S

A SKETCH ON INFINITESIMAL CALCULUS

Benedict Sestini SJ

GEORGETOWN
J·V·G and Co
LITHOGRAPHERS
1863

1 Finite infinitesimal and infinite quantities

We call finite quantity any quantity whose magnitude is measurable or contained within measurable limits.

A quantity so small that it escapes all measurements and thereby all numerical value however small, is called *infinitesimal*

A quantity whose magnitude exceeds all measurement and cannot, consequently, be represented by any numerical value however great, is called *infinite*

We know from geometry that a regular polygon circumscribed about a circle or inscribed in it and whose sides constantly increase in number, would finally coincide with the circle. The number of sides of such coinciding polygon surpassing all measure presents an example of the mathematical infinite: the length of each side of the same polygon, being less than any assignable measure, presents an example of the infinitesimal.

An infinitesimal would cease to be a quantity if it would be destitute of all magnitude: an infinite number and thereby a quantity actually infinite involves repugnance; hence the infinite and the infinitesimal, notwithstanding their not being susceptible of measurement, necessarily admit of a magnitude within definite limits. Hence infinite and infinitesimal quantities may be of different orders.

Let $\underline{a}$ be any finite quantity, $\underline{\alpha}$ an infinitesimal and $\underline{\beta}$ an infinite quantity; the numerical value of $\underline{\alpha}$ being smaller than any assignable fraction and that of $\underline{\beta}$ greater than any assignable number, it follows first that the product $a \cdot \alpha$ is infinitesimal and the product $a \cdot \beta$ is infinite, secondly the ratios $\frac{1}{\alpha}, \frac{a}{\alpha}$ are infinite and the ratios $\frac{1}{\beta}, \frac{a}{\beta}$ are infinitesimal.

Since

$$\frac{1}{\alpha} = \frac{\alpha}{\alpha^2} = \frac{\alpha^2}{\alpha^3} = \cdots = \frac{\alpha^{m-1}}{\alpha^m}$$

$$\frac{1}{\beta} = \frac{\beta}{\beta^2} = \frac{\beta^2}{\beta^3} = \cdots = \frac{\beta^{m-1}}{\beta^m},$$

it follows that $\underline{\alpha}^2$ bears the same relation to $\underline{\alpha}$ and $\underline{\alpha}^3$ to $\underline{\alpha}^2$ &c as α does to 1 or to $\underline{a}$. Therefore taking $\underline{\alpha}$ as an infinitesimal of the first order, $\underline{\alpha}^2$ is an infinitesimal of the second order $\underline{\alpha}^3$ of the third order and $\underline{\alpha}^m$ of the m^{th} order. In like manner $\beta^2, \beta^3, \cdots \beta^m$ are infinites of the second of the third of the $\underline{m}^{th}$ order relatively to β.

It is plain that two infinitesimals $\underline{\alpha}$ & α' and two infinites $\underline{\beta}$ & $\underline{\beta}'$, whose ratios $\frac{\alpha}{\alpha'}, \frac{\beta}{\beta'}$ are finite, are of the same order.

Let $\underline{a}_1', \underline{a}_2'', \underline{a}_3$ &c be finite quantities. from the fact that the magnitude of the infinitesimal is less than any assignable quantity it follows that

$$a \text{ and } a + a_1\alpha + a_2\alpha^2 + \cdots$$
$$a_1\alpha \text{ and } a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots$$
$$a_2\alpha^2 \text{ and } a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4 + \cdots$$

may be considered as equal and however questionable in strict mathematical precision might be the accuracy of the equations

$$a = a + a_1\alpha + a_2\alpha^2 + \cdots$$
$$a_1\alpha = a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots$$
$$a_2\alpha^2 = a_2\alpha^2 + a_3\alpha^3 + \cdots \quad \&c - ;$$

they are certainly admissible in physical questions without any perceptible error resulting from them.

Making $\beta = \frac{1}{\alpha_1}$

$$a\beta + a_1\beta^{m-1} + a_2\beta^{m-2} + \cdots + a_{m-1}\beta + a_m =$$
$$\frac{1}{\alpha_1^m}\left(a + a_1\alpha_1 + a_2\alpha_1^2 + \cdots + a_{m-1}\alpha^{m-1} + a_m\alpha_1^m\right)$$

now α_1 being infinitesimal

$$a = a + a_1\alpha_1 + a_2\alpha_1^2 + \cdots + a_m\alpha_1^m$$

consequently

$$a\beta^m = a\beta^m + a_1\beta^{m-1} + a_2\beta^{m-2} + \cdots + a_{m-1}\beta + a_m$$

we obtain in like manner

$$a_1\beta^{m-1} = a_1\beta^{m-1} + a_2\beta^{m-2} + \cdots + a_m$$
$$a_2\beta^{m-2} = a_2\beta^{m-2} + \cdots + a_m$$

finally

$$a_{m-1}\beta = a_{m-1}\beta + a_m$$

All these equations may be epitomized in the following principles

Infinitesimal quantities can be suppressed without error when added to or taken from finite quantities, or when added to or taken from infinitesimals of inferior orders.

Finite quantities added to or taken from infinite quantities and infinite quantities added to or taken from infinite of higher orders can also be suppressed without error

2 Explicit and Implicit functions. An expression composed of constant and variable quantities is called a function of the variables. If the value of one of the variables depends on that of the others, the dependent variable is called a function of the others: thus in the equation $y = ax + cz + d$, in which y, x, z are variable and the value of y depends on those given to x, and z, the variable y is a function of the independent variables x and z. The same y being given in the form of a resolved equation, is called an explicit function. When the expression containing the dependent and independent variables does not present the form of a resolved equation relatively to the dependent variable the function is called implicit for example in the equation $y + mx = az + b$, y is called an implicit function of x and z.

If we wish to express that y is an explicit function of $x, z, \cdots$ without writing the form of the function we adopt the notation

$y = F(x, z, \cdots)$ or some similar as for example $y = f(x, z, \cdots)$, $y = \varphi(x, z, \cdots)$.

3 Differential of the variable and of function — Examples

An infinitesimal change made in the independent variables is called the differential of the variables: the change of the function attending that of the variables is called the Differential of the function. Let

$$y = f(x)$$

be any function of x: the differential of x is expressed by dx and that of y by dy: hence

$$dy = f(x+dx) - f(x)$$

dy results differently for different functions: let us see some examples

1st Example $\quad y = ax + C$

we shall have

$$dy = a(x+dx) + C - (ax+C)$$

hence $\quad dy = a\,dx$

2d Example $\quad y = \frac{a}{x} + C$

from this

$$dy = \frac{a}{x+dx} + C - \left(\frac{a}{x} + C\right)$$

$$= \frac{a}{x+dx} - \frac{a}{x}$$

$$= -\frac{a\,dx}{x^2 + x\,dx}$$

an expression in which the denominator is equivalent to x^2 on account of the infitesimal $x\,dx$. hence

$$dy = -\frac{a\,dx}{x^2}$$

3^d Example. Set

$$y = x^a + C$$

and consequently

$$dy = (x+dx)^a + C - (x^a + C)$$
$$= (x+dx)^a - x^a$$

In the supposition that a is a whole and positive number

$$(x+dx)^a = x^a + ax^{a-1}dx + \frac{a(a-1)}{2}x^{a-2}dx^2 + \dots + dx^a$$

therefore

$$dy = ax^{a-1}dx$$

4^th Example From

$$y = \sin x + C$$

we obtain

$$dy = \sin(x+dx) + C - (\sin x + C)$$
$$= \sin(x+dx) - \sin x$$

now

$$\sin(x+dx) - \sin x = 2\cos\tfrac{1}{2}((x+dx)+x)\sin\tfrac{1}{2}dx$$

but $\sin\frac{1}{2}dx = \frac{1}{2}dx$ and $\cos\frac{1}{2}(x+dx+x) = \cos x$

therefore

$$dy = \cos x\, dx.$$

5^th Example Let

$$y = \cos x + C$$

from which

$$dy = \cos(x+dx) + C - (\cos x + C)$$
$$= \cos(x+dx) - \cos x$$
$$= 2\sin\tfrac{1}{2}(x+x+dx)\sin\tfrac{1}{2}(-dx)$$

therefore

$$dy = -\sin x \cdot dx$$

6^th Example Set $y = tg x + C$

hence

$$dy = tg(x+dx) + C - (tgx + C)$$
$$= tg(x+dx) - tgx$$
$$= \frac{\sin(x+dx)}{\cos(x+dx)} - \frac{\sin x}{\cos x}$$
$$= \frac{\sin(x+dx)\cos x - \cos(x+dx)\sin x}{\cos(x+dx)\cos x}$$

now $\sin(x+dx)\cos x - \sin x\cos(x+dx) = \sin dx = dx$ and $\cos(x+dx)\cos x = \cos^2 x$, therefore.

$$dy = \frac{dx}{\cos^2 x}$$

7[th] Example Let

$$y = \log x + C$$

from which

$$dy = \log(x+dx) + C - (\log x + C)$$
$$= \log(x+dx) - \log x$$
$$= \log\frac{x+dx}{x}$$
$$= \log\left(1+\frac{dx}{x}\right) :$$

observe that $\log\left(1+\frac{dx}{x}\right) = \frac{dx}{x}\log\left(1+\frac{dx}{x}\right)^{\frac{x}{dx}}$ the exponent $\frac{x}{dx}$ being infinite, its numerical value may be taken as having the form of a whole number for the difference of a fraction in the infinite disappears. Taking besides dx of the same sign of x, we have

$$\left(1+\frac{dx}{x}\right)^{\frac{x}{dx}} = 1^{\frac{x}{dx}} + \frac{x}{dx}\cdot\frac{dx}{x} + \frac{1}{2}\frac{x}{dx}\left(\frac{x}{dx}-1\right)\left(\frac{dx}{x}\right)^2$$
$$+ \frac{1}{2\cdot 3}\frac{x}{dx}\left(\frac{x}{dx}-1\right)\left(\frac{x}{dx}-2\right)\left(\frac{dx}{x}\right)^3 + \cdots$$
$$= 2 + \frac{1}{2}\left(\frac{x}{dx}\right)^2\left(1-\frac{dx}{x}\right)\left(\frac{dx}{x}\right)^2 +$$
$$\frac{1}{2\cdot 3}\left(\frac{x}{dx}\right)^3\left(1-\frac{dx}{x}\right)\left(1-2\frac{dx}{x}\right)\left(\frac{dx}{x}\right)^3 + \cdots$$

consequently

$$\left(1+\frac{dx}{x}\right)^{\frac{x}{dx}} = 2+\frac{1}{2}\left(1-\frac{dx}{x}\right)+\frac{1}{2\cdot3}\left(1-\frac{dx}{x}\right)\left(1-2\frac{dx}{x}\right)+\cdots$$
$$= 2+\frac{1}{2}+\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots\cdots$$
$$= 2,718281828\cdots\cdots$$

Thus

$$dy = \frac{dx}{x}\log 2,7182818\cdots$$

and, taking 2,718281… for base of logarithms, (This base is designated by $\underline{e}$)

$$dy = \frac{dx}{x}$$

8th <u>Example</u> Let

$$y = a^x + C$$

from which

$$dy = a^{x+dx} + C - (a^x + C)$$
$$= a^{x+dx} - a^x$$
$$= da^x$$

Now $da^x = a^x d\log(a^x)$

$$= a^x d(x\log a)$$
$$= a^x \log a\cdot dx$$

hence

$$dy = a^x \log a\cdot dx$$

4 Remarks on the preceding examples

It is easy to see from all the preceding examples, that the same differentials would be obtained by suppressing in the given functions the constant $\underline{C}$

We obtain from the fourth example

$$dx = \frac{dy}{\cos x}$$
$$= \frac{d\sin x}{\cos x}$$
$$= \frac{d\sin x}{\sqrt{1-\sin^2 x}}$$

and representing by z the sine of the arc x, the same equation may be also expressed as follows

$$d\,\text{arc}(\sin = z) = \frac{dz}{\sqrt{1-z^2}}$$

an equation the first member of which is read, differential of the arc whose sine is z.

The fifth example gives

$$dx = -\frac{d\cos x}{\sin x}$$
$$= -\frac{d\cos x}{\sqrt{1-\cos^2 x}}$$

and representing by z the cosine of x

$$d\,\text{arc}(\cos = z) = -\frac{dz}{\sqrt{1-z^2}}$$

From the sixth example we have

$$dx = \cos^2 x\, d\,tg\,x$$
$$= \frac{d\,tg\,x}{\sec^2 x}$$
$$= \frac{d\,tg\,x}{1+tg^2 x}$$

hence, calling z the tangent of x

$$d\,\text{arc}(tg = z) = \frac{dz}{1+z^2}$$

5 Differential coefficients — Their different orders

The ratio $\frac{dy}{dx}$ is called *Differential coefficient*: it is commonly a new function of x called *derivative function* which is expressed by $f'(x)$. We subjoin in the following table the primitive functions of the preceding examples with their corresponding differential coefficients or derivative functions, retaining the order of the same examples

Primit. funct.	Deriv. funct.
1. $y = f(x) = ax + C$	$\frac{dy}{dx} = \cdots\cdots a$
2. $y = f(x) = \frac{a}{x} + C$	$\frac{dy}{dx} = f'(x) = -\frac{a}{x^2}$
3. $y = f(x) = x^a + C$	$\frac{dy}{dx} = f'(x) = ax^{a-1}$
4. $y = f(x) = \sin x + C$	$\frac{dy}{dx} = f'(x) = \cos x$
5. $y = f(x) = \cos x + C$	$\frac{dy}{dx} = f'(x) = -\sin x$
6. $y = f(x) = tg\,x + C$	$\frac{dy}{dx} = f'(x) = \frac{1}{\cos^2 x}$
7. $y = f(x) = \log x + C$	$\frac{dy}{dx} = f'(x) = \frac{1}{x}$
8. $y = f(x) = a^x + C$	$\frac{dy}{dx} = f'(x) = a^x \log a$

In all these equation the differential dx and generally the differentials of independent variables are constant although arbitrary.

As from $\quad y = f(x)$

we infer $\quad dy = f'(x)dx$

so from this we deduce

$$ddy = f''(x)dxdx$$

and again from the last

$$dddy = f'''(x)dxdxdx$$

and so on, f'', f''', &c. being new derivative functions called of the second of the third order and so on, relatively to f' which is the derivative function of the first order. ddy, $dddy$, ... are represented by d^2y d^3y, &c. . ; thus the differentials of $y = f(x)$ succeding $dy = f'(x)dx$ are expressed by

$$d^2y = f''(x)dx^2$$

$$d^3y = f'''(x)dx^3 \quad \&c. .$$

from which the differential coefficients of the second of the third order and so on

$$\frac{d^2y}{dx^2} = f''(x)\ , \quad \frac{d^3y}{dx^3} = f'''(x)\ , \quad \&c \dots$$

To give an example of consecutive derivative functions let us again take $y = x^a$. As from $f(x) = x^a$ we obtain $f'(x) = ax^{a-1} = \frac{dy}{dx}$ we also infer that

$$f''(x) = a(a-1)x^{a-2} = \frac{d^2y}{dx^2}$$
$$f'''(x) = a(a-1)(a-2)x^{a-3} = \frac{d^3y}{dx^3}$$
$$\&c \dots$$

Supposing $\underline{a} = 3$, $f'''(x)$ would be equal to $a(a-1)(a-2)$ and $d^3y = a(a-1)(a-2)dx^3$ from which (it being a constant) $d^4y = 0$; hence $f^{iv}(x)$ also is in this case equal to zero

6 Functions of different independent variables

Let

$$\mu = f(x, z, u, \dots)$$

represent a function of any number of independent variables $x, z, u, \dots$ If we regard only one of these quantities as variable and the others as constant, the differentials of the function $\underline{\mu}$ will be obtained in the same manner as those of functions of one variable only. Such differentials are called *partial* and represented as follows

$$d_x\mu\ , \ d_x^2\mu\ , \dots$$
$$d_z\mu\ , \ d_z^2\mu\ , \dots$$
$$\&c \dots$$

i.e. $d_x\mu$, $d_x^2\mu$, ... are the differentials of the first of the second order &c of μ

relatively to x, $d_z\mu$, $d_z^2\mu$, ... are the differentials of the different orders of the same μ relatively to z and so on. The partial derivative functions are expressed by

$$\frac{d_x\mu}{dx}, \quad \frac{d_x^2\mu}{dx^2}, \ldots$$

$$\frac{d_z\mu}{dz}, \quad \frac{d_z^2\mu}{dz^2}, \ldots$$

&c — — — — —

But more simply by

$$\frac{d\mu}{dx}, \quad \frac{d^2\mu}{dx^2}, \ldots$$

$$\frac{d\mu}{dz}, \quad \frac{d^2\mu}{dz^2}, \ldots$$

&c — — — —

The same derivative functions are also designated by

$$f_x'(x, z, u, \ldots), \quad f_x''(x, z, u, \ldots), \ldots$$

$$f_z'(x, z, u, \ldots), \quad f_z''(x, z, u, \ldots), \ldots$$

&c — — — — —

The total differential of μ is deduced from the partial ones. For

$$f(x+dx, z, u \ldots) - f(x, z, u, \ldots) = f_x'(x, z, u, \ldots)dx$$

$$f(x+dx, z+dz, u, \ldots) - f(x+dx, z, u, \ldots) = f_z'(x+dx, z, u, \ldots)dz = f_z'(x, z, u, \ldots)dz$$

$$f(x+dx, z+dz, u+du, \ldots) - f(x+dx, z+dz, u, \ldots) = f_u'(x+dx, z+dz, u, \ldots)du = f_u'(x, z, u, \ldots)du$$

&c — —

These partial differentials added together, give

$$f(x+dx, z+dz, u+du, \cdots) - f(x, z, u, \cdots) =$$
$$f'_x(x, z, u, \cdots) + f'_z(x, z, u, \cdots) + \cdots\cdots$$

Or $\quad d\mu = d_x\mu + d_z\mu + d_u\mu + \cdots$

The successive differentials of μ relatively to two or more of the independent variables are designated by

$$d_z d_x \mu, \quad d_u d_z d_x \mu, \quad \cdots$$

7 Integral calculus — Examples

As when a function is given, its differential may be found, so vice-versa when a differential is given we may occasionally find the function from which it is derived. To find this primitive function forms the object of integral calculus

Let $f(x)$ be the derivative function of $F(x)$ i.e. let $dFx = f(x)dx$; $F(x)$ or $F(x)+C$ is the integral of $f(x)dx$ it is called indefinite integral. To express that $F(x)+C$ is the integral of $f(x)dx$ we write

$$\int f(x)dx = F(x) + C \quad \cdots (a)$$

an equation which is read: integral of $f(x)dx$ equal to $F(x)+C$

The algorithm $\int$ signifies sum; in reality the integration is nothing but a sum adding to the given function as much as required to obtain the primitive function of which it is differential. This sum is in some cases obviously indicated by the form of the given differential, as in the following examples

$$(a+1)x^a dx,\quad a^{cx}\log(a)\,dcx,$$

$$\frac{dx}{\sqrt{1-x^2}},\quad \frac{dx}{1+x^2}$$

The first of these expressions is (3 Ex 3) the differential of $x^{a+1}+C$, hence

$$\int (a+1)x^a dx = x^{a+1}+C$$

and also $$\int x^a dx = \frac{x^{a+1}}{a+1}+C$$

The second is (3 Ex 8) the differential of $a^{cx}+C$, hence

$$\int a^{cx}\log(a)\,dcx = a^{cx}+C$$

also $$\int a^{cx}dx = \frac{a^{cx}}{c\log(a)}+C$$

The third is (4) the differential of $\operatorname{arc}(\sin = x)+C$; therefore

$$\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc}(\sin = x)+C$$

The last is (4) the differential of $\operatorname{arc}(\operatorname{tang} = x)+C$ (4), hence

$$\int \frac{dx}{1+x^2} = \operatorname{arc}(tg = x)+C.$$

In other cases when the differential has not the form of any known differential, it may be transformed into one of them by means of some substitutions. Let, for example,

$$\frac{1}{1+a^2x^2}dx,\quad \frac{1}{x^2+a^2}dx,\quad -\frac{1}{\sqrt{a^2-x^2}}dx$$

be the given differentials, the first of which is easily changed into the equivalent $\dfrac{adx}{a(1+a^2x^2)} = \dfrac{d(ax)}{a(1+(ax)^2)}$. But this is (4) the differential of $\frac{1}{a}\operatorname{arc}(tg = ax)+C$, therefore

$$\int \frac{1}{1+a^2x^2}dx = \frac{1}{a}\operatorname{arc}(tg = ax)+C$$

The third differential also can be equally modified and changed into the equivalent $-\dfrac{\frac{1}{a}dx}{\frac{1}{a}\sqrt{a^2-x^2}} = -\dfrac{d\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}$; which is the differential of arc$(\cos = \frac{x}{a}) + C$, hence

$$\int -\frac{1}{\sqrt{a^2-x^2}}dx = \text{arc}(\cos = \frac{x}{a}) + C$$

Observe here that from $d(F(x)+C) = -f(x)dx$ we deduce $d-1(F(x)+C) = f(x)dx$, hence $\int -f(x)dx = F(x)+C$, $\int f(x)dx = -(F(x)+C)$ and $-\int f(x)dx = F(x)+C$, therefore $\int -f(x)dx = -\int f(x)dx$ consequently

$$-\int \frac{1}{\sqrt{a^2-x^2}}dx = \text{arc}(\cos = \frac{x}{a}) + C$$

8 *Definite integrals* In the indefinite integral (α) give to $\underline{x}$ the value of x_n, then that of x_0; from the first we have $\int f(x_n)dx = F(x_n)+C$ from the second $\int f(x_0)dx = F(x_0)+C$ and this being subtracted from the other we obtain the difference $F(x_n) - F(x_0)$, which does not contain the constant $\underline{C}$. Such a difference is called the *definite* integral of the differential $f(x)dx$ taken from $x = x_0$ to $x = x_n$. It is represented by $\int_{x_0}^{x_n} f(x)dx$ i.e.

$$\int_{x_0}^{x_n} f(x)dx = F(x_n) - F(x_0)$$

Let for example $f(x) = x^a$, from which $f(x)dx = x^a dx$ and (γ) $\int x^a dx = \frac{x^a}{a+1} + C$, and let the two values given to x be 1 & 0. The preceding equation will then assume the form

$$\int_0^1 x^a = \frac{1}{a+1}$$

Take $f(x)dx = \frac{dx}{x^2+a^2}$ since (7) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} arc\left(tg = \frac{x}{a}\right) + C$, making first $x = a$ then $x = 0$, we shall have

$$\int_0^a \frac{dx}{x^2+a^2} = \frac{1}{a} arc(tg = 1) - \frac{1}{a} arc(tg = 0)$$

now $arc(tg = 1) = 45° = \frac{1}{4}\pi$ and $arc(tg = 0) = 0$ or $= \pi$, & excluding the second value

$$\int_0^a \frac{dx}{x^2+a^2} = \frac{\pi}{4a}$$

If instead of x_n we take any value for $\underline{x}$ the definite integrals becomes variable; for the equation

$$\int_{x_0}^{x} f(x)dx = F(x) - F(x_0)$$

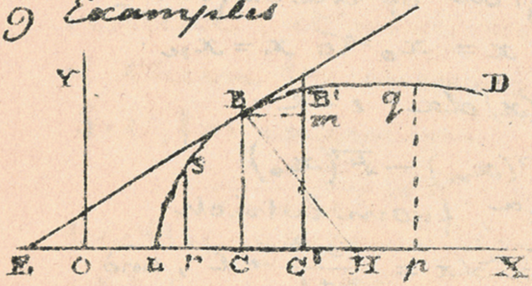
depends on $\underline{x}$. This definite integral does not differ from the indefinite $\int f(x)dx = Fx + C$, The constant C however may be different $-F(x_0)$, and in this case let $C = -F(x_0) + C'$ we shall then have

$$\int f(x)dx = F(x) - F(x_0) + C'$$

and consequently

$$\int f(x)dx = \int_{x_0}^{x} f(x)dx + C'$$

9 Examples



Let $x, x_1, x_2, \cdots$ be different values of $\underline{x}$ to which correspond $y, y_1, y_2, \cdots$ in the equation $y = f(x)$. Let also OX, OY be two rectangular axes and take on OX Oc, Op, $\cdots = x, = x_1, = \cdots$; draw $\underline{CB}$, pq, $\cdots$ parallel to OY and take CB, pq, $\cdots = y, = y_1 = \cdots$; the extremities B, q, $\cdots$ form a series of points generally ranged on a curved line. x and y are called <u>coordinates</u> the different values of x abscissas, those of y ordinates.

Suppose LBD to be the curved line corresponding to the equation $y = f(x)$ and let $\underline{CC'}$ represent an infinitesimal increment of the abcissa $OC\ (=x)$; i.e. let $CC' = dx$:

Let also $C'B'$ be the ordinate corresponding to $x + dx$: draw from $\underline{B}$, $B\underline{m}$ parallel to $\underline{OX}$, thus we obtain

$$B'm = dy\ ,\ Bm = CC' = dx.$$

Let E be the point of the axis of abscissas met by the tangent of the point $\underline{B}$ and H the point of the same axis met by the normal of $\underline{B}$. The arc BB' being infinitesimal may be considered as coinciding with the tangent and the triangle BmB' may be regarded as rectilinear. Thus we have three similar triangles ECB, BCH and the infinitesimal BmB', from which we infer

$$\frac{BC}{EC} = \frac{B'm}{Bm} \text{ from which}$$
$$tg\,B'Bm = \frac{dy}{dx}$$
$$\frac{CE}{CB} = \frac{Bm}{B'm} \text{ from which}$$
$$CE = \frac{y\,dx}{dy}$$
$$\frac{CH}{CB} = \frac{B'm}{Bm} \text{ from which}$$
$$CH = \frac{y\,dy}{dx}$$

CE is called the Subtangent, CH the

Subnormal of the point B of the curve

Differential of the area terminated by a curve & its definite integral

The area BLC terminated by the curve LBD varies with $\underline{x}$; hence it is a function of this variable and may be represented b $F(x)$. The differential of BLC is the infinitesimal area $BCC'B'$ a trapezoid whose parallel sides are y & $y+dy$ and their distance is dx hence

$$BB'C'C = \frac{dx}{2}(y+y+dy)$$
$$= ydx + \frac{dxdy}{2}$$
$$= ydx$$

but $y = f(x)$ and $BB'C'C = dF(x)$, hence

$$dF(x) = f(x)dx$$

Call now x_0 the abcissa of a given point $\underline{s}$ of the curve, we shall have

$$\int_{x_0}^{x} f(x)dx = \int_{x_0}^{x} ydx = srCB :$$

the area $srCB$ is equivalent to that of a rectangle having $rC = x - x_0$ for base, and for height an ordinate y_m intermediate between rs & CB i.e. between y_0 & y; hence

$$\int_{x_0}^{x} ydx = (x - x_0)y_m$$

or

$$\int_{x_0}^{x} f(x)dx = (x - x_0)f(x_m)$$

The same area may be regarded as the sum of an infinite number of infinitesimal rectangles like $y \cdot dx$. In this supposition

$$srCB = y_0 dx + (y_0 + dy)dx + (y_0 + 2dy)dx +$$

$+ \cdots + (y-dy)dx$

Or $\int_{x_0}^{x} f(x)dx = f(x_0)dx + f(x_0+dx)dx + f(x_0+2dx)dx + \cdots + f(x-dx)dx$

Differential of the arc of a curve, and its definite integral

As the area BLC varies with x so also does its curvilinear limit LSB; and it is consequently a function of x which we may represent by $\varphi(x)$.

Call ds the infinitesimal increment or differential BB' of LSB. Since $\overline{BB'}^2 = \overline{Bm}^2 + \overline{B'm}^2 = dx^2 + dy^2 = dx^2(1+(f'(x))^2)$; we have ds or

$$d\varphi(x) = dx\sqrt{1+f'^2(x)}$$

Now $\int_{x_0}^{x} d\varphi(x) = \varphi(x) - \varphi(x_0) = SB$

and calling s this arc

$$\int_{x_0}^{x} dx\sqrt{1+f'^2(x)} = s$$

Given differential equations to be integrated

Let the differential equation $dy - cydx = f(x)dx$ be given to be integrated. Its integration may be reduced to that of $\frac{f(x)dx}{e^{cx}}$

Observe that the variable y can be made equal to the product uz of two other variables, in which case we have

$$dy = zdu + udz \cdots\cdots (c)$$

where the differential dz may be any at pleasure, for instance $dz = czdx$.

In this supposition

$$\left.\begin{array}{l} \frac{1}{c}\cdot\frac{dz}{z} = dx \\ udz - czudx = 0 \end{array}\right\} \cdots\cdots (c_1)$$

From the first of which two equations

we have (3 · 7th)

$$\left.\begin{aligned} cx &= \log(z) \\ z &= e^{cx} \end{aligned}\right\} \cdots\cdots (c_2)$$

Then placing in the given differential equation uz for y we have (c)

$$zdu + udz - czudx = f(x)dx$$

and with $dz = czdx$

$$zdu = f(x)dx$$

$$du = \frac{1}{z} f(x)dx.$$

Hence from the second (c_2)

$$du = \frac{f(x)dx}{e^{cx}}$$

and consequently

$$u = \int \frac{f(x)dx}{e^{cx}} + C$$

and this $\underline{u}$ being inferred from the given differential equation is necessarily the one, which multiplied by another variable whose differential is $czdx$ gives the y fulfilling the same equation. Now from $y = zu$ we have

$$y = e^{cx}\left[\int \frac{f(x)dx}{e^{cx}} + C\right].$$

Hence the integration of the given differential formula may be reduced to that of $\dfrac{f(x)dx}{e^{cx}}$.

Next let the following differential equation of the second order be given

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Its integration may be reduced to that of two equations of the first order as follows.

Take two such quantities k & k' that

$$k+k'=a \quad , \; kk'=b$$

and the equations

$$\frac{dy'}{dx}-ky'=0, \quad \frac{dy}{dx}-k'y=y':$$

then observe that by substituting in the first of these equations the value of y' given by the second it becomes the given differential equation. Therefore the y in the second equation is the same as that of the given differential equation. Now the same two equations may be reduced to the following form

$$dy'-ky'dx=0 \quad , \; dy-k'ydx=y'dx$$

which is the same as that of the differential equation of the preceding example. Hence from the first of them we have

$$y'=Ce^{kx} \; ,$$

from the second

$$y=e^{k'x}\left[\int\frac{y'dx}{e^{k'x}}+C'\right]$$

and therefore

$$y=e^{k'x}\left[\int\frac{Ce^{kx}dx}{e^{k'x}}+C'\right]$$

$$=e^{k'x}\left[\int Ce^{(k-k')x}dx+C'\right]$$

Now $e^{(k-k')x}dx=e^{(k-k')x}\ell e\, dx$

$$=\frac{1}{k-k'}e^{(k-k')x}\ell e\, d(k-k')dx$$

and (3. Ex 8)

$$e^{(k-k')x}\ell e\, d(k-k')dx=d(e^{(k-k')x}+C'')$$

hence

$$\int Ce^{(k-k')x}dx=\frac{C}{k-k'}e^{(k-k')x}+C''$$

and $y = e^{k'x}\left(\frac{C}{k+k'}e^{(k-k')x} + C''\right.$

$= \frac{C}{k-k'}e^{kx} + C''e^{k'x}$.

Representing the constant $\frac{C}{k-k'}$ by D and the constant C'' by D'

$$y = De^{kx} + D'e^{k'x}$$

If instead of the preceding, the differential equation of the second order were

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x),$$

its integration would be reduced to that of the two following

$$\frac{dy'}{dx} - ky' = f(x), \quad \frac{dy}{dx} - k'y = y'$$

and with the process used in the preceding case we find

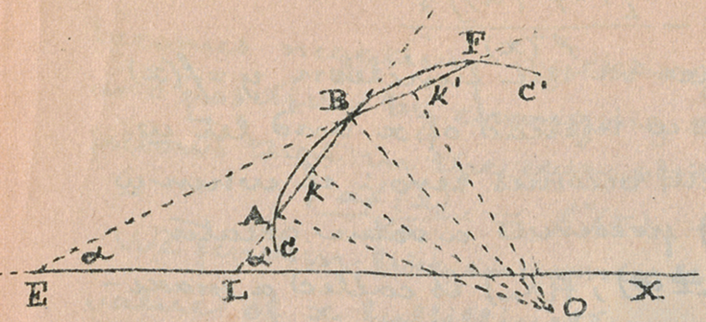
$$y' = e^{kx}\left[\int\frac{f(x)\,dx}{e^{kx}} + C\right]$$

$$y = e^{k'x}\left[\int\frac{y'dx}{e^{k'x}} + C'\right]$$

10 Osculating circle – determination of its radius

Conceive a curved line albgc cutting the circle afbgc in three points a, b, c. The chords drawn from a to b and from b to c are thus common to the curve and to the circle. If the same chords be infinitesimal the circle is called osculating

Let CC' be a plane curve referred to rectangular axes, and let $y=f(x)$ be its equation. Let also BA, BF two equal infinitesimal chords, from the middle



points K & K' of which draw to them the perpendiculars KO, $K'O$. O is the centre and OB the radius of the osculating circle corresponding to the point B of the curve.

Let now ds be the infinitesimal arc $\underline{AB}$ or $\underline{BF}$ and call $\underline{i}$ the infinitesimal angle EBL formed by the chords; the ratio $\frac{ds}{i}$ gives the length of the radius of the osculating circle. For observe that representing by ds' the infinitesimal arcs of this circle subtended by the chords $\underline{AB}$, $\underline{BF}$ we have $ds' = ds$. Now $ds' = BO \cdot BOA = BO \cdot KOK' = BO \cdot i$; hence

$$BO = \frac{ds}{i}$$

But (9) $ds = \left(\sqrt{1 + f'^2(x)}\right)dx$ and representing by α the angle BEX and by α' the angle BLX, which the chords form with the axis of abscissas, we obtain $i = \alpha' - \alpha$ $= -(\alpha - \alpha') = -d\alpha = -d\,\mathrm{arc}\left(tg = \frac{dy}{dx}\right)$ (9)

Hence, calling $\underline{r}$ the radius $\underline{BO}$

$$r = \frac{\left(\sqrt{1 + f'^2(x)}\right)dx}{-d\,\mathrm{arc}\left(tg = \frac{dy}{dx}\right)}$$

$$= -\frac{\left(\sqrt{1 + f'^2(x)}\right)dx}{d\,\mathrm{arc}\left(tg = f'(x)\right)}$$

Again (4)

$$d\,\mathrm{arc}\left(tg = f'(x)\right) = \frac{df'(x)}{1 + f'^2(x)} = \frac{f''(x)\,dx}{1 + f'^2(x)}$$

hence
$$r = -\frac{[1+f'^2(x)]^{\frac{3}{2}}}{f''(x)}$$

11 Maxima & minima values

Place in the function $y = f(x)$ $x_n \pm \omega$ instead of x and let $\underline{\omega}$ diminish until it becomes zero: if when $\underline{\omega}$ is near zero $f(x_n)$ preserves a value greater than that of $f(x_n \pm \omega)$, $f(x_n)$ is called a maximum, on the contrary if $f(x_n)$ preserves a value less than that of $f(x_n \pm \omega)$, $f(x_n)$ is called a minimum. Hence in the case of the maximum, dy in the equation
$$y_n + dy = f(x_n + dx)$$
will be negative whatever be the sign of $\underline{dx}$. Therefore with dx negative or < 0
$$\frac{dy}{dx} = f'(x) > 0$$
with dx positive or > 0
$$\frac{dy}{dx} = f'(x) < 0 .$$
In the case of the minimum dy will be positive whatever be the sign of dx. Hence when in this case $dx < 0$, also
$$\frac{dy}{dx} = f'(x) < 0$$
and when $dx > 0$ also
$$\frac{dy}{dx} = f'(x) > 0$$
It follows therefore that the values of x which render $f(x)$ eiter a maximum or a minimum may be found among the roots of the equation
$$f'(x) = 0$$
Again since when the value of $\underline{x}$ which fulfills thisequation belongs to ~~is~~ a maximum, $f'(x)$

becomes negative by changing x into $x+dx$ and positive by changing x into $x-dx$; it follows that in the case of the maximum

$$\frac{df'(x)}{dx} = f''(x) < 0$$

We find in like manner that when the value of x fulfilling the equation $f'(x)=0$ belongs to a minimum,

$$\frac{df'(x)}{dx} = f''(x) > 0$$

12 Evolution of a given function into a series

Let $f(x)$ be any function of x and make

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

i.e. suppose $f(x)$ developed into a series. In order to determine the values of the constant coefficients $a_0, a_1, a_2, \ldots$ observe that

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$f''(x) = 2a_2 + 2\cdot 3 a_3 x + 3\cdot 4 a_4 x^2 + \cdots$$
$$f'''(x) = 2\cdot 3 a_3 + 2\cdot 3\cdot 4 a_4 x + 3\cdot 4\cdot 5 a_5 x^2 + \cdots$$
&c — — — — — — —

Making in all these equations $x=0$ we obtain

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \tfrac{1}{2} f''(0),\ a_3 = \tfrac{1}{2\cdot 3} f'''(0), \ldots$$

hence

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{2\cdot 3} f'''(0) + \cdots$$

1st Example

Let us see some examples and first let

$$f(x) = e^x.$$

From this (3 · Ex. 8)

$$f'(x) = f''(x) = f'''(x) = \cdots = e^x$$

and making $x=0$, we obtain

$$f(0) = f'(0) = f''(0) = \cdots = 1$$ hence

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \cdots$$

2d Example Secondly let $f(x) = \sin x$

We have from (3. Ex 4, 5)

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ \ldots$$

hence $$f(0) = f''(0) = f^{iv}(0) = \cdots = 0$$

$$f'(0) = f'''(0) = f^{v}(0) = \cdots = 1$$

therefore

$$\sin x = x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \cdots$$

3d Example Lastly let $f(x) = \cos x$

From this

$$f'(x) = -\sin x,\ f''(x) = -\cos x,\ f'''(x) = \sin(x),\ \ldots$$

and $$f(0) = -f''(0) = f^{iv}(0) = \cdots = 1$$

$$f'(0) = f'''(0) = f^{v}(0) = \cdots = 0$$

hence

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4} - \frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$

Taking $\pm x\sqrt{-1}$ instead of x, the first example gives us

$$e^{\pm x\sqrt{-1}} = 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4} - \frac{x^6}{2\cdot 3\cdot 4\cdot 5 6} + \cdots$$
$$\pm \left(x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \cdots\right)\sqrt{-1}$$

But from the second and third examples we have $1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4} - \cdots = \cos x$ and $x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \cdots = \sin x$; hence

$$e^{\pm x\sqrt{-1}} = \cos x \pm \sin x\sqrt{-1}$$