo, qui fit ex D, in E; atque is, qui fit ex D, in E, inter se primi. 30. septimi Rursus quia duo numeri simul, uidelicet compositus ex ijs, qui fiunt ex D, & E, in se ipsos, atque is, qui fit ex D, in E, ad aliquem ipsorum, ut ad eum, qui fit ex D in E, primi sunt, ut ostensum est; erunt etiam duo illi, nimirum compositus ex ijs, qui fiunt ex D, & E, in se ipsos, atque is, qui fit ex D, in E, inter se primi. 30. septimi Cum igitur ex D, & E, in se ipsos fiant A, & C; Item ex D, in E, fiat B; erũt A, & C, simul compositi, primi ad B. Quam ob rem, si tres numeri deinceps proportionales, &c. Quod erat ostendendum.

SCHOLION.

CAMPANVS hoc theorema aliter demonstrat de quotcunque numeris continue proportionalibus minimis, hoc modo ipsum proponens.

SI quotcunque numeri deinceps proportionales, fuerint minimi omnium eandẽ cum ipsis rationem habentium: Ad quemlibet eorũ reliqui omnes simul compositi, erunt primi.

SINT continue proportionales minimi quotcunque numeri A, B, C, D. Dico ad quemlibet eorum reliquos omnes simul compositos, esse primos; Videlicet A, B, C, simul ad D; & A, B, D, simul ad C; & A, C, D, simul ad B; & B, C, D, simul ad

Figure 51: Book Nine, Proposition 15 (continued)

Book Ten, Theorem 93/Proposition 117

It is proposed that we show the diameter of a square figure to be incommensurable with the length of its side.

Let there be square ABCD, in which the diameter is AC. I declare the diameter AC to be incommensurable to the length of the side AB. For if it is not incommensurable, it will be commensurable to the length, and thus, AC and AB would have a proportion between them which is that of a number to a number (X.5). Let AC have to AC the proportion like that of number EF to number G, and numbers EF and G are the least of all numbers which have that proportion (VI.20 and VIII. 11).

Propositum sit nobis ostendere, in quadritis figuris diametrum lateri incommensurabilem esse longitudine.

Sit quadratum ABCD, in quo diameter AC. Dico diamterum AC, longitudine incommensurabilem esse lateri AB. Si enim non est incommensruabilis, commensurabilis erit longitudine; ac propterea AC, AB, proportionem habebunt, quam numerus ad numerum (5 decimi). Habeat AC, ad AB, proportionem, quam numerus EF, ad numerum G; sintque numeri EF, & G, minimi omnium eandem proportionem habentium. (20 sexti & 11 octavi).

Therefore because AC is to AB as the number EF is to the number G, it will be that the square of AC is to the square of AB as the square of the number EF is to the square of the number G (VI.20 and VIII.8). (For the squares have a proportion between them double that of their sides; moreover if the sides have equal proportions, the proportions of the squares are equal since they are the doubles of equals.) But the square on AC is double to the square of AB, by that which we showed in the scholion to proposition 47 in Book 1. Therefore, the square of the number EF, will be double to the square of the number G. And thus, because the square of the number EF has a half, and thus can be divided into two parts, it is even from the definition.

Quoniam ergo est ut AC, ad AB, ita numerus EF, ad numerum G. Erit quoque ut quadratum ex AC, ad quadratum ex AB, ita quadratus numerus ex EF ad quadratum numerum ex G. (Cum enim quadrata habeant suorum laterum proportionem duplicatam latera autem aequales habeant proportiones; erunt proportiones quadratorum aequales etiam, eum sint aequalium duplicatem.) Sed quadratum ex AC, duplum est quadrati ex AB, per ea, quae in scholio propos. 47. lib. 1 ostendimus. Igitur & quadratus numerus ex EF, duplus erit quadrati numeri ex G; Ac propterea quadratus numerus ex EF, cum dimidium habeat, atque adeo bisariam possit dividi, par erit, ex defin.

Therefore, EF itself and that which it produces are even. (For if it were odd, when it multiplied itself to produce its square, it and the square would be odd, which is because an odd number multiplied by and odd number produces an odd number (IX.29) Which is absurd, so it is shown to be even.) Indeed since EF and G are the smallest numbers in their proportion they are prime to each other, and since EF is shown to be even, G will be odd. (For if it were even, both EF and G would be measured by 2, and thus would not be prime to each other. Which is absurd.) Now let E be divided in half at H. Therefore the number EF is double of the number EH, and since squares have a proportion between them double to the proportion between the sides (VIII.11),

Igitur & ipse EF, illum producens par erit. (Si namque impar esset, cum se ipsum multiplicans producat suum quadratum, esset & quadratus ipse impar; eo quod impar imparem multiplicans imparem procreet (29 noni). Quod est absurdum. Ostensus est enim par.) Quia vero EF, & G in sua proportione minimi inter se primi sunt (24 septimi), & EF ostensus est par, erit G, impar. (Si enim par etiam esset, metiretur utrunque EF, & G, binaries, atque adeo non essent inter se primi. Quod est absurdum.) Dividatur iam par numerus EF, bifariam in H. Quia igitur numerus EF, duplus est numeri EH; & quadrati haben proportionem laterum duplicatam (11 octavi);

the square of EF is quadruple the square of EH. (For quadruple is the proportion double to the proportion duple, as is apparent in the number 4, 2, 1.) Thus since the square of EF is double the square of G and quadruple the square of EH, which condition is that if the area of the square of EF is 4, that of the square of G is 2 and that of the square of EH is 1. The square of G is double the square of EH, as those are in the proportion of 2 to 1, and this, as we said above about the number EF, the square of G has a half, and thus it and G are even. But G has been shown to be odd. Which is absurd. Thus, the diameter AC is not commensurable with the length of the side of the square AB. Therefore, it is incommensurable with the length.

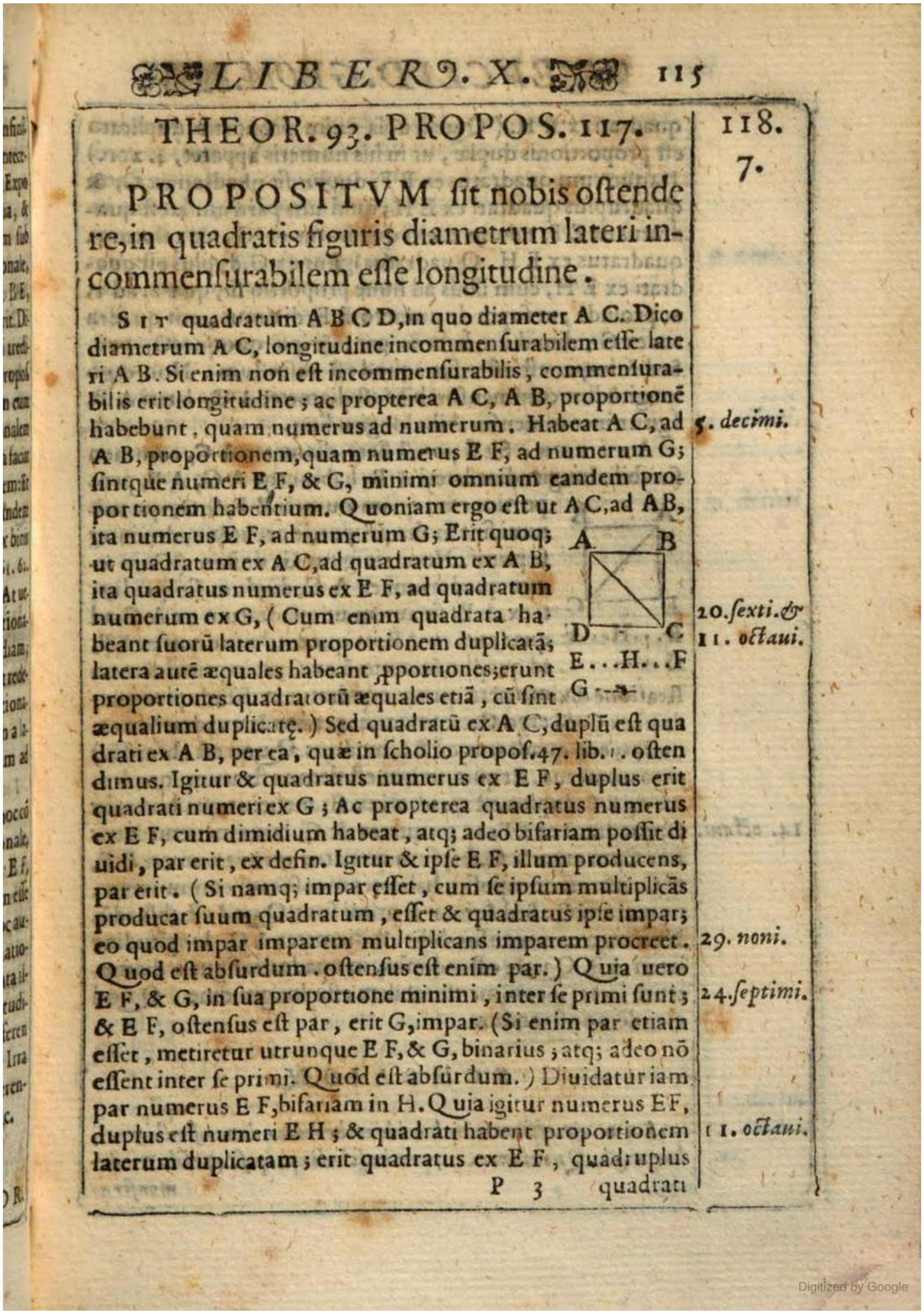
erit quadratus ex EF, quadruplus quadrati ex EH. (Quadrupla enim proportio duplicata est proportionis duplae, ut in his numeris apparet, 4. 2. 1.) Itaque cum quadratus ex EF, duplus sit quadrati ex G; & quadruplus quadrati ex EH; qualium partium 4 est quadratus ex EF, talium 2. Erit quadratus ex G, & talium 1 quadratus ex EH. Quadratus igitur ex G, duplus est quadrati ex EH, cum illius ad hunc proportion sit, quae 2 ad 1. Ac proinde ut supra de numero EF, diximus, erit quadratus ex G, dimidium habens par, & ipse quoque G, par. Sed & impar est ostensus. Quod est absurdum. Non ergo longitudine commensurabilis es diameter AC, lateri AB. Igitur incommensurabilis longitudine.

ALTERNATIVELY. If it can happen that the diameter AC is commensurable with the length of the side AB, and AC and AB have a proportion between them, like that between the numbers EF & G, which are the least numbers in that proportion and are, therefore, prime to each other. Therefore, G is not unity. (Because the square of AC is double to the square of AB, however, the square of AC is to the square of AB as the square of the number EF is the to square of the number G, as we said in the previous demonstration. Likewise, the square of EF is double the square of G. Therefore, if G is unity, and, thus the square of it is unity, the square of EF will be 2. Which is absurd.) Therefore, G is a number.

ALITER. Sit si potest fieri, diameter AC, commnsurabilis longitudine lateri AB, habeantque AC, AB, proportionem, quam numeri EF, &G, qui mnimi sint in sua proportione, atque adeo inter se primi. Non erit igitur G, unitas. (Cum enim quadratum ex AC, duplum sit quadrati ex AB; sit autem ut quadratum ex AC, ad quadratum ex AB, ita quadratum numerus ex EF, ad quadratum numerum ex G, ut in priori demonstratione diximus; erit quoque quadratus ex EF, duplus quadrati ex G. Si ergo G, est unitas, atque adeo & quadratus ex ea, unitas; erit quadratus ex EF, binarius. Quod est absurdum.) ergo numerus.

And since, as has already been demonstrated, the square of EF is double to the square of G, the square of G measures the square of EF, and thus the side of G measures the side of EF (VIII.14). And because G measures itself, the numbers EF and G are composite to one another. But they are prime to one another. Which is absurd. Therefore, the diameter AC is not commensurable to the length of the side AB. Thus we have shown that in a square figure the diameter is incommensurable with the length of the side. Which was to be demonstrated.

Et quia, ut iam est demonstratum, quadratus ex EF duplus est quadrati ex G, quadratum ex EF, ac propterea & G latus metietur latus EF. Cum ergo & G se ipsum metiatur; erunt numeri EF, & G, inter se compositi, habentes mensuram communem, numerum G: Sed & inter se primi sunt. Quod est absurdum. Non ergo commensurabilis est diameter AC, longitudine lateri AB. Quare ostendimus, in quadratis figuris diametrum lateri incommensurabilem esse longitudine. Quod erat demonstrandum.



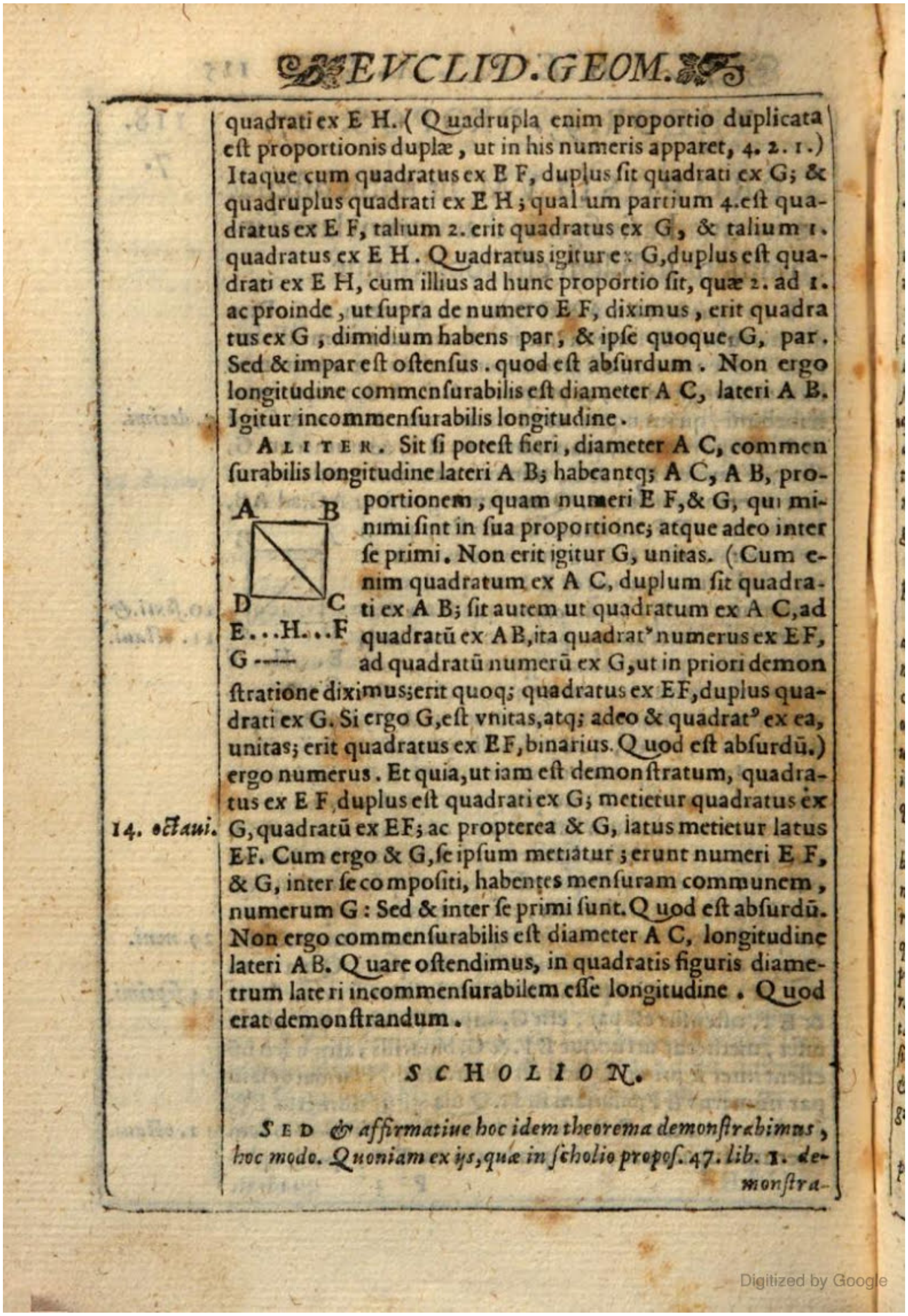
THEOR. 93. PROPOS. 117. 118. 7.

PROPOSITVM ſit nobis oſtendere, in quadratis figuris diametrum lateri incommenſurabilem eſſe longitudine.

Sit quadratum A B C D, in quo diameter A C. Dico diametrum A C, longitudine incommenſurabilem eſſe lateri A B. Si enim non eſt incommenſurabilis, commenſurabilis erit longitudine; ac propterea A C, A B, proportionē habebunt, quam numerus ad numerum. Habeat A C, ad *5. decimi.* A B, proportionem, quam numerus E F, ad numerum G; ſintque numeri E F, & G, minimi omnium eandem proportionem habentium. Quoniam ergo eſt ut A C, ad A B, ita numerus E F, ad numerum G; Erit quoq; ut quadratum ex A C, ad quadratum ex A B, ita quadratus numerus ex E F, ad quadratum numerum ex G, (Cum enim quadrata ha- *20. ſexti. & 11. octaui.* beant ſuorū laterum proportionem duplicatā; latera autē æquales habeant ꝓportiones; erunt proportiones quadratorū æquales etiā, cū ſint æqualium duplicatę.) Sed quadratū ex A C, duplū eſt quadrati ex A B, per ea, quæ in ſcholio propoſ. 47. lib. 1. oſtendimus. Igitur & quadratus numerus ex E F, duplus erit quadrati numeri ex G; Ac propterea quadratus numerus ex E F, cum dimidium habeat, atq; adeo bifariam poſſit diuidi, par erit, ex defin. Igitur & ipſe E F, illum producens, par erit. (Si namq; impar eſſet, cum ſe ipſum multiplicās producat ſuum quadratum, eſſet & quadratus ipſe impar; eo quod impar imparem multiplicans imparem procreet. *29. noni.* Quod eſt abſurdum. oſtenſus eſt enim par.) Quia uero E F, & G, in ſua proportione minimi, inter ſe primi ſunt; *24. ſeptimi.* & E F, oſtenſus eſt par, erit G, impar. (Si enim par etiam eſſet, metiretur utrunque E F, & G, binarius; atq; adeo nō eſſent inter ſe primi. Quod eſt abſurdum.) Diuidatur iam par numerus E F, biſariam in H. Quia igitur numerus E F, duplus eſt numeri E H; & quadrati habent proportionem *11. octaui.* laterum duplicatam; erit quadratus ex E F, quadruplus

P 3 quadrati

Figure 52: Book Ten, Theorem 93/Proposition 117



quadrati ex E H. (Quadrupla enim proportio duplicata eſt proportionis duplæ, ut in his numeris apparet, 4. 2. 1.) Itaque cum quadratus ex E F, duplus ſit quadrati ex G; & quadruplus quadrati ex E H; qualium partium 4. eſt quadratus ex E F, talium 2. erit quadratus ex G, & talium 1. quadratus ex E H. Quadratus igitur ex G, duplus eſt quadrati ex E H, cum illius ad hunc proportio ſit, quæ 2. ad 1. ac proinde, ut ſupra de numero E F, diximus, erit quadratus ex G, dimidium habens par, & ipſe quoque G, par. Sed & impar eſt oſtenſus. quod eſt abſurdum. Non ergo longitudine commenſurabilis eſt diameter A C, lateri A B. Igitur incommenſurabilis longitudine.

ALITER. Sit ſi poteſt fieri, diameter A C, commenſurabilis longitudine lateri A B; habeantq; A C, A B, proportionem, quam numeri E F, & G, qui minimi ſint in ſua proportione; atque adeo inter ſe primi. Non erit igitur G, unitas. (Cum enim quadratum ex A C, duplum ſit quadrati ex A B; ſit autem ut quadratum ex A C, ad quadratũ ex A B, ita quadrat⁹ numerus ex E F, ad quadratũ numerũ ex G, ut in priori demonſtratione diximus; erit quoq; quadratus ex E F, duplus quadrati ex G. Si ergo G, eſt vnitas, atq; adeo & quadrat⁹ ex ea, unitas; erit quadratus ex E F, binarius. Quod eſt abſurdũ.) ergo numerus. Et quia, ut iam eſt demonſtratum, quadratus ex E F, duplus eſt quadrati ex G; metietur quadratus ex

14. octaui. G, quadratũ ex E F; ac propterea & G, latus metietur latus E F. Cum ergo & G, ſe ipſum metiatur; erunt numeri E F, & G, inter ſe compoſiti, habentes menſuram communem, numerum G: Sed & inter ſe primi ſunt. Quod eſt abſurdũ. Non ergo commenſurabilis eſt diameter A C, longitudine lateri A B. Quare oſtendimus, in quadratis figuris diametrum lateri incommenſurabilem eſſe longitudine. Quod erat demonſtrandum.

SCHOLION.

SED & affirmatiue hoc idem theorema demonſtrabimus, hoc modo. Quoniam ex iis, quæ in ſcholio propoſ. 47. lib. 1. demonſtra-

Figure 52: Book Ten, Theorem 93/Proposition 117 (continued)

Book Twelve, Problem 2/Proposition 17

Two spheres around the same circle being given, to inscribe in the greater sphere a solid polyhedron such that it does not touch the surface of the smaller sphere.

Duabus sphaeris circa idem centrum existentibus, in maiori sphaera solidum polyedrum inscribere, quod non tangat minoris sphaerae superficiem.

Let there be two spheres – ABCD and EFGH – around the same center, I, and it is required to inscribe in the larger ABCD a solid polyhedron, or multi-sided figure, such that it does not touch the smaller sphere EFGH. Let both spheres be cut in any plane through the center and the common sections of the plane and spheres make ABCD and EFGH which are circles, from the description of a sphere, having the same center as the spheres, I.

Sint duae sphaerae ABCD, EFGH, circa idem centrum I, oporteatque in maiori ABCD, inscribere solidum polyedrum, seu multilaterum, quod non tangat minorem sphaeram EFGH. Secentur ambae sphaerae plano aliquot per centrum, sintque communes sectiones factae in sphaeris plana ABCD, EFGH, quae circuli erunt, ex descriptione sphaerae, habentes idem centrum sphaerarum I.

For the semicircles, by whose circular revolution the spheres are described, when led around, fall on sections ABCD and EFH. Therefore, those sections can be named circles. Or truly, since all the straight lines falling from I to the edges of the sections are equal, because they are drawn from the center of a sphere to its surface, the same sections are circles from the definition of a circle. Let there be drawn in these circles diameters AC and BD, cutting each other at right angles at the center I, so that there are quadrants, AB, BC, CD, DA, etc. Then in the greater circle ABCD, inscribe a polygon that does not touch the smaller circle EFGH (XII.16). Which indeed, so that the whole claim may be more easily demonstrated, is done in this manner.

Nam semicirculi, ad quorum circumuloutionem sphaerae describuntur, circunducti congruent sectionibus ABCD, EFGH. Quare dictae sectiones circuli erunt, Vel certe, quia omnes lineae rectae cadentes ex I, ad peripherias sectionum sunt aequales, cum ducantur ex centro sphaerarum ad earum superficiem, erunt ipsae sections, criculi, ex definitione circuli. Ducantur in his circulis diametri AC, BD, se se in centro I, secantes ad angulos rectos, ut sint quadrantes AB, BC, CD, DA etc. Deinde in maiori circulo ABCD inscribatur polygonum non tangens minorem circulum EFGH (16 duodec.). Quod quidem, ut facilius omnia demonstrentur in hunc modum efficiatur.

From G to EG draw a perpendicular Gγ, just to the circumference of the circle ABCD which touches circle EFGH only in G, from the corollary to proposition sixteen in book 3. And let the straight line Gγ connected to circle ABCD, be equal to the straight line Aε, and thus if the arc Cγ is understood to subtend the straight line that makes the triangle GCγ (IV.1), the side Cγ, is opposite the greatest angle, namely the right angle, and is greater than the side Gγ, which is opposite to a smaller angle (I.19), evidently an acute angle, and so straight line Cγ is greater than straight line Aε; and thus the arc Cγ will be greater than the arc Aε, as follows from the scholion to proposition 28 in book 3. Let there be introduce therefore arc Cδ, equal to arc Aε.

Ex G, ad EG, ducatur perpendicularis Gγ, ad circunferentiam usque circuli ABCD, quae circulum EFGH, tanget in G, ex coroll. Propos. 16. Lib. 3. Et rectae Gγ, applicetur in circulo ABCD, recta aequalis Aε. Quia vero, si arcui Cγ, intelligatur subtendi recta, ut fiat triangulum GCγ, latus Cγ, oppositum maiori angulo, nempe recto, maius est latere Gγ, quod minori angulo opoonitur, nimirum acuto; erit quoque recta Cγ, maior recta Aε; ac proinde arcus Cγ, arcu Aε maior erit, ut constat ex scholio poropos. 28. Lib. 3. Ablcindatur ergo argus Cδ, arcui Aε aequalis.

If from the quadrant CD half DL is taken, and from the remaining CL half LK, and so forth, there remains thus the smaller arc Cδ, that is the arc Aε (X.1). Therefore the arc CK is less, and the line CK subtending the arc is less than the line Aε, that is than Gγ from the scholion to proposition 29 in book 3. I say therefore that the straight line CK is one side of the inscribed equilateral polygon. For because the straight line that subtends arc Cδ which is smaller than the arc Cγ, it does not touch the circle EFGH, as is clear from the demonstration of the preceding proposition, much less could the straight line CK, subtending an arc smaller than Cδ, touch the same circle.

Quod si ex quadrante CD, dimidium auferatur DL, & ex reliquo CL, dimidium LK, & sic deiceps; relinquetur tandem arcus minor arcu Cδ, seu Aε (1 decimi). Sit ergo iam arcus CK, minor; Eritque recta CK, subtensa minor quam recta Aε, hoc est, quam Gγ, ex scholio propos. 29. Lib. 3. Dico igitur, rectam CK, esse unum latus polygoni aequilateri incribendi. Nam cum recta subtendens arcum Cδ, minorem arcu Cγ, non tangat circulum EFGH, ut ex demonstratione praecedntis propos. patet, multo minus recta CK, subtendens arcum minorem arcu Cδ, eundem circulum tanget.

In turn, draw the diameter KN, and raise up IO from the center I, perpendicular to the plane of the circles ABCD, EFGH, and meeting the surface of the greater sphere in O (XI.12). And between OI, and AC, and OI and KN let there be drawn planes, which are perpendicular to the circle ABCD (XI.18), and make common sections of circles, as has already been said, whose semicircles are AOC and NOK. Since angles OIC and OIK are right, from the third definition of book 11, OC and OK are quadrants, and thus because the circles ABCD, AOC, and NOK are equal, which is because their diameters are all diameters of the greater sphere, the quadrants CD, OC and OK are also equal.

Rursus, ducta diametro KN, erigatur ex centro I, ad plana circulorum ABCD, EFGH, perpendicularis, IO, occurrens superficei sphaerae maioris in O (12 undec.); Et per rectas OI, AC & OI, KN, plana ducantur, quae ad circulum ABCD, recta erunt (18 undec.), efficientque communes sectiones, circulos, ut iam dictum est, quorum semicirculi sint AOC, NOK. Quia vero anguli OIC, OIK, recti sunt, ex defin. 3. Lib. 11, quadrantes erunt OC, OK; atque adeo, cum circuli ABCD, AOC, NOK, aequales sint, quod eorum diametric sint & sphaerae maioris diamteri, erunt quoque quadrantes CD, OC, OK, aequales.

Let therefore the arc DL be divided into as many equal parts as arc CL, and the quadrants OC and OK are divided into arcs equal in number and magnitude to those in the quadrant CD, the straight lines that subtend all of the arcs are equal, evidently CK, KL, LM, MD; CP, PQ, QR, RO; KS, ST, TV, and VO are equal (III.29). Morever, let them be joined by the straight lines PS, QT and RV, and let perpendiculars to the plane of the circle ABCD, PX and SY, fall from P and S, so that they fall on the common sections AC and NK and are parallel to each other (XI.38 and XI.6). Therefore the angle PXC and SYK of the triangles PCX and SKY are right,

Si igitur arcus DL, in tot partes equales distribuatur, in quot divisus fuit arcus CL; Et quadrantes OC, OK, in arcus numero & magnitudine aequales arcubus quadrantis CD; Erunt rectae his omnibus arcubus aequaliubs subtensae, nimirum CK, KL, LM, MD; CP, PQ, QR, RO, KS, ST, TV, VO, aequales (29 tertii). Coniunctis autem rectis PS, QT, RV, demittantur ex P, & S, ad planum circuli ABCD, perpendiculares PX, SY, quae in communes sectiones AC, NK, cadent; eruntque inter se parallelae (38 & 6 undec.) Quoniam igitur triangulorum PCX, SKY, anguli PXC, SYK, recti sunt, from the definition 3 of book 11, and the angles PCS and SKY are equal, which is because the edges AOP and NOS about which they stand are equal (III.27). (For if from the semicircles AOC, and NOK equal arcs CP and KS are taken, the remaining arcs AOP and NOS are also equal.) Because the two angles PCX and PXC of the triangle PCX are equal to the two angles SKY and SYK of the triangle SKY, the sides PC and SK opposite the right angles are also equal (I.26). Therefore, the remaining sides PX and XC are equal to the remaining sides SY and YK. Thus because PX and SY are equal and parallel, if X and Y are connected by a straight line, the lines PS and XY are also equal and parallel (I.33). And therefore, the straight lines CK and XY are parallel

ex defin. 3. Lib. 11. & anguli PCX, SKY, aequales, quod & aequales sint perphaeriae AOP, NOS, quibus insistunt (27 tertii); (Nam si ex semicirculis AOC, NOK aequalibus demantur arcus aequales CP, KS; reliqui arcus AOP, NOS, aequales quoque erunt.) Erunt duo anguli PCX, PXC, trianguli PCX, aequales duobus angulis SKY, SYK, trianguli SKY: Sunt autem & latera PC, SK, rectis angulis opposita, aequalia (26 primi). Igitur reliqua latera PX, XC, reliquis lateribus SY, YK, aequalia erunt. Quare cum rectae PX, SY, aequales sint & parallelae; si connectatur recta XY; aequales quoque erunt & parallelae PS, XY (33 primi). At quia rectae CK, XY, parallelae sunt, because the sides IC and IK are cut proportionally (VI.2). (For if from the semidiameters IC and IK equal lines CX and KY are taken, the remaining IX and IY will be equal. And IX will be in the same proportion to XC as IY is to YK. (XI.9)). PS and CK are also parallel to each other, because both are parallel to XY, and therefore the joining lines, CP and KS, exist in the same plane as those lines (XI.7). Therefore, the whole quadrilateral CKSP is in a single plane. And if from Q and T lines perpendicular to the plane of circle ABCD are dropped and connected to the straight line QC and TK, we can similarly show CK and QT to be parallel and, thus, PS and QT to be parallel to each other, which is because they are both parallel to the same line CK.

quo latera IC, IK, proportionaliter secta sint (2 sexti). (Si enim ex semidiametris IC, IK, aequalibus demantur aequales rectae CX, KY, relinquentur & IX, IY, aequales; Ac proinde erit, ut IX, ad XC, ita IY, ad YK. (9 undec.)) Erunt parallelae quoque: PS, CK, inter se cum utraque parallele sit ipsi XY; ideoque eas coniungentes rectae CP, KS, in eodem cum ipsis plano existent (7 undec). Totum igitur quadrilaterum CKSP, in uno erit plano. Quod si ex Q, & T demittantur ad planum circuli ABCD, perpendiculars, & connectantur rectae QC, TK, ostendemus similiter CK, QT, esse parallelas; atque adeo ipsas PS, QT, inter se parallelas esse cum eidem CK, sint parallelae

And the whole of quadrilateral PSTQ is in one plane. By the same reason, the quadrilateral QTVR will be in one plane. Moreover, the triangle RVO is in one plane (XI.2). If therefore the same construction is produced above the remaining sides KL, LM, and MD, drawn of course in the quadrants OL, OM, and OD, and also in the remaining three quarters and the remaining hemisphere, so that the whole of the greater sphere is completely filled with quadrilaterals and triangles which are similar to the above discussed constructed in between OC and OK and above the side CK, there will have been inscribed in the greater sphere a solid polyhedron which is bounded by the described quadrilateral and triangles. This I say does not touch the smaller sphere, EFGH.

totumque quadrilaterum PSTQ, in uno esse plano. Eadem ratione in uno erit plano quadrilaterum QTVR: Est autem & triangulum RVO, in uno plano (2 undec). Si igitur eadem construction exhibeatur super reliqua latera KL, LM, MD, ductis scilicet quadrantibus OL, OM, OD; necnon in reliquis tribus quartis, ac reliquo hemisphaerio; ut tota sphere maior repleatur quadrilateris & triangulis, quae similia sint praedictis inter quadrantes OC, OK, super latus CK, constructis; inscriptum erit in sphaera maiori solidum polyedrum circunscriptum dictis quadrilateris, atque triangulis. Hoc ergo dico non tangere sphaeram minorem EFGH.

Let there be drawn from I to the plane CKSP, the perpendicular line IZ, and let Z and C and Z and K be connected by straight lines. Thus, since from definition 3 of book 11, the angles IZC and IZK are right, the square on IC will be equal to the sum of the squares on IZ and ZC, and the square on IK will be equal to the sum of the squares on IZ and ZK (I.47). Thus, because the squares of the right lines IC and IK are equal, the sums of the squares on IZ and ZC and the squares on IZ and ZK are equal. And since the square on IZ is common to both, the remaining squares on the lines ZC and ZK are equal. Similarly, we can show the straight lines which are drawn from Z to P and S to be equal to each other and to the straight lines ZC and ZK.

Ducatur ex I, ad planum CKSP, perpendicularis IZ, connectanturque rectae ZC, ZK. Quoniam igitur, ex defin. 3 lib. 11 anguli IZC, IZK, recti sunt; erit quadratum rectae IC, quadratis rectarum IZ, ZC & quadraturm rectae IK quadratis recarum IZ, ZK, aequale (47 primi.) Cum ergo quadrata rectarum aequalium IC, IK, aequalia sint, erunt & quadrata rectarum IZ, ZC, quadratis rectarum IZ, ZK aequalia; Ac proinde dempto communi quadrato IZ, reliqua quadrata rectarum ZC, ZK aequalia erunt, ideoque & ipse rectae ZC, ZK aequlaes. Similiter ostendemus rectas, quae ex Z, ad P, S, ducentur aequales esse & inter se, & rectis ZC, ZK;

Hence a circle described on Z with the interval ZC will pass through the four points C, K, S, P. And by the same reasons we can deomstrate that circles can be described around quadrilaterals PSTQ, QTVR, and triangle RVO. Since, as we will later show, the angle CZK is obtuse, the square on the straight line CK will be greater than the sum of the squares on the straight lines ZC and ZK (II.12), and thus because those squares are equal, the square on the straight line CK is greater than double the square on the straight line ZC.

Let there be drawn from K to the straight line AC the perpendicular line Kα. Since AC is double to AI and Aα is greater than AI, AC is less than double of Aα.

Quare circulus ex Z, ad intervallum ZC, descriptus per quatuor puncta C, K, S, P, transibit. Eademque ratione circa reliqua quadrilatera PSTQ, QTVR, & triangulum RVO, circulos describi posse, demonstrabimus. Quoniam vero, ut postea ostendemus, angulus CZK, obtusus est erit quadratum rectae CK, maius quadratis rectarum ZC, ZK (12 secundi); ideoque cum haec quadrata aequalia sint, maius erit quadratum rectae CK, duplo quadrati rectae ZC.

Ducatur ex K, ad rectam AC, perpendicularis Kα. Cum igitur AC, dupla sit ipsius AI, & Aα maior sit, quam AI, erit AC, minor duplo ipsius Aα.

Since it is the case that as AC is to Aα, so the rectangle contained under AC and αC is to the rectangle contained under Aα and αC, because the bases of those rectangles are AC and Aα and the altitudes are the same αC (VI.1), the rectangle contained under AC and αC will be less than double the rectangle under Aα and αC. Moreover, the rectangle under AC and αC is equal to the square on the straight line CK and the rectangle under Aα and αC is equal to the square on the straight line Kα (VI.17) because the straight line CK is a mean proportional between AC and αC and the straight line Kα is a mean proportional between Aα and αC, from the corollary to proposition 8 in book 6. (For if they are connected by the straight line AK they make the right triangle AKC.)

Quam ob rem cum sit ut AC, ad Aα ita rectangulum sub AC, αC ad rectangulum sub Aα, αC, quod bases horum rectangulorum sint AC, Aα, & eadem altitudo αC (1 sexti); erit quoque rectangulum sub AC, αC, minus duplo rectanguli sub Aα, αC. Est autem rectangulum sub Aα, αC, aequale quadrato rectae CK; & rectangulum sub Aα αC, aequale quadrato rectae Kα (17 sexti); quod rectae CK, inter AC, αC sit media propotionalis; & recta Kα, inter Aα, αC, ex coroll. propos. 8 lib. 6. (Si enim connecteretur recta AK, fieret triangulum rectangulum AKC.)

And therefore, the square on the straight line CK is less than double the square on the straight line Kα. And thus, since the square on the straight line has been shown to be greater than double the square on the line ZC, the square on Kα is greater than the square on ZC. Since, truly, the square on the straight line IC is equal to the sum of the squares on the straight lines IZ and ZC and the square of the straight line IK is equal to the sum of the squares on Iα and αK (I.47), and the squares on the straight lines IC and IK are equal, the squares on lines IZ and ZC are equal to the squares on lines Iα and αK. If therefore, from one of these you take the greater square, truly the square on αK, and from the other the lesser, which is clearly the square on AC, the remaining square on IZ will be greater than the remaining square on Iα;

Igitur & quadraturm rectae CK, minus erit duplo quadrati rectae Kα. Ac propterea cum quadratum rectae CK, ostensum sit maius esse duplo quadrati rectae ZC; erit quadratum rectae Kα, maius quadrato rectae AC. Quoniam vero quadratum rectae IC aequale est quadratis recarum IZ, ZC, & quadrtum rectae IK, quadratis rectarum Iα, αK (47 primi): Suntque aequalia quadrata rectarum aequalum IC, IK; Erunt & quadrata rectarum IZ, ZC, aequalia quadratis rectarum Iα, αK. Si ergo ex his dematur quadratum maius nempe rectae αK; & ex illis minus videlicet rectae ZC, erit reliquum quadratum rectae IZ, maius quadrato reliquo rectae Iα;

and thus the straight line IZ is greater than the straight line Iα. For what reason, the point α does not touch the smaller sphere EFGH because by the corollary of the preceding proposition, Kα is entirely outside of the same sphere. Much less does point Z at a longer distance touch the same sphere. And thus, because, as we will soon show, all of the other points in the plane CKSP are farther removed from the sphere EFGH than point Z, the plane CKSP does not touch the sphere EFGH.

But we can show that the plane CKSP does not touch the smaller sphere, EFGH, more quickly almost from the construction of the figure itself, if as earlier the straight line Iγ is drawn, in this way.

ideoque recta IZ, maior quam recta Iα. Quapropter cum punctum α, non tangat sphaeram minorem EFGH, quod per coroll. propos. Praecedentis recta Kα, tota sit extra dictam sphaeram; multo minus punctum Z, longius distans, eandem sphaeram contingent. Ac proinde cum omnia alia puncta plani CKSP, longius absint a sphaera EFGH, quam punctum Z, ut mox ostendemus, non tanget planum CKSP, sphaeram EFGH.

Sed & expeditious ex ispa fere constructione figurae ostendemus, planum CKSP, non tangere sphaeram minorem EFGH, si prius ducatur recta Iγ, hoc modo.

Since from the construction it made clear that the straight line CK is less than the straight line Gγ, and, moreover, that CK is greater than AC, because, as will soon be demonstrated, the angle CZK is obtuse (I.19). Gγ will then be much greater than ZC. And thus, the square on the straight line Gγ is much greater than the square on the straight line ZC. Since, truly, the square on the straight line Iγ is equal to the sum of the squares on the straight lines IG and Gγ, and the square on the straight line IC is equal to the sum of the squares on the straight lines IZ and ZC (I.47), and the squares on Iγ and IC are equal to equals, the sum of the squares on IG and Gγ are equal to the sum of the squares on IZ and ZC.

Quoniam ex constructione ostensum fuit, rectam CK, minorem esse recta Gγ: Est autem CK, maior quam ZC, quod angulus CZK, obtusus sit (19 primi), ut mox demonstrabitur; Multo maior erit Gγ quam ZC; Ac proinde quadratum rectae Gγ maius quadrato rectae ZC. Quia vero quadratum rectae Iγ, aequale est quadratis rectarum IG, Gγ; & quadratum rectae IC, quadratis rectarum IZ, ZC (47 primi); sunt autem quadrata rectarum Iγ, IC aequalium aequalia; erunt & quadrata rectarum IG, Gγ, quadratis rectarum IZ, ZC, aequalia.

Therefore, if the square of the line Gγ is taken away from that pair, and the square of the line ZC is taken away from the other pair, the remaining square on the line IG is less than the square on IZ. And thus, the line IG is less than IZ. On account of which, because IG is the semidiameter of the smaller sphere EFGH, the point Z is outside of that same sphere. And it follows, as before, that the plane CKSP never touches the sphere EFGH.

Again, let a perpendicular line Iβ be drawn from I to the plane PSTQ. And β will be the center of a circle described around PSTQ as has been demonstrated. Moreover, having joined β to P and I to P with straight lines, in that the angle IβP is right, from the third definition of the eleventh book, the square on the straight line IP will be equal to the sum of the squares

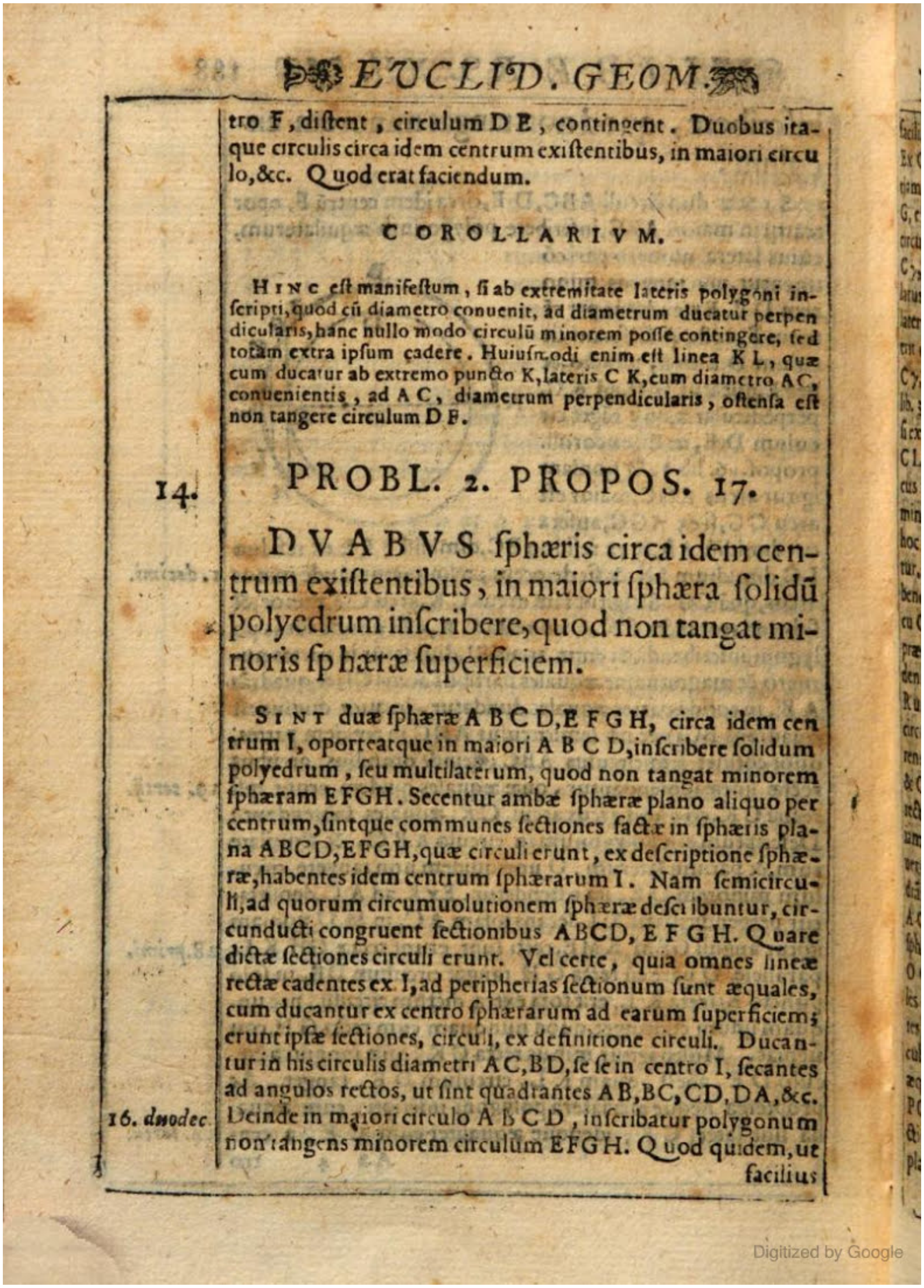
Dempto ergo illinc quadrato rectae Gγ, & hinc quadrato rectae ZC; relinquetur quadratum rectae IG, minus quadrato rectae IZ; Ac propterea recta IG, minor quam IZ. Quam ob rem, cum IG, sit sphaerae minoris EFGH semidiamter; existet punctum Z, extra eandem sphaeram. Et proinde, ut prius, planum CKSP, sphaeram EFGH, nequaquam contingent.

Ducatur rursus ex I, ad planum PSTQ, perpendicularis Iβ, eritque β, centrum circuli circa PSTQ, descripti, ut demonstratum est: Connexis autem rectis βP, IP, cum angulus IβP, rectus sit, ex 3 defin. lib. 11 erit quadratum rectae IP, aequale quadratis on Iβ and βP (I.47). And because the square on the straight line IC (which is equal to the quare on the straight line IP because the straight lines IC and IP are equal) is equal to the sum of the squares of IZ and ZC, the sum of the squares on the straight lines Iβ and βP is equal to the sum of the squares on IZ and ZC. Truly the square on line ZC is greater than the square on the line βP because the line ZC is greater than the line βP, as we will show later. Therefore, the remaining square on Iβ is greater than the remaining square on IZ. And, thus, the line Iβ is greater than the line IZ: And therefore, the point β is much farther outside of the sphere EFGH than the point Z, and because of that much less than the plane CKSP does the plane PSTQ touch the smaller sphere EFGH.

rectarum Iβ, βP (47 primi). Quia vero & quadratum rectae IC, (quod aequale est quadrato rectae IP, ob aequalitatem rectarum IC, IP) aequale est quadratis rectarum IZ, ZC; erunt quadrata rectarum Iβ, βP, quadratis rectarum IZ, ZC, aequalia: est vero quadratum rectae ZC, maius quadrato rectae βP, quod & linea ZC, maior sit, quam linea βP, ut postea ostendemus. Reliquum igitur quadratum rectae Iβ, reliquo quadrato rectae IZ, maius erit; ideoque & linea Iβ, maior quam linea IZ: Ac proinde multo magis punctum β, extra sphaeram EFGH, existet, quam punctum Z, propterea que multo minus planum PSTQ quam CKSP, tanget sphaeram minorem EFGH.

In the same way we can demonstrate that none of the remaining planes can touch the said sphere. Thus, given two spheres around the same conter, we have inscribed a solid polyhedron in the larger sphere that does not touch the surface of the smaller sphere. Which was to be done.

Eodem modo demonstrabimus, quod neque reliqua plana sphaeram dictam contingere possint. Quodcirca, duabus sphaeris circa idem centrum existentibus, in maiori sphaera solidum polyedrum inscripsimus, quod non tangat minoris sphaerae spuerficiem. Quod erat faciendum.



EVCLID. GEOM.

tro F, distent, circulum D E, contingent. Duobus itaque circulis circa idem centrum existentibus, in maiori circulo, &c. Quod erat faciendum.

COROLLARIVM.

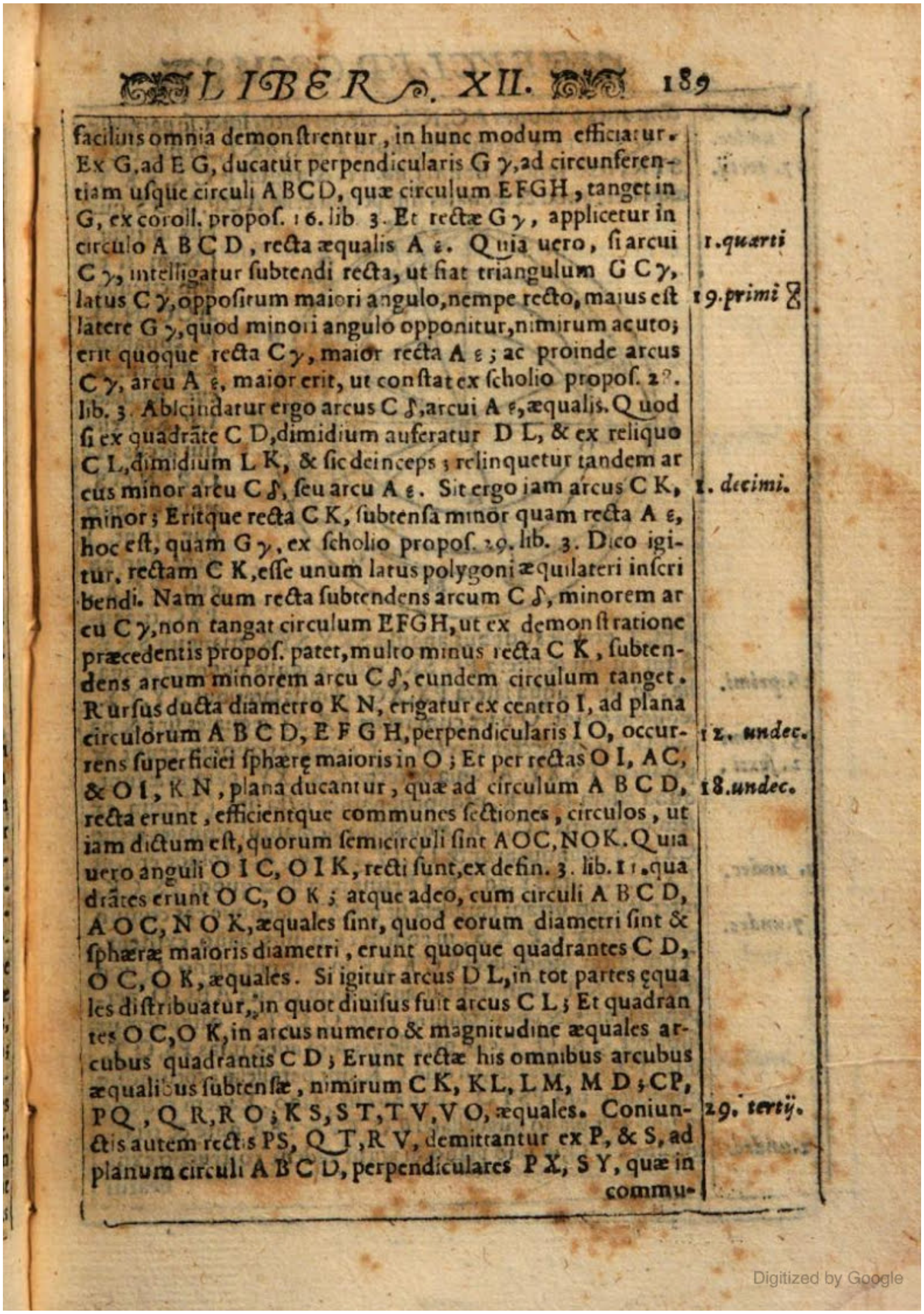
HINC est manifestum, si ab extremitate lateris polygoni inscripti, quod cũ diametro conuenit, ad diametrum ducatur perpendicularis, hanc nullo modo circulũ minorem posse contingere, sed totam extra ipsum cadere. Huiusmodi enim est linea K L, quæ cum ducatur ab extremo puncto K, lateris C K, cum diametro A C, conuenientis, ad A C, diametrum perpendicularis, ostensa est non tangere circulum D F.

14. PROBL. 2. PROPOS. 17.

DVABVS sphæris circa idem centrum existentibus, in maiori sphæra solidũ polyedrum inscribere, quod non tangat minoris sphæræ superficiem.

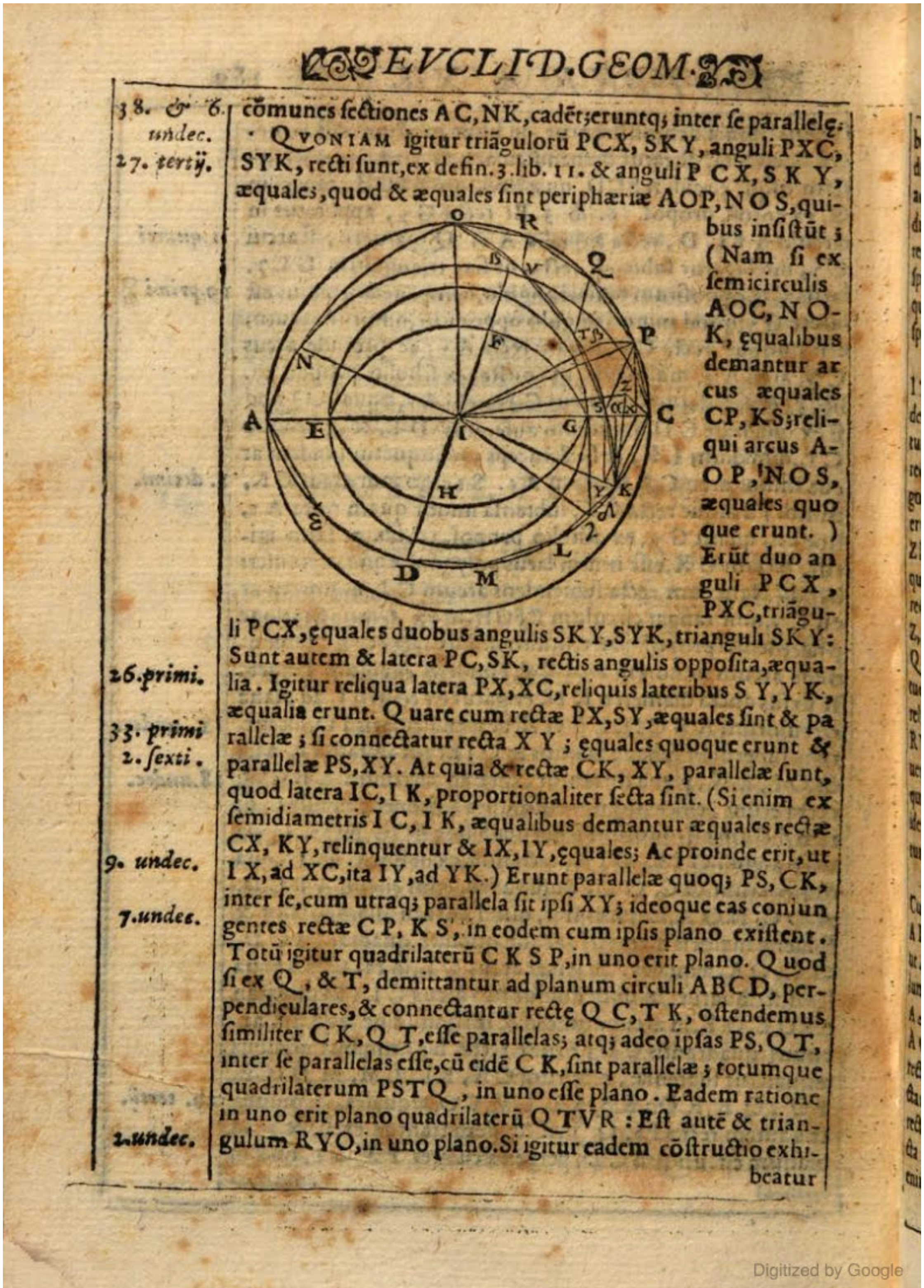
SINT duæ sphæræ A B C D, E F G H, circa idem centrum I, oporteatque in maiori A B C D, inscribere solidum polyedrum, seu multilaterum, quod non tangat minorem sphæram EFGH. Secentur ambæ sphæræ plano aliquo per centrum, sintque communes sectiones factæ in sphæris plana ABCD, EFGH, quæ circuli erunt, ex descriptione sphæræ, habentes idem centrum sphærarum I. Nam semicirculi, ad quorum circumuolutionem sphæræ describuntur, circumducti congruent sectionibus ABCD, E F G H. Quare dictæ sectiones circuli erunt. Vel certe, quia omnes lineæ rectæ cadentes ex I, ad peripherias sectionum sunt æquales, cum ducantur ex centro sphærarum ad earum superficiem; erunt ipsæ sectiones, circuli, ex definitione circuli. Ducantur in his circulis diametri AC, BD, se se in centro I, secantes ad angulos rectos, ut sint quadrantes AB, BC, CD, DA, &c. 16. duodec. Deinde in maiori circulo A B C D, inscribatur polygonum non tangens minorem circulum EFGH. Quod quidem, ut facilius

Figure 53: Book Twelve, Proposition 17



facilius omnia demonstrentur, in hunc modum efficiatur.
Ex G, ad E G, ducatur perpendicularis G γ, ad circunferen-
tiam usque circuli A B C D, quæ circulum E F G H, tanget in
G, ex coroll. propos. 16. lib. 3. Et rectæ G γ, applicetur in
circulo A B C D, recta æqualis A ε. Quia uero, si arcui 1. quarti
C γ, intelligatur subtendi recta, ut fiat triangulum G C γ,
latus C γ, oppositum maiori angulo, nempe recto, maius est 19. primi
latere G γ, quod minori angulo opponitur, nimirum acuto;
erit quoque recta C γ, maior recta A ε; ac proinde arcus
C γ, arcu A ε, maior erit, ut constat ex scholio propos. 28.
lib. 3. Abscindatur ergo arcus C δ, arcui A ε, æqualis. Quod
si ex quadrãte C D, dimidium auferatur D L, & ex reliquo
C L, dimidium L K, & sic deinceps; relinquetur tandem ar
cus minor arcu C δ, seu arcu A ε. Sit ergo iam arcus C K, 1. decimi.
minor; Eritque recta C K, subtensa minor quam recta A ε,
hoc est, quam G γ, ex scholio propos. 29. lib. 3. Dico igi-
tur, rectam C K, esse unum latus polygoni æquilateri inscri
bendi. Nam cum recta subtendens arcum C δ, minorem ar
cu C γ, non tangat circulum E F G H, ut ex demonstratione
præcedentis propos. patet, multo minus recta C K, subten-
dens arcum minorem arcu C δ, eundem circulum tanget.
Rursus ducta diametro K N, erigatur ex centro I, ad plana
circulorum A B C D, E F G H, perpendicularis I O, occur- 12. undec.
rens superficiei sphæræ maioris in O; Et per rectas O I, A C,
& O I, K N, plana ducantur, quæ ad circulum A B C D, 18. undec.
recta erunt, efficientque communes sectiones, circulos, ut
iam dictum est, quorum semicirculi sint A O C, N O K. Quia
uero anguli O I C, O I K, recti sunt, ex defin. 3. lib. 11. qua
drãtes erunt O C, O K; atque adeo, cum circuli A B C D,
A O C, N O K, æquales sint, quod eorum diametri sint &
sphæræ maioris diametri, erunt quoque quadrantes C D,
O C, O K, æquales. Si igitur arcus D L, in tot partes æqua
les distribuatur, in quot diuisus fuit arcus C L; Et quadran
tes O C, O K, in arcus numero & magnitudine æquales ar-
cubus quadrantis C D; Erunt rectæ his omnibus arcubus
æqualibus subtensæ, nimirum C K, K L, L M, M D; C P,
P Q, Q R, R O; K S, S T, T V, V O, æquales. Coniun- 29. tertii.
ctis autem rectis P S, Q T, R V, demittantur ex P, & S, ad
planum circuli A B C D, perpendiculares P X, S Y, quæ in
commu-

Figure 53: Book Twelve, Proposition 17 (continued)



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38. & 6. undec. cõmunes sectiones AC, NK, cadẽt; eruntq; inter se parallelę.

27. tertij. QVONIAM igitur triãgulorũ PCX, SKY, anguli PXC, SYK, recti sunt, ex defin. 3. lib. 11. & anguli PCX, SKY, æquales, quod & æquales sint periphæriæ AOP, NOS, quibus insistũt; (Nam si ex semicirculis AOC, NOK, ęqualibus demantur arcus æquales CP, KS; reliqui arcus AOP, NOS, æquales quoque erunt.) Erũt duo anguli PCX, PXC, triãguli PCX, ęquales duobus angulis SKY, SYK, trianguli SKY: Sunt autem & latera PC, SK, rectis angulis opposita, æqualia.

26. primi. Igitur reliqua latera PX, XC, reliquis lateribus SY, YK, æqualia erunt. Quare cum rectæ PX, SY, æquales sint & parallelæ;

33. primi si connectatur recta XY; ęquales quoque erunt &

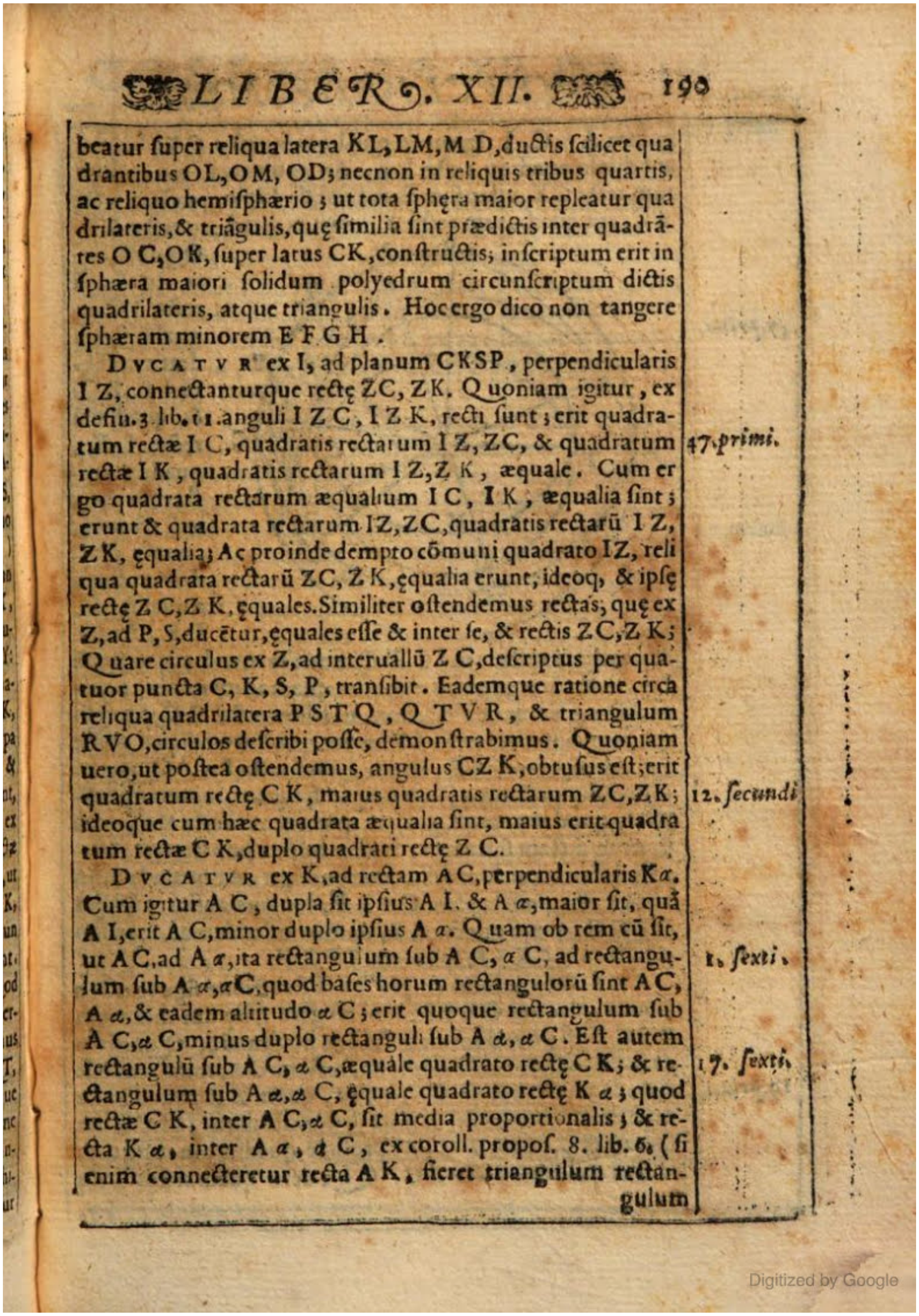
2. sexti. parallelæ PS, XY. At quia & rectæ CK, XY, parallelæ sunt, quod latera IC, IK, proportionaliter secta sint. (Si enim ex semidiametris IC, IK, æqualibus demantur æquales rectæ CX, KY, relinquentur & IX, IY, ęquales; Ac proinde erit, ut

9. undec. IX, ad XC, ita IY, ad YK.) Erunt parallelæ quoq; PS, CK, inter se, cum utraq; parallela sit ipsi XY; ideoque eas coniun-

7. undec. gentes rectæ CP, KS, in eodem cum ipsis plano existent. Totũ igitur quadrilaterũ CKSP, in uno erit plano. Quod si ex Q, & T, demittantur ad planum circuli ABCD, perpendiculares, & connectantur rectę QC, TK, ostendemus similiter CK, QT, esse parallelas; atq; adeo ipsas PS, QT, inter se parallelas esse, cũ eidẽ CK, sint parallelæ; totumque quadrilaterum PSTQ, in uno esse plano. Eadem ratione in uno erit plano quadrilaterũ QTVR: Est autẽ & trian-

2. undec. gulum RYO, in uno plano. Si igitur eadem cõstructio exhibeatur

Figure 53: Book Twelve, Proposition 17 (continued)



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beatur super reliqua latera KL, LM, MD, ductis scilicet quadrantibus OL, OM, OD; necnon in reliquis tribus quartis, ac reliquo hemisphærio; ut tota sphera maior repleatur quadrilateris, & triāgulis, que similia sint prædictis inter quadrātes OC, OK, super latus CK, constructis; inscriptum erit in sphæra maiori solidum polyedrum circunscriptum dictis quadrilateris, atque triangulis. Hoc ergo dico non tangere sphæram minorem EFGH.

Dvcatvr ex I, ad planum CKSP, perpendicularis IZ, connectanturque recte ZC, ZK. Quoniam igitur, ex defin. 3. lib. 11. anguli IZC, IZK, recti sunt; erit quadratum rectæ IC, quadratis rectarum IZ, ZC, & quadratum 47. primi.
rectæ IK, quadratis rectarum IZ, ZK, æquale. Cum ergo quadrata rectarum æqualium IC, IK, æqualia sint; erunt & quadrata rectarum IZ, ZC, quadratis rectarū IZ, ZK, equalia; Ac proinde dempto cōmuni quadrato IZ, reliqua quadrata rectarū ZC, ZK, equalia erunt; ideoq; & ipse recte ZC, ZK, equales. Similiter ostendemus rectas, que ex Z, ad P, S, ducētur, equales esse & inter se, & rectis ZC, ZK; Quare circulus ex Z, ad interuallū ZC, descriptus per quatuor puncta C, K, S, P, transibit. Eademque ratione circa reliqua quadrilatera PSTQ, QTVR, & triangulum RVO, circulos describi posse, demonstrabimus. Quoniam uero, ut postea ostendemus, angulus CZK, obtusus est; erit quadratum recte CK, maius quadratis rectarum ZC, ZK; 12. secundi
ideoque cum hæc quadrata æqualia sint, maius erit quadratum rectæ CK, duplo quadrati recte ZC.

Dvcatvr ex K, ad rectam AC, perpendicularis Kα. Cum igitur AC, dupla sit ipsius AI. & Aα, maior sit, quā AI, erit AC, minor duplo ipsius Aα. Quam ob rem cū sit, ut AC, ad Aα, ita rectangulum sub AC, αC, ad rectangu- 1. sexti.
lum sub Aα, αC, quod bases horum rectangulorū sint AC, Aα, & eadem altitudo αC; erit quoque rectangulum sub AC, αC, minus duplo rectanguli sub Aα, αC. Est autem rectangulū sub AC, αC, æquale quadrato recte CK; & re- 17. sexti.
ctangulum sub Aα, αC, equale quadrato recte Kα; quod rectæ CK, inter AC, αC, sit media proportionalis; & recta Kα, inter Aα, αC, ex coroll. propos. 8. lib. 6. (si enim connecteretur recta AK, fieret triangulum rectangulum

Figure 53: Book Twelve, Proposition 17 (continued)

gulum A K C.) Igitur & quadratum rectæ CK, minus erit duplo quadrati rectæ K α. Ac propterea cum quadratum rectę C K, oſtenſum ſit maius eſſe duplo quadrati rectę Z C; erit quadratum rectæ K α, maius quadrato rectæ Z C. Quoniã uero quadratũ
47.primi. rectæ I C, ęquale eſt quadratis rectarũ I Z, Z C; & quadratũ rectæ I K, quadratis rectarum I α, α K: Suntq; ęqualia quadrata rectarũ æqualiũ I C, I K; Erunt & quadrata rectarum I Z, Z C, æqualia quadratis rectarũ I α, α K. Si ergo ex his dematur quadratum maius, nempe rectæ α K; & ex illis minus, uidelicet rectæ Z C, erit reliquum quadratum rectæ I Z, maius quadrato reliquo rectæ I α; ideoque recta I Z, maior quam recta I α. Quapropter cum punctum α, non tangat ſphęram minorem E F G H, quod per coroll. propoſ. pręcedentis recta K α, tota ſit extra dictam ſphæram; multo minus punctum Z, longius diſtans, eandem ſphæram continget. Ac proinde cum omnia alia puncta plani C K S P, longius abſint a ſphęra E F G H, quam punctum Z, ut mox oſtendemus, non tanget planum C K S P, ſphæram E F G H.

SED & expeditius ex ipſa ſere cõſtructione figurę oſtendemus, planum C K S P, non tangere ſphęram minorem E F G H, ſi prius ducatur recta I γ, hoc modo. Quoniam ex conſtructione oſtenſum fuit, rectam C K, minorem eſſe
19.primi. recta G γ: Eſt autem C K, maior, quam Z C, quod angulus C Z K, obtuſus ſit, ut mox demonſtrabitur; Multo maior erit G γ, quam Z C; Ac proinde quadratum rectę G γ, maius quadrato rectæ Z C. Quia uero quadratum rectę I γ,

Figure 53: Book Twelve, Proposition 17 (continued)

I γ, æquale eſt quadratis rectarum I G, G γ; & quadratum rectæ I C, quadratis rectarum I Z, Z C; ſunt autem quadrata rectarum I γ, I C, æqualium æqualia; erunt & quadrata rectarum I G, G γ, quadratis rectarum I Z, Z C, æqualia. Dempto ergo illinc quadrato rectæ G γ, & hinc quadrato rectæ Z C; relinquetur quadratum rectæ I G, minus quadrato rectæ I Z; Ac propterea recta I G, minor, quam I Z. Quam ob rem, cum I G, ſit ſphæræ minoris E F G H, ſemidiameter; exiſtet punctum Z, extra eandem ſphæram; Et proinde, ut prius, planum C K S P, ſphæram E F G H, nequaquam continget. 47. primi

DVCATVR rurſus ex I, ad planum P S T Q, perpendicularis I β, eritque β, centrum circuli circa P S T Q, deſcripti, ut demonſtratum eſt: Connexis autem rectis β P, I P, cum angulus I β P, rectus ſit, ex 3. defin. lib. 11. erit quadratum rectæ I P, æquale quadratis rectarum I β, β P. Quia uero & quadratum rectæ I C, (quod æquale eſt quadrato rectæ I P, ob æqualitatẽ rectarũ I C, I P) æquale eſt quadratis rectarũ I Z, Z C; erunt quadrata rectarũ I β, β P, quadratis rectarum I Z, Z C, æqualia: eſt uero quadratum rectæ Z C, maius quadrato rectę β P, quod & linea Z C, maior ſit, quã linea β P, ut poſtea oſtendemus. Reliquum igitur quadratũ rectæ I β, reliquo quadrato rectę I Z, maius erit; ideoq; & linea I β, maior quam linea I Z: Ac proinde multo magis punctum β, extra ſphæram E F G H, exiſtet, quam punctum Z; proptereaque multo minus planum P S T Q, quam C K S P, tanget ſphæram minorem E F G H. Eodem modo demonſtrabimus, quod neque reliqua plana ſphæram dictam contingere poſſint. Quocirca, duabus ſphæris circa idem centrum exiſtentibus, in maiori ſphæra ſolidum polyedrum inſcripſimus, quod non tangat minoris ſphæræ ſuperficiem. Quod erat faciendum. 47. primi.

COROLLARIVM.

Ex iis, quæ demonſtrata ſunt, manifeſtum eſt, ſi in quauis alia ſphæra deſcribatur ſolidum polyedrum ſimile prædicto ſolido polyedro, proportionem polyedri in una ſphæra ad polyedrum in altera ſphæra eſſe triplicatam eius, quam habent ſphærarum diametri. Nam ſi ex centris ſphærarum ad omnes angulos baſium dictorum polyedrorum rectæ lineæ ducantur, diſtribuentur polyedra in

Figure 53: Book Twelve, Proposition 17 (continued)

Book Twelve, Problem 2/Proposition 17, Additional Proof

Note: In Chapter 5, I discuss the last of the three items that Clavius promised to demonstrate after the proof for this proposition (XII.17). He followed through on that promise in the scholion. The proof for the third claim, which states that ZC is greater than the line βP is as follows.

ZC is greater than the line βP.

Third, and the last thing that needs to be shown, the straight line ZC is greater than the straight line βP. Which is closely tied to the prior demonstration that the straight line PS is greater than the straight line QT. Let there be described therefore the part of the above figure, that one may see, which contains the semidiamters IC and IO and the quadrants OC and OK, etc. Then let fall from Q and T to the plane of the circle ABCD, in which the triangle ICK is, perpendiculars Qμ and Tξ, which fall on the common sections

ZC maior quam βP.

Tertio, ac ultimo probandum est, rectam ZC, maiorem esse recta βP. Quod ut aptius fiat, demonstrandum prius erit, rectam PS, maiorem esse recta QT. Describatur igitur pars superioris figurae, ea videlicet, quae continetur semidiamteris IC, IO, & quadrantibus OC, OK, &c. Demittantur deinde ex Q, & T, ad planum circuli ABCD, in quo est triangulum ICK, perpendiculars Qμ,, Tξ quae in communes sectiones

IC and IK and are parallel to each other, as was said for the straight lines PX and SY (XI.38 and XI.6). If the straight line μξ is added, QT and μξ are parallel and equal, which can be shown in the same way that PS and XY were shown to be parallel and equal. And μξ is parallel to CK itself, and because the sides IC and IK are cut proportionally in μξ, just as we said of the straight line XY, XY and μξ are also parallel (VI.2, I.30). Hence, by the corollary to the fourth proposition of book 6, IY will be to YX as Iξ is to ξμ. Moreover, IY is greater than Iξ. Therefore, YX is greater than ξμ. And, thus, PS, which is equal to XY, is greater than QT, which is equal to ξμ.

IC, IK, cadent, eruntque inter sese parallelae, ut de rectis PX, SY, dictum est (38 undec. 6 undec.) Quod si adiungatur recta μξ, erunt QT, μξ, parallela & aequales, quemadmodum ostensum fuit parallelas esse & aequales PS, XY. Quia vero μξ, ipsi CK, parallela est, quod latera IC, IK proportionaliter sint secta in μ & ξ, veluti diximus de recta XY; erunt quoque μξ, XY, parallela (2 sexti, 30 primi). Quare erit, ex coroll. propos. 4 lib. 6 ut IY, ad YX, ita Iξ ad ξμ; est autem IY, maior quam Iξ. Igitur & YX, maior erit quam ξμ; Ac proinde & PS, quae aequlis est ipsi XY, maior erit quam QT, quae aequalis est ipsi ξμ.

Therefore, for this demonstration let there be described circles from centers Z and β around the quadrilaterals CK, SP, PSTQ, and lead straight lines ZC, ZK, ZX, ZP; βP, βS, βT, and βQ from the centers Z and β. If therefore ZC is not believed to be greater than βP, it will be either equal or smaller. Let it first be equal. Therefore, because the sides ZK and ZC are set to be equal to the sides βS and βP and the base KC is greater than the base PS, the angle KZC will be greater than the angle SβP (I.25). By the same reason, angle SZP will be greater than angle TβQ. And since the bases KS and CP are equal to the bases ST and PQ, the angles KZS and CZP are equal to the angles SβT and PβQ (I.8). Therefore, the four angles on Z are greater than the four angles on β.

Hoc ergo demonstrato, describantur ex centris Z, β, circa quadrilatera CKSP, PSTQ, circuli egredianturque e centris rectae ZC, ZK, ZS, ZP, βP, βS, βT, βQ. Si igitur ZC, non credatur maior, quam βP, erit vel aequlis, vel minor. Sit primum aequalis. Quia ergo latera ZK, ZC aequal ponuntur lateribus βS, βP; & basis KC, maior est base PS; erit angulus KZC, maior angulo SβP (25 primi). Eadem ratione maior erit anguls SZP, angulo TβQ. At quoniam bases KS, CP, basibus ST, PQ, sunt aequales; erunt anguli KZS, CZP, angulis SβT, PβQ, aeuales (8 primi). Igitur quatuor anguli ad Z, maiores erunt quatuor angulis ad β

However they are also equal because these, like those, are equal to four right angles from corollary 2 to proposition 15 of book 1. Which is absurd. Therefore, straight line ZC is not equal to straight line βP.

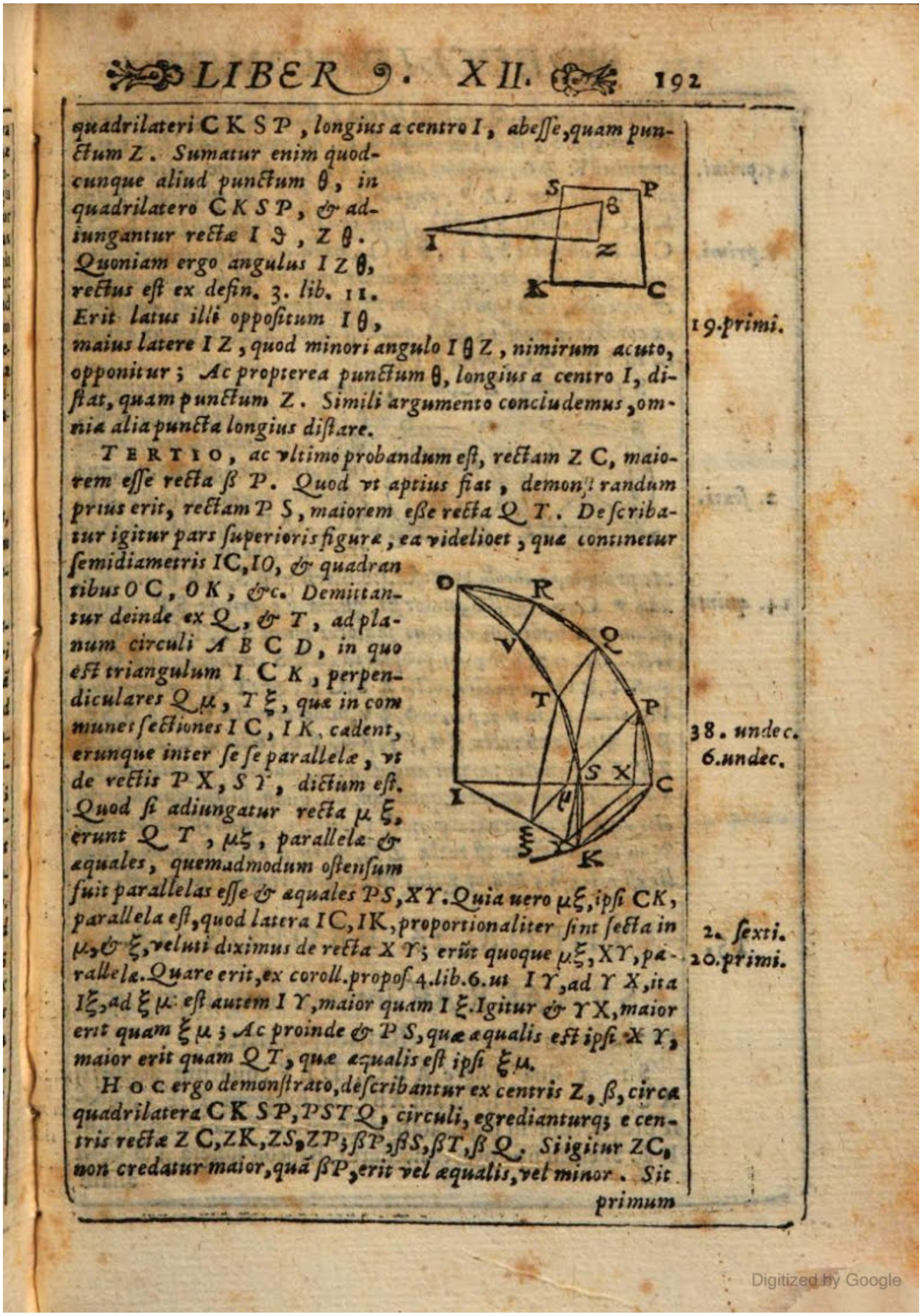
Second, let ZC be smaller than βP. And take βπ, βρ, βω, βφ, to be equal to ZC, ZK, ZS, and ZP, and draw let them be connected by the straight lines πρ, ρω, ωφ, and φπ, which are parallel to the straight lines PS, ST, TQ, and QP. This is because the straight lines from the center are cut proportionally (VI.2). And thus from the corollary to the fourth proposition in Book 6, βS will be to SP as βρ is to ρπ. Because, therefore, βS is greater than βρ, SP is greater than ρπ (V.14). By the same reason ST, TQ, and QP are greater than ρω, ωφ, and φπ.

Sunt autem & aequals cum tam hi quam illi quatuor recti sint aequales, ex coroll. 2. Propos. 15 lib 1. Quod est absurdum. Non igitur aequlis est recta ZC, recta βP.

Sit secundo ZC, minor quam βP. Et abscindantur βπ, βρ, βω, βφ, ipsis ZC, ZK, ZS, ZP, aequales, connectanturque rectae πρ, ρω, ωφ, φπ, quae parallelae erunt rectis PS, ST, TQ, and QP, eo quod rectae ex centris sectae sint proportionaliter (2 sexti); Ac proinde, ex coroll. propos. 4 lib. 6 erit ut βS, ad SP, ita βρ ad ρπ. Cum ergo βS, maior sit, quam βρ, erit & SP, maior quam ρπ (14 quinti). Eademque ratione maiores erunt ST, TQ, QP, rectis ρω, ωφ, φπ;

And, thus, because PS is smaller than CK, and ST, PQ are equal to the straight lines KS, CP, and TQ is smaller than PS, the straight lines πω, ρω, ωφ, and φπ are smaller than the straight lines CK, KS, SP, and PC. Hence, because the straight lines βπ, βρ, βω, βφ are equal to the straight lines ZC, ZK, ZS, and SP, the angles on Z are greater than the angles on β. However, they are also equal, because all of those, like all of these, are equal to four right angles from the second corollary to proposition 15 in book 1. Which is absurd. Therefore, the straight line ZC is not smaller than the straight line βP. But it was shown that it also was not equal. Therefore it is greater. Which was to be shown.

Ac propterea cum PS, minor sit quam CK, & ST, PQ, aequles rectis KS, CP; & TQ, minor quam PS, erunt rectae πρ, ρω, ωφ, φπ, minors rectis CK, KS, SP, PC. Quare cum rectae βπ, βρ, βω, βφ, rectis ZC, ZK, ZS, ZP, sint aequales; erunt anguli ad Z, maiores anguils at β: Sunt autem & aequales, quod tam illi, quam hi sint quatuor rectis aequles ex coroll. 2 propos. 15 lib. 1. Quod est absurdum. Non igitur minor est recta ZC, quam βP: Sed neque aequalis est ostens; Maior igitur est. Quod erat ostendendum.



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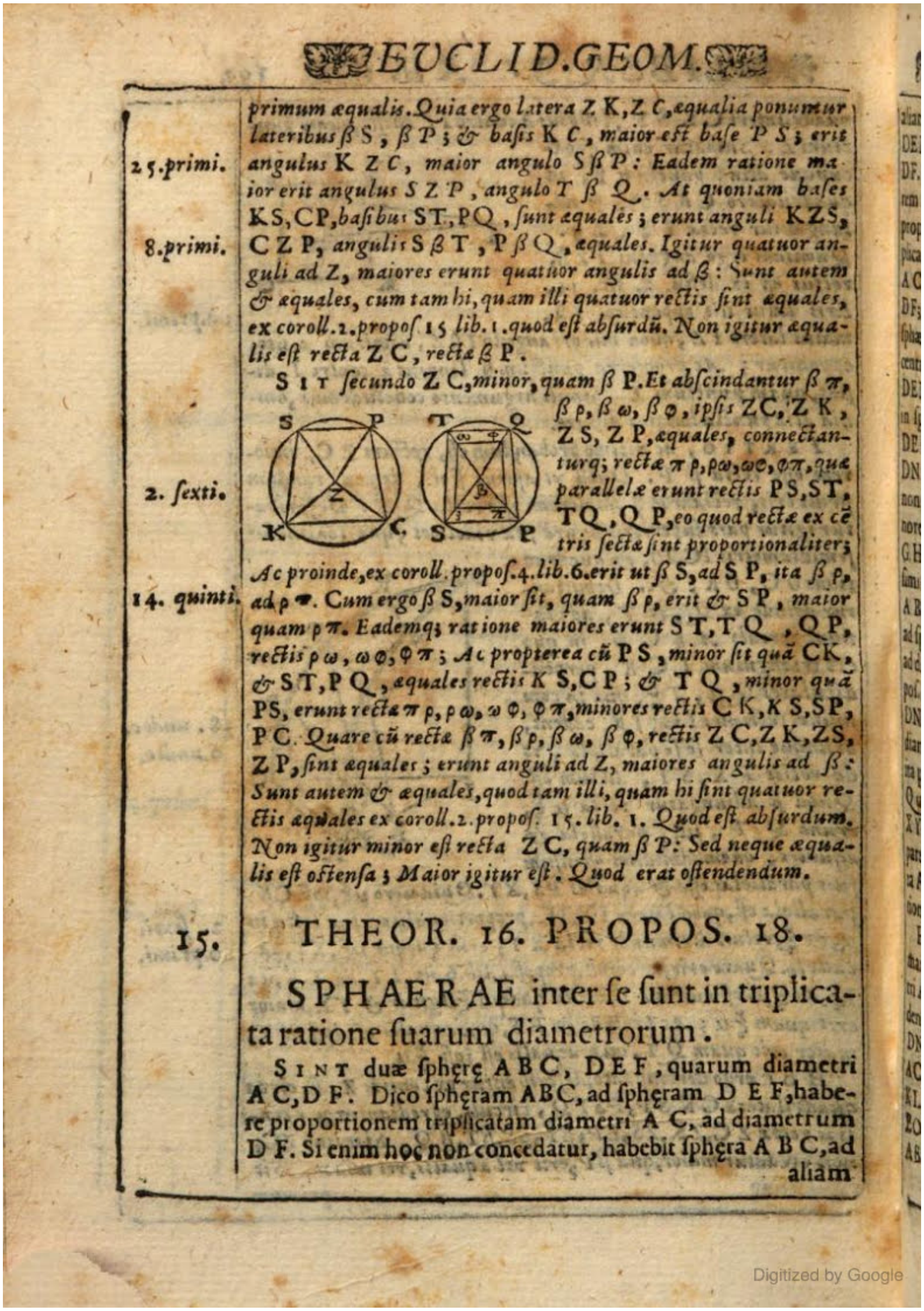
quadrilateri C K S P, longius a centro I, abeſſe, quam punctum Z. Sumatur enim quodcunque aliud punctum θ, in quadrilatero C K S P, & adiungantur rectæ I θ, Z θ. Quoniam ergo angulus I Z θ, rectus eſt ex defin. 3. lib. 11. Erit latus illi oppoſitum I θ, 19.primi.
maius latere I Z, quod minori angulo I θ Z, nimirum acuto, opponitur; Ac propterea punctum θ, longius a centro I, diſtat, quam punctum Z. Simili argumento concludemus, omnia alia puncta longius diſtare.

TERTIO, ac vltimo probandum eſt, rectam Z C, maiorem eſſe recta β P. Quod vt aptius fiat, demonſtrandum prius erit, rectam P S, maiorem eſſe recta Q T. Deſcribatur igitur pars ſuperioris figuræ, ea videlicet, quæ continetur ſemidiametris IC, IO, & quadrantibus O C, O K, &c. Demittantur deinde ex Q, & T, ad planum circuli A B C D, in quo eſt triangulum I C K, perpendiculares Q μ, T ξ, quæ in communes ſectiones I C, I K, cadent, 38. undec.
erunque inter ſe ſe parallelæ, vt 6.undec.
de rectis P X, S Υ, dictum eſt. Quod ſi adiungatur recta μ ξ, erunt Q T, μ ξ, parallelæ & æquales, quemadmodum oſtenſum fuit parallelas eſſe & æquales P S, X Υ. Quia uero μ ξ, ipſi C K, parallela eſt, quod latera I C, I K, proportionaliter ſint ſecta in 2. ſexti.
μ, & ξ, veluti diximus de recta X Υ; erũt quoque μ ξ, X Υ, pa- 20.primi.
rallelæ. Quare erit, ex coroll. propoſ 4. lib. 6. ut I Υ, ad Υ X, ita I ξ, ad ξ μ: eſt autem I Υ, maior quam I ξ. Igitur & Υ X, maior erit quam ξ μ; Ac proinde & P S, quæ æqualis eſt ipſi X Υ, maior erit quam Q T, quæ æqualis eſt ipſi ξ μ.

HOC ergo demonſtrato, deſcribantur ex centris Z, β, circa quadrilatera C K S P, P S T Q, circuli, egrediunturq; e centris rectæ Z C, Z K, Z S, Z P; β P, β S, β T, β Q. Si igitur Z C, non credatur maior, quã β P, erit vel æqualis, vel minor. Sit

primum

Figure 54: Book Twelve, Proposition 17, Additional Proof



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primum æqualis. Quia ergo latera Z K, Z C, equalia ponuntur lateribus β S, β P; & baſis K C, maior eſt baſe P S; erit
25.primi. angulus K Z C, maior angulo S β P: Eadem ratione maior erit angulus S Z P, angulo T β Q. At quoniam baſes KS, CP, baſibus ST, PQ, ſunt equales; erunt anguli KZS,
8.primi. C Z P, angulis S β T, P β Q, equales. Igitur quatuor anguli ad Z, maiores erunt quatuor angulis ad β: Sunt autem & equales, cum tam hi, quam illi quatuor rectis ſint equales, ex coroll. 2. propoſ. 15 lib. 1. quod eſt abſurdũ. Non igitur equalis eſt recta Z C, rectæ β P.

Sit ſecundo Z C, minor, quam β P. Et abſcindantur β π, β ρ, β ω, β φ, ipſis ZC, Z K, Z S, Z P, equales, connectanturq; rectæ π ρ, ρω, ωφ, φπ, quæ
2. ſexti. parallelæ erunt rectis PS, ST, TQ, Q P, eo quod rectæ ex cẽtris ſectæ ſint proportionaliter; Ac proinde, ex coroll. propoſ. 4. lib. 6. erit ut β S, ad S P, ita β ρ,
14. quinti. ad ρ π. Cum ergo β S, maior ſit, quam β ρ, erit & S P, maior quam ρ π. Eademq; ratione maiores erunt S T, T Q, Q P, rectis ρ ω, ω φ, φ π; Ac propterea cũ P S, minor ſit quã CK, & S T, P Q, equales rectis K S, C P; & T Q, minor quã PS, erunt rectæ π ρ, ρ ω, ω φ, φ π, minores rectis C K, K S, SP, P C. Quare cũ rectæ β π, β ρ, β ω, β φ, rectis Z C, Z K, ZS, Z P, ſint equales; erunt anguli ad Z, maiores angulis ad β: Sunt autem & æquales, quod tam illi, quam hi ſint quatuor rectis aquales ex coroll. 2. propoſ. 15. lib. 1. Quod eſt abſurdum. Non igitur minor eſt recta Z C, quam β P: Sed neque æqualis eſt oſtenſa; Maior igitur eſt. Quod erat oſtendendum.

15. THEOR. 16. PROPOS. 18.

SPHAERAE inter ſe ſunt in triplicata ratione ſuarum diametrorum.

Sint duæ ſphęrę A B C, D E F, quarum diametri A C, D F. Dico ſphęram ABC, ad ſphęram D E F, habere proportionem triplicatam diametri A C, ad diametrum D F. Si enim hoc non concedatur, habebit ſphęra A B C, ad aliam

Figure 54: Book Twelve, Proposition 17, Additional Proof (continued)

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