Who steals this book

Who steals this book, steals trash!

Who steals this book, steals trash!

Why are the students of [illegible] like horses?
Because they have no [illegible] - no)
Because they leave their "marks" in their stalls
Because they are [illegible] with lines

Keats Frash

Theodore Jenkins

Feb 24th 1855

Why ought a humble man
never to marry? Because
the wife he's bound to take the
bridle (bridal)

Theo Jenkins

Theo. Jenkins

T. J.

Another night has passed
I wish it were my last
That in dust I could rest where my forefathers
That everlasting scream
Still haunts me in my dream
The cat-a-wauling Tom Cat Grey
Chorus
Old Tom Cat ever hateful
Kicks can not drive him away
Most vicious of his kind
So old he's nearly blind
The cat-a-wauling Tom Cat Grey

He plagues me night & morn
My occupation's gone
All that I can do will not keep him away
My ideas to have flown
My room is not my own
I'm banished by Old Tom Cat Grey
Chorus
Old Tom Cat ever hateful
Bricks can not drive him away
Will I never never find
Some one who'll be so kind
As to rid me of this Tom Cat Grey

When e'er I see him pass
On me his eyes he'll cast
I know if he could speak he would bore me [illegible]
But, as this he can not do
He thinks he'll go and mew
The cat-a-wauling Tom Cat Grey

Chorus
Old Tom cat ever hateful
Nothing I can drive him away
He's a devil of his kind
I know I'll never find
A worse friend than Tom Cat Grey

A TREATISE

OF

A parody on "Old Dog Tray."

ANALYTICAL GEOMETRY,

PROPOSED

BY REV. BENEDICT SESTINI, S. J.,

PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY,

IN

GEORGETOWN COLLEGE.

"Who steals this book steals trash."

"'Tis good, i' faith, and not less good than true
But less than ~~trash when~~ someone stealeth you"

WASHINGTON:
GIDEON AND CO., PRINTERS.
1852.

1 J. D. Dougherty
2 W. Choice
3 J. R. Randall
4 W. Blandford
5 Henry Bowling
6. Edgar D. Aguin
7 Lincius Northrop
8 Chas B. Kenny
9 Frank Lancaster
10 Theo. Jenkins
11 Wm J. Hill
12 John Bowling
13 B. R. Riordan

Order of the Examination
July 2nd 1855

Examiners { A. Van den Heuvel
Rev. Benedict Sertimer

1st
Spherical Trigonometry

art
" 1st 2 first formula's, 1st of each
" 2nd Napier's circular parts
3rd Solution of a right angle Spherical triangle

2nd
Mensuration

1st Mensuration of surfaces to find area of a square, rectangle or parallelogram
2nd find area of triangle when base & altitude given.
3rd find area of triangle when 3 sides given
4th " " " " " 2 " & include and included angle are given.
5th find area of trapezoid
6th find circumference of circle diameter given or diameter when circumference "
7th find area of a circle

PREFACE.

This treatise of Analytical Geometry, which in progress of time may form a part of a complete course of mathematics, although destined at present for a class in Georgetown College, is offered also to those students who cultivate this branch of science in other public institutions. To all these the present introduction is addressed, together with the following treatise, divided into four parts or books: the first of which treats of *co-ordinates*, and *geometrical loci on a plane;* the second of *co-ordinates*, and *geometrical loci in space;* the third treats of *lines of the second order;* and the fourth of *surfaces* of the same order. The first and second are nothing more than an introduction to the principal object of this part of analysis which is exclusively treated of in the third and fourth books. The learner will probably find in our method something not entirely conformable to that usually adopted in other similar works; thus, for instance, in the third and fourth books he will easily remark that the questions are reduced to some principal heads, from which, as from a nucleus, we derive the theory of the lines and surfaces of the second order. Nay, more, all the properties of the lines, as well as of the surfaces, are altogether derived from the discussion of the simple quadrinomial formula $mx^2 + nx + p = q$ (*); or from the trinomial $mx^2 + nx$

(*) Book III, § 44, (i_3). Book IV, § 111, final remark.

$= d$. For this ingenious simplification we are indebted to Baron A. L. Cauchy, who is deservedly considered not only as one of the best mathematicians of the present time, but not inferior to any of those who flourished in preceding ages. The compendious style used by this celebrated author would probably not be intelligible to the incipient learner; our endeavors, therefore, were especially devoted to develop and explain, in a manner suitable to students, the analysis which the French author first offered to the scientific world; (*) yet, notwithstanding this labor, some perhaps will object that the present treatise still requires, on the part of the student, a certain penetration of mind. This we readily admit; but nobody, we trust, will condemn us for supposing some penetration of mind in those who give themselves to the study of the sciences; and if, in some instances, notwithstanding this supposed aptitude, the student could not overcome by himself some difficult point, we take it for granted that works of this character are not only to be studied in private, but are also to be explained by the teacher. This necessarily supposes the students in general not to be able to overcome all the difficulties by themselves, even in the most elementary treatise, unless the school be considered as a mere formality. Let us even remark, that difficulties in some cases are not inherent in the method but in the object, and to diminish them nothing contributes more than simplicity and order. Order, moreover, excludes all the difficulties which are not inherent in the matter, diminishes the

(*) Exercises de Mat., par M. A. Cauchy, (troisieme annēe.)

labor of the teacher as well as of the student, and is the only means by which the mind can be enriched by a really scientific knowledge and comprehension of the subject. After this, to exclude order from geometry, and in general from mathematical works, is to expel the owner from his own house. Another observation, probably, might be made, namely, that the present treatise is rather defective in point of familiar examples. We neither deny nor grant it; allow us only to remark, that we may here suppose two species of applications or examples, those taken from analytical geometry itself, or those taken from branches of natural philosophy to which this analysis is applicable. The second class is evidently extraneous to our subject; and as to the former, we thought it enough to give only a few of them, which, affording the illustration of some peculiar point, could be at once a model for many others which the teacher and even the student can form for himself illustrative of the same or of other points.

The index which we subjoin, especially the part which belongs to the third and fourth books, may perhaps give to the reader, who should desire it, a more complete idea of the plan and character of the treatise. The parts of this treatise having such connexion and dependence upon each other, we have been compelled to make use of frequent references. We know well that some writers of works of this kind avoid as much as possible such references, and some also, even eminent, exclude them altogether; the reader being, as they allege, thus stopped and disturbed on his way. Consulting, however, our own experience, and the assistance frequently offered by these references, although accompanied with some trouble, we preferred to follow the example of many others likewise eminent and equally experienced in teaching. And for the sake of some of our friends, to whom we are indebted for the remarks made on the treatise before its publication, and who incline to the exclusion, or at least diminution, of the references, we observe, that although such references are not all and at all times profitable for each reader in particular, the book being written for a great variety of readers, it is not improbable that the number of references be rather deficient than too copious. And, finally, whenever recourse to some of the preceding questions is indispensable, (and they must necessarily form part of the demonstration at hand,) in such cases, and even generally, a reference is either useful to the reader or not; if it is useful, there is no reason of complaint; if not useful, the reader can easily go on without noticing it, not being compelled by a mere reference to interrupt his course.

INDEX.

BOOK I.

CO-ORDINATES AND GEOMETRICAL LOCI.

BOOK II.

CO-ORDINATES AND GEOMETRICAL LOCI IN SPACE.

BOOK III.

LINES OF THE SECOND ORDER.

BOOK IV.

SURFACES OF THE SECOND ORDER.

BOOK I.

CO-ORDINATES AND GEOMETRICAL LOCI ON A PLANE.

Preliminary Propositions.

PROPOSITION I.

The sum of the projections of the sides of any polygon is equal to zero.

1. Let (fig. 1) ABC . . . be any polygon whatever, either in a plane or with its sides in different planes, that is, let the plane determined by the sides AB, BC, for instance, be different from that determined by the sides BC, CD . . .

2. The projection of one straight line on another is determined by the interval comprised between two planes perpendicular to the latter, and passing through the extremities of the former line. For example, the projection of the side AB on the *axis* PX will be HK, supposing that two planes passing through the extremities A and B of the side and perpendicular to the axis, cut PX in H and K. Now the extremity B of the second side is common with that of the first, and consequently the projection of the second side will begin from the very same point K, and end in another L, either towards X or towards P; and considering the first direction positive and the second negative, in the first case the projection of the two sides will be equal to the sum, in the second to the difference of the partial projections. But in every case the interval HL comprised between the planes, passing through the first extremity of the first side and the last extremity of the second, will be equal to the algebraic sum of the two projections. Likewise we may demonstrate that the interval comprised between

the planes passing by the first extremity of the first side and the last extremity of the third, gives the algebraic sum of the partial projections of the three sides, and so on. Therefore the sum of the projections of all the sides will be determined by the planes passing through the first extremity of the first side, and the last of the last side; but these extremities are in the same point, therefore the length or sum of the projections of all the sides of any polygon whatever is equal to naught.

Corollary I. It is known from trigonometry that the projection of a straight line on another is equal to the co-sine of the angle which the same line makes with the corresponding axis of projection, or with a line parallel to it, multiplied by the side itself: hence, supposing Aa, Bb, Cc, &c., parallel to the axis PX and calling the partial projections $p, p', p'', \ldots$; since

$$p + p' + p'' + \ldots = 0$$

and $$p = \text{AB} \cdot \cos \text{BA}a, p' = \text{CB} \cos \text{CB}b, \ldots$$

we will also have

$$\text{AB} \cdot \cos \text{BA}a + \text{CB} \cos \text{CB}b + \ldots = 0$$

It is here to be observed, that as the angle which the first side BA makes with Aa is taken with reference to the positive direction of the axis, so the other angles must be taken in the same direction.

Corollary II. Supposing moreover any side, for instance AB, parallel to the axis, then cos BAa becomes equal to unity, and the preceding equation will be converted into the following

$$\text{AB} = - \text{BC co-sin CB}b - \text{DC co-sin DC}c - \ldots$$

that is to say, any side of a polygon is equal to the negative sum of the remaining sides, each multiplied by the co-sine of the angle which it makes with the first side, or with a line parallel to the same side.

PROPOSITION II.

The projection of the plane area of a polygon is equal to the given area multiplied by the co-sine of the angle which the plane of the area makes with the plane of projection.

3. Let ABCD (fig. 2) be the given plane area, and from the angles A, B, . . . draw the perpendicular lines Aa, Bb, ... on the plane P of projection, and let the points a, b, . . . met by the perpendicular lines be joined, so as to form a polygon, which is the projection of ABCD. Now if we suppose another plane P′ to pass through A parallel to the plane P, by producing the perpendicular lines so as to meet the new plane, the same projection will be renewed on the new plane P′, and as the inclination of the plane CDAB with P′ is the same as with P, every relation between the given area and the first projection will be the same as that between the given area and the second projection.

Let us now call β the angle which the plane ABCD makes with P′, and let the given area be represented by α and the area of the projection by π. Let the perpendicular Aa be produced to A′, and let all the other lines be likewise produced first to B′, C′, D′, and then to r', n', m', so as to give BB′, CC′, DD′, rr', mm', nn', all equal to AA′. Thus we will have two prisms, the solidities of which are equal, because, besides the common solid $AB'rmC'$ the remaining solid $C'm'D'r'$ of the one is equal to the remaining solid $CmDr$ of the other. But supposing that $A'p$ is a perpendicular drawn to the plane of the given area, we know from geometry that the solidity of the prism AD′B′D is equal to $\alpha \times A'p$ and the solidity of $Am'r'm$ is equal to $\pi \times AA'$, therefore

$$\pi \times AA' = \alpha \times A'p \quad (o).$$

But according to geometry the angle made by two lines respectively perpendicular to two planes, is equal to the angle of the planes; hence

$$AA'p = \beta$$

Again, from the right-angle triangle $A'pA$ we have $A'p = AA'$ $\cos AA'p = AA' \cos \beta$, therefore we shall derive from (o)

$$\pi = a \cos \beta$$

DEFINITIONS.

4. Let (fig. 3) AX and AY be two straight lines drawn at any angle to each other on a plane, and let K be any point on the same plane. From K let us draw KH parallel to AY and KL parallel to AX; these two parallels evidently determine the position of the point K with reference to AX and AY, because the parallels drawn from any point which is not K will either both, or at least one, be different from KH and KL. Now KL is equal to AH, consequently the position of the point K with reference to AX and AY may be determined by KH, KL, as well as by KH, AH; that is to say, by the line parallel to AY, drawn from K to AX, and the portion of AX comprised between the intersection A and the point H of which the parallel KH meets AX. The portion AH of AX is termed the *abscissa* of the point K. HK is termed the *ordinate* of the same point; both taken together are called *co-ordinates.* The point A is called the *origin* of the co-ordinates, AX the *axis* of abscissas, and AY the *axis* of ordinates. The axis are called *rectangular* or *orthogonal*, if at right angles to each other; otherwise *oblique.*

REMARKS.

5. Suppose (fig. 3) the directions of the axes AX, AY to be considered as positive, the opposite direction AX′, AY′ must then be considered as negative. Consequently if, instead of the point K, situated within the angle XAY, the point K′, within XAY′, be referred to the axes, the abscissa AH will remain positive, but the ordinate H′K′ shall be negative. And if the point referred to the axes, suppose K″, is within the angle YAX′, the ordinate K″ H″ will be positive, and the abscissa AH″ negative. Finally, if the point K‴ be within the angle X′AY′, both the abscissa AH‴ and the ordinate H‴ K‴ will be negative.

Of the relative position of the points.

REMARKS.

6. Let us imagine (fig. 4) two systems of axes, (AX, AY) and (A′X′, A′Y′); the point K will be referred to the first system, by means of the co-ordinates KH, HA, and to the second by the co-ordinates KH′, H′A′. Now the axis AX of the abscissas is commonly indicated by X, the axis AY of the ordinates by Y, and the whole system by (XY). The abscissas, moreover, of any point whatever are represented by x, and the ordinates by y, observing that when several co-ordinates occur in the same equation, their symbols are to be distinguished by some mark, for instance, x_0, y_0 : x_1, y_1, &c. The same is to be said of any system of axes; hence (X′Y′) will represent the system (A′X′, A′Y′), x', y' the co-ordinates, and so on. To these indications, which are generally adopted, we may add another very useful in the transformation of co-ordinates. The angle which the axis of abscissas forms with the corresponding axis of ordinates will be occasionally represented by (xy) : the angle which the axis X forms with the axis X′ of another system will be exhibited by (xx'); and likewise the angles which X forms with Y′, and Y with X′, and Y′ will be represented by (xy'), (yx'), (yy').

Formulas for passing from one system of axes to another.

7. Besides the ordinates (fig. 5) KH′, KH, let us draw from the point K to the axis X the perpendicular KB, which we will produce towards r. From the angles H′, A′, N of the polygon BKH′A′NB let the lines H′r', A′r'', Nr''' be drawn parallel to the perpendicular KB. Again, if the angle which the side H′K of this polygon forms with Kr be called α', and the angles formed by the other sides and the lines parallel to KB be called α'', α''', α^{iv}; we shall have from the first proposition (2 Cor. II)

$$\mathrm{KB} = -\mathrm{H'K}\cos\alpha' - \mathrm{A'H'}\cos\alpha'' - \mathrm{NA'}\cos\alpha''' - \mathrm{BN}\cos\alpha'''' \quad (m)$$

But from trigonometry $KB = KH \cos HKB$; and on account of the right-angled triangle KBH

$\cos HKB = \sin KHB = \sin KHX = \sin (xy)$; moreover

KH is the ordinate y of K, hence

$$KB = y \sin (xy).$$

Again: $\cos \alpha' = \cos H'Kr = -\cos H'KB$; and on account of the right-angled triangle KBS,

$\cos H'KB = \sin KSB = \sin Y'oX = \sin (y'x)$; moreover $H'K = y'$;

hence $$-H'K \cos \alpha' = y' \sin (y'x).$$

Again: $\cos \alpha'' = \cos A'H'r' = -\cos A'H'l$, and on account of the right-angled triangle $H'ln$, $\cos A'H'l = \sin H'nl = \sin (xx')$; and since $A'H' = x'$, we will have

$$-A'H' \cos \alpha'' = x' \sin (xx').$$

Likewise: $\cos \alpha''' = NA'r'' = -\cos NA'q$, and on account of the right-angled triangle $A'qN$, $\cos NA'q = \sin A'Nq = \sin A'NX = \sin (xy)$. Moreover, if the co-ordinates of A' with regard to (XY) be termed $x_\circ$, $y_\circ$, we will have $A'N = y_\circ$; therefore

$$-A'N \cos \alpha''' = y_\circ \sin (xy).$$

Finally: $\cos \alpha'''' = \cos BNr''' = \cos 90^\circ = o$: consequently

$$-BN \cos \alpha'''' = o$$

Substituting, now, the values thus found in (m), we will have

$$y \sin (xy) = y' \sin (y'x) + x' \sin (x'x) + y_\circ \sin (xy)$$

from which

$$y = y_\circ + \frac{x' \sin (xx') + y' \sin (xy)}{\sin (xy)}$$

By a similar construction and process we obtain

$$x = x_\circ + \frac{x' \sin (yx') + y' \sin (yy')}{\sin (xy)}$$

And these are the most general formulas for the transformation of co-ordinates; from which it appears that when the co-ordinates of any point are given with reference to the system $(X' Y')$, in order to have the co-ordinates of the same point with reference to the system (XY) it is necessary to know the co-ordinates $x_\circ$, $y_\circ$ of the origin of the system $(X' Y')$, and also the value of the angles (xy), (xx'), (xy'), (yx'), (yy').

Corollary I. Let us suppose that the axes X, Y are at right angles; $\sin (xy)$ becomes equal to **1**, and $(x'y) = 90° \pm x'x)$, $(y'y) = 90° \pm (y'x)$; hence $\sin (x'y) = \cos (x'x)$, $\sin (y'y) = \cos (y'x)$; therefore the preceding equations become

$$y = y_\circ + x' \sin (xx') + y' \sin (xy')$$

$$x = x_\circ + x' \cos (xx') + y' \cos (xy').$$

Corollary II. If the origin A' of the system $(X' Y')$ coincides with A, then $x_\circ = y_\circ = o$ and the general equations will be converted into the following

$$y = \frac{x' \sin (xx') + y' \sin (xy')}{\sin (xy)}$$

$$x = \frac{x' \sin (yx') + y' \sin (yy')}{\sin (xy')}$$

REMARKS.

8. Before we leave these relations between the co-ordinates of different systems, it is to be observed that, according to the general formulas, any power of the variables x, y, for instance x^n, y^n, will be given by equal and inferior powers of the variables x', y', which will be plain by observing that the same formulas may be modified in the following manner:

$$y = y_\circ + \frac{\sin (xx')}{\sin (xy)} x' + \frac{\sin (xy')}{\sin (xy)} y'$$

$$x = x_\circ + \frac{\sin (yx')}{\sin (xy)} x' + \frac{\sin (yy')}{\sin (xy)} y'$$

9. If we conceive on the plane of (XY) a series of points ranged either in a straight line or in a curve, it is evident that each one of these points mày be referred to the axes by means of the corresponding co-ordinates. Now it happens, as the following examples demonstrate, that the relation between the abscissa and ordinate of one point of the series is the same as that of the co-ordinates of every other point of the series; so that this relation being given, by assigning different values to the abscissa, we will be enabled to find the corresponding ordinates. This relation of formula, which shows how the ordinates are to be deduced from the corresponding abscissas, is called *equation.*

Equation of a straight line.

10. Let the given line be B′F (fig. 6). From any point K draw the ordinate KH or y, which, supposing the axes rectangular, will be perpendicular to X, AH will be the corresponding abscissa or x. The ordinate of the point E, in which the given line cuts the axis Y, is AE, which we will represent by $y_\circ$, and to which corresponds an abscissa equal to zero.

Now, from the similarity of the triangles BAE and BHK we derive the proportion

$$\mathrm{AE} : \mathrm{BA} :: \mathrm{HK} : \mathrm{BA} + \mathrm{AH}$$

That is to say,

$$y_\circ : \mathrm{BA} :: y : \mathrm{BA} + x.$$

From which

$$y = \frac{y_\circ(\mathrm{BA} + x)}{\mathrm{BA}}$$

But from trigonometry the triangle ABE gives $\frac{\mathrm{AE}}{\mathrm{BA}} = tg\,\mathrm{EBA}$; hence $\mathrm{BA} = \frac{\mathrm{AE}}{tg\,\mathrm{EBA}} = \frac{y_\circ}{tg\,\mathrm{EBA}}$, and if the tangent of the angle under the given line and the positive direction of X be, for

$x = tx + y_0$
$x = ay + b$

brevity's sake, called t; $\text{BA} = \frac{y_0}{t}$, which value substituted in the preceding equation will give

$$y = tx + y_0 .$$

The required equation between the co-ordinates x , y of any point K, and the constant quantities t and y_0, that is, the equation of the given line, which may be transformed into the following

$$x = ay + b$$

by making $\frac{1}{t} = a$, and $-\frac{y_0}{t} = b$.

Scholium. Let E′ F′ be another straight line parallel to BF, the angle of this line with the positive axis X must necessarily be equal to that of the first line, but the ordinate AE′ corresponding to the origin of the axes is different from y_0; hence, if we denote by $\underline{y_0}$ that ordinate, the equation of E′ F will be

$$y = tx + \underline{y_0}\ (1) .$$

In the supposition of the line passing through the origin of the co-ordinates, $\underline{y_0} = o$, and consequently in this case the equation will be

$$y = tx$$

Corollary I. Let us suppose now that BF passes through a point of which x_1 and y_1 are the co-ordinates, these two values must fulfil the equation of the line, and we will have

$$y_1 = tx_1 + y_0$$

which subtracted from the general equation between the co-ordinates of any point, will give

$$y - y_1 = t\,(x - x_1)$$

in which nothing is variable but the co-ordinates x , y of any point of the line; hence it exhibits the constant relation between the co-ordinates, and consequently it is the general equation of the straight line referred to rectangular axes and passing through a given point.

Corollary II. If besides the line BF the line CE′ perpendicular to the first be referred to the axes X , Y (fig. 7), its equation will be $y = tg\,\text{DCX} \cdot x + \text{AE}'$; but $\text{OCX} = \text{COB} + \text{OBC} = 90° + \text{OBC}$; hence $tg\,\text{OCX} = tg\,(90° + \text{OBC}) = -\cot . \text{OBC} = -\frac{1}{tg\,\text{OBC}} = -\frac{1}{t}$; therefore the equation of E′ C will be

$$y = -\frac{1}{t}\,x + \text{AE}'.$$

By comparing, now, the coefficient t of the abscissa in the equation of EF with that of the same abscissa in equation (1), it is easy to perceive that the product between these two coefficients is equal to the negative unity; and since CE , the perpendicular to B′ F can alone admit of the coefficient $-\frac{1}{t}$ we may conclude that if

$$y = hx + c$$

$$y = kx + d$$

be the equations of two given lines, these lines are at right angles to each other, whenever

$$hk + 1 = o$$

Equations of the periphery of the circle with reference to two systems of rectangular axes.

11. Let us first suppose the origin of the co-ordinates at the extremity A (fig. 8) of the diameter $\text{AB} = 2r$ of the circle AFB. From any point E draw ED parallel to Y. Now, it is known from geometry that the square of ED perpendicular to AX is equal to the product of AD and DB, that is, $\text{ED}^2 = \text{AD} \times \text{DB}$.

But ED $= y$, AD $= x$, DB $=$ AB $-$ AD $= 2r - x$; hence

$$y^2 = 2rx - x^2$$

The equation of the circle with reference to the axes AX, AY.

If the origin of the co-ordinates be situated at the centre of the given circle, and AX, AY be (fig. 9) the rectangular axes; from any point F draw FD perpendicular to AK, we will have $FD^2 = MD \cdot DN$; and since FD $= y$, MD $=$ MA $+$ AD $= r + x$, DN $=$ NA $-$ AX $= r - x$;

$$y^2 = r^2 - x^2$$

The equation of the circle referred to a system of rectangular axes having their origin at the centre of the same circle.

Analysis or discussion of equations.

EQUATIONS OF THE CIRCLE.

12. The first of the preceding equations of the circle corresponds to the following:

$$y = \pm \sqrt{[2rx - x^2]}$$

from which we perceive that when x is positive and less than $2r$, to every value of the abscissa correspond two values of y, the one positive and the other negative, and both equal in length. When x is positive, but greater than $2r$, then the difference $2rx - x^2$ is negative, and no real value can be found for y. And when x is less than $2r$, but negative, the same difference likewise results negatively, and for the same reason no real value for the ordinate is given.

The second equation corresponds to

$$y = \pm \sqrt{[r^2 - x^2]}$$

from which we see that to every value of x, positive as well as negative, corresponds the double value of y equal in length and contrary in sign, provided x be less than r, because with $x > r$, y never can have a real value.

REMARK.

13. From the preceding analysis it plainly appears that we can, not only from a given series of points, derive the equation, but that from a given equation we can derive the corresponding line with its properties. Such lines, and, in general, every series of points deduced from a given equation, are called *geometrical loci*, and the deduction of these loci and of their properties constitutes the so termed discussion of equations, the principal object of analytical geometry. To have some examples of this discussion let us take the equations of those curve lines of which we will speak more fully afterwards.

EXAMPLE I.

Discussion of the equation $y^2 = px$.

14. In the given equation let p be a known and positive quantity. As the equation can be reduced to the following

$$y = \pm \sqrt{px};$$

it is evident that every negative value of x gives an imaginary one for y, therefore AX, AY being (fig. 10) the positive direction of the rectangular axis, no point of the geometrical locus corresponding to the given equation can be found from A towards X'. Again, from the same equation it follows that to every positive value of x correspond two equal values for y, the one positive and the other negative; so that, substituting for x a value equal to Aa and taking on the perpendicular bb', two portions, ab, ab' equal to y deduced from the equation, the two extremities b and b' will be points of the locus. Again, the same equation shows that the values of y increase with those of x; consequently both branches of the curve at the same distance from the axis Y are equally distant from the axis X and extend indefinitely from both axes. Likewise we can deduce that the axis X cuts the curve at the origin of the co-ordinates, and the axis Y is a tangent to the curve at that point.

EXAMPLE II.

Discussion of the equation $y^2 = p\,(a^2 - x^2)$.

15. Let the given equation be transformed into the following

$$y = \pm \sqrt{[p\,(a^2 - x^2)]}$$

and supposing p a known and positive quantity ; to every value of x, either positive or negative, will correspond two equal values for y with contrary signs, provided x be less than a. When x is equal to a, then y will be equal to zero, and when it is greater than a, the value of y becomes imaginary. We perceive, therefore, that the geometrical locus (fig. 11) corresponding to the given equation is a curve which cuts the axis X at a distance equal to a from the origin A and on both sides of the axis. Secondly, the superior branch of the curve is equal to the inferior; and finally, beyond the points of intersection the curve no longer exists.

EXAMPLE III.

Discussion of the equation $y^2 = p\,(x^2 - a^2)$.

16. Let the given equation be reduced to the following form :

$$y = \pm \sqrt{[p\,(x^2 - a^2)]}$$

in which let p be positive. It appears that, as far as x shall be less than a, no real value can be obtained for y, whether x be positive or negative. Again, with x equal to $+ a$ or $- a$, y shall be equal to zero, and to all the other abscissas taken (fig. 12) on the positive direction of X as well as on the negative, shall correspond two equal ordinates with different signs always increasing together with x. Therefore the corresponding locus shall cut the axis X at a distance equal to $+ a$ and $- a$ from the origin of the axes, and then it shall extend itself with four uniform and indefinite branches.

REMARKS.

17. It is to be observed that the geometrical loci come under the same denomination of the corresponding equations. Now two kinds of equations are to be distinguished, the algebraical and the transcendental. Every equation in which the operations to be performed with regard to the variable quantities, are nothing but algebraical addition and subtraction, multiplication and division, or elevation to a fixed power, either whole or fractional, belongs to the algebraical kind. The others of whatever quality are transcendental. Hence it follows that the equations of the preceding examples are algebraical.

Again, the algebraical equations and corresponding loci are distributed in different orders, according to the degree of the equation. So the straight line is the locus of the first order, because the corresponding equation is of the first degree ; the circular line is of the second order, because the corresponding equation is of the second degree.

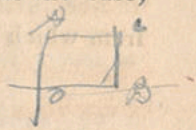
Lucius B Northrop

Transcendental Lines.

EXAMPLE I.

18. Let us suppose the plane of the circle CL (fig. 13) to be the same as that in which is the line CD tangent to the same circle. Let the point of contact C be marked with a dot, and the circle being rolled like a wheel along the tangent line CD, the marked point will describe on the plane a curvilinear path CAD called *cycloid.* The circle by whose revolution the cycloid is traced out is called the *generating circle ;* and supposing D to be the point where it will have completed one revolution, the line CD or the *base* of the curve will be exactly equal to the circumference of the generating circle. The perpendicular AB on the middle of the base is its axis. Now, in order to know if such a curve is transcendental, we must find the equation. To this end suppose the origin of the co-ordinates in A and the axis AX of the

abscissas coincident with the axis AB of the curve, and the axis AY of the ordinates perpendicular to AX. The co-ordinates of any point M shall be MN $= y$, NA $= x$. Let us now transpose the describing point of the generating circle on M by means of a corresponding revolution. The diameter C′ L′ drawing from the point C′ of the actual contact shall be perpendicular to the base, and consequently parallel to the axis X; hence



$$\left.\begin{aligned} x &= \mathrm{L}q \\ y &= \mathrm{M}q + q\mathrm{N} = \mathrm{M}q + \mathrm{C'B} \end{aligned}\right\} \quad (o)$$

To determine the values of these elements draw from the center o the radius oM, and let the angle MoL or the corresponding arc be termed α, and supposing the radius of the circle equal to r, since from trigonometry we have Mq = Mo sin MoL and Lq = Mo . vers-sin MoL = Mo (1 — cos MoL) we shall obtain

$$\mathrm{M}q = r \sin\alpha\,,\ \mathrm{L}q = r\,(1 - \cos\alpha)\,.$$

With regard to the value of C′ B observe that C′ B = BC — CC′, but BC = C′ ML and CC′ = C′ M, hence C′ B = ML. Now it is known from geometry that the arc of a circle is in proportion to the radius, therefore the arc ML (= α) with relation to the corresponding arc of the circle, whose radius is 1, is to be expressed by $r\alpha$, and so

$$\mathrm{C'B} = r\,\alpha\,.$$

Substituting, now, these values in (o), we shall have for the co-ordinates

$$x = r\,(1 - \cos\alpha)$$

$$y = r\,\alpha + r \sin\alpha$$

But from these equations

$$x - r \quad = -\, r \cos\alpha$$

$$y - r\,\alpha = \quad r \sin\alpha$$

and $$(x - r)^2 = r^2 \cos^2 \alpha$$

$$(y - r\alpha)^2 = r^2 \sin^2 \alpha$$

therefore, since from trigonometry $\sin^2 \alpha + \cos^2 \alpha = 1$

$$(x - r)^2 + (y - r\alpha)^2 = r^2$$

from which

$$y = r\alpha + \sqrt{[r^2 - (x - r)^2]}$$

or $$y = r\alpha + \sqrt{[2rx - x^2]}.$$

Let us now observe, that since $r\alpha$ is an arc, to have the second member of the equation rectilinear, we must rectify the first term. Secondly, since the arc depends on the abscissa Lq, we must examine what operation is to be performed with regard to x to have the required rectification, and the whole second member immediately dependent on x. To this end it is to be remembered that $\mathrm{M}q = r \sin \alpha$, and consequently $\sin \alpha = \frac{\mathrm{M}q}{r}$: and from geometry $\overline{\mathrm{M}q}^2 = \mathrm{C}'q \cdot q\mathrm{L}'$, hence $\mathrm{M}q = \sqrt{\mathrm{C}'q \cdot q\mathrm{L}'} = \sqrt{(\mathrm{C}'\mathrm{L}' - \mathrm{L}'q)\, q\mathrm{L}'} = \sqrt{(2r - x)\, x} = \sqrt{2rx - x^2}$;

therefore

$$\sin \alpha = \frac{\sqrt{2rx - x^2}}{r}$$

From which, by subjecting x to trigonometrical operation, we are enabled to derive the rectification of α. Hence the preceding equation between the co-ordinates xy of any point of the cycloid is transcendental, therefore the curve is also transcendental.

EXAMPLE II.

19. Let us examine now the inverse case of the preceding, by deducing from a given equation the corresponding locus; and let

$$y = a^x$$

be the equation in which a is a positive and constant quantity, the remaining are variable, and on account of the variable exponent x the equation is transcendental. Now, it is plain, that by making x equal to zero the corresponding y becomes equal to 1. And if x increases in arithmetical proportion, y increases in a geometrical one, supposing x either positive or negative: and in the second supposition, since $a^{-x} = \frac{1}{a^x}$; the value of y will be represented by $\frac{1}{a^x}$ and will diminish when x increases; yet it will never be equal to zero as long as x preserves a finite value. Therefore let (fig. 14) AX, AY be the axes. The curve will cut AY in K at a distance from A equal to 1, and towards the positive direction of X will depart more and more from that axis; towards the negative direction it will continually approach to the same axis without ever reaching it, but at an infinite distance from A. This curve, on account of the relation between the co-ordinates, is called *Logarithmic:* and the axis or straight line to which it continually approaches is called *Asymptote.* The asymptotes are common to several curves. J. Perkins

Polar co-ordinates.

20. It happens, frequently, that the use of the polar co-ordinates is preferred to that of the rectilinear co-ordinates, hence follows the necessity of passing from one system to the other. To ascertain the relation between such co-ordinates let (fig. 15) AX, AY, be the rectangular axes, and from the origin A of these axes let the straight line AM be drawn to any point M of the curve LL′. If the length of AM and the angle which it forms with AX be known, the position of the point M will be ascertained with regard to A. These two elements, by means of which the position of the point is determined, are called *polar co-ordinates.* Now let the variable angle which MA makes with AX be represented by ω and MA (called either *radius* or *radius vector*) by ρ. From the same point M, let the perpendicular MP be drawn to X; we shall have $AP = x$, $MP = y$, but

$$PM = AM \cdot \sin MAP \, , \; AP = AM \cdot \cos MAP \, .$$

Therefore

$$y = \rho \cdot \sin \omega \, , \; x = \rho \cdot \cos \omega$$

The required relations in order to pass from the rectangular system to that of the polar co-ordinates, or vice versa.

Functions of curvilinear loci.

21. *Tangent* of the point M (fig. 16) of the curve LM is that part of the geometrical tangent which is limited between the point of contact and the point T of the axis X, that is, MT.

The *Normal* is that part of the line MT′ perpendicular to the tangent, comprised between the point of contact and T′, the point of intersection with the axis X.

The *sub-tangent* is the line TP below the tangent TM, that is, the distance between the ordinate MP of the point of contact and the tangent, on the axis X.

The *sub-normal* is the line PT under the normal MT′, that is, the distance between the ordinate MP and the normal, on the axis X.

John E. Plater

BOOK II.

CO-ORDINATES AND GEOMETRICAL LOCI IN SPACE.

Relative position of the points in space.

Two

~~Three~~ different manners of ascertaining the co-ordinates of the same point.

22. As any series of points on a plane can be referred to any system of two axes on the same plane ; so any series of points situated in space can be referred to three axes in as many planes, and their position may be determined with relation to the system of axes. Let M (fig. 17) be a point in space, and let AX, AY, AZ or X, Y, Z, designate three straight lines or axes in space. If these lines are at right angles to each other, AZ will be perpendicular to the plane XAY, and AX perpendicular to the plane YAZ, and of course the planes XAZ, YAZ perpendicular to the third XAY, and XAZ perpendicular to YAZ. Now, to have the position of the point M with reference to any system of axes, let us draw from M the line MK or z, parallel to the axis Z, so as to meet the plane XAY in K. Let another line KH or y be drawn from K parallel to the axis Y. Let AH be represented by x. It is evident that the position of the point M with reference to the axes X, Y, Z, is determined by the co-ordinates x, y, z ; because the same three co-ordinates cannot belong at the same time to any other point. It is unnecessary to repeat that the signs of the co-ordinates correspond to those of the axes, and if taken from the positive part AX, AY, AZ, they are positive ; if from the negative AX$'$, AY$'$, AZ$'$, they are negative.

23. In the same manner as from M we drew to the plane XAY, MK parallel to Z, from the same point let us draw MR to the

plane XAZ and parallel to Y, and MS to the plane ZAY and parallel to X. Now, since MS is parallel to X and MK to Z, we know from geometry, first, that the plane SMK is parallel to ZAX; secondly, that since MR and KH are parallel to Y, and consequently parallel to each other, and both being comprised between two parallel planes, they are equal to each other. In the same manner we may prove that AH is equal to MS. Therefore, the co-ordinates of any point M situated in space are determined by three lines MK, MR, MS, parallel to the axes, and drawn from the point to the planes. The denomination of each co-ordinate depends on the axis to which it is parallel; that is to say:

$$MK = z,\ MR = y,\ MS = x.$$

24. The same co-ordinates can be determined in a third manner, because let H, L, G be the points at which the axes are met by the planes RMK, SMR, SMK. The parallel lines AL, MK : SM, AH : AG, MR shall be equal. Therefore, the co-ordinates x, y, z of the point M are determined by AH, AG, AL. Now, let us join M with L, H, G. ML shall be a line parallel to the plane YAX. Therefore, to ascertain the co-ordinate z of the point M, it is sufficient to draw from this point to the axes Z, ML parallel to the plane YAX, and the portion AL of the axes between the origin and the point L, shall be the required co-ordinate. Likewise, the portions AH, AG of the axes X and Y, between the origin and the points met by MH, MG, parallel to the planes YAZ, XAZ, shall be the co-ordinates x and y of the same point M. It should be observed, that if the axes be at right angles, the lines ML, MH, MG shall be perpendicular to the corresponding axes, as well as MK, MR, MS to the corresponding planes.

The distance between two points in space determined by the co-ordinates of each point.

25. Let x, y, z be the co-ordinates of the point M, and x', y', z', the co-ordinates of the point M′, (fig. 18.) Draw from M and

M′ MK, M′K′ to the plane XAY and parallel to Z and KH, K′H′ parallel to Y, we will have

$$AH = x, KH = y, KM = z.$$

$$AH' = x', K'H' = y', K'M' = z'.$$

Again, draw MM′, KK′, and K′Q parallel to X, and M′N parallel to KK′, and let us suppose the axes at right angles. The triangle K′QK is then rectangular in Q and M′MN in N; hence

$$\overline{MM'}^2 = \overline{M'N}^2 + \overline{MN}^2$$

$$\overline{KK'}^2 = \overline{KQ}^2 + \overline{K'Q}^2$$

But M′N = KK′ hence

$$\overline{MM'}^2 = \overline{KK'}^2 + \overline{NM}^2 = \overline{NM}^2 + \overline{KQ}^2 + \overline{K'Q}^2$$

Now $NM = MK - NK = MK - M'K' = z - z'$

$KQ = KH - QH = KH - K'H' = y - y'$

$K'Q = HH' = AH - AH' = x - x'$

Therefore,

$$\overline{MM'}^2 = (z - z')^2 + (y - y')^2 + (x - x')^2$$

Which formula gives the distance MM′ between the points M and M′ by the co-ordinates of the same points.

Corollary I. If the point M′ be transferred to the origin of the axes, the co-ordinates z', y', x' will be equal to zero, and the preceding formula shall become

$$\overline{AM}^2 = z^2 + y^2 + x^2$$

that is to say, the square of the straight line drawn from the origin of the axes to any point M, is equal to the sum of the squares of the co-ordinates of that point. It is plain that to the preceding formula we may add the following

$$\overline{AM'}^2 = z'^2 + y'^2 + x'^2$$

Corollary II. Let the angle MAM′ be called β. We know from trigonometry that $\overline{MM'}^2$ is equal to the sum of the squares $\overline{AM}^2$, $\overline{AM'}^2$ minus 2 AM . AM′ . cos β. From this equation is deduced the following

$$\cos\beta = \frac{\overline{AM}^2 + \overline{AM'}^2 - \overline{MM'}^2}{2\,AM\,.\,AM'}$$

In which, substituting the values before determined, we will find

$$\cos\beta = \frac{xx' + yy' + zz'}{AM\,.\,AM'}$$

or

$$\cos\beta = \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}\sqrt{x'^2 + y'^2 + z'^2}}$$

That is, the value of the angle formed by two straight lines AM, AM′, passing through the origin of the axes, is given by the co-ordinates of M and M′.

Corollary III. Let the angles which AM makes with the axes X, Y, Z be represented by X, Y, Z, and the angles which AM′ makes with the same axes by X', Y', Z'; if from M and M′ be drawn the lines MH, M′H′ : Mk, M′k' : Mm, M′m' perpendicular to the axes, we shall have (24)

$$AH = x,\ Ak = y,\ Am = z$$

$$AH' = x',\ Ak' = y',\ Am' = z'$$

but from trigonometry

$$AH = AM\ \cos X,\ Ak = AM\ \cos Y,\ Am = AM\ \cos Z$$

$$AH' = AM' \cos X',\ Ak' = AM' \cos Y',\ Am' = AM' \cos Z'$$

hence,

$$xx' = AM\,.\,AM'\,.\cos X \cos X'$$

$$yy' = \text{AM} \cdot \text{AM}' \cdot \cos Y \cos Y'$$

$$zz' = \text{AM} \cdot \text{AM}' \cdot \cos Z \cos Z'$$

Therefore, from these values, and from the formula of the precedent corollary, we shall find

$$\cos \beta = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z'$$

The same cosine of the angle β given by the cosines of the angles which each line makes with the axes.

Corollary IV. If MAM′ is equal to 90°, then $\cos \beta = o$. Therefore when two straight lines passing through the origin of the coordinates are perpendicular to each other, the relation between the cosines of the angles, and consequently between the angles formed by each line with the axes, is given by the following formula :

$$\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' = o$$

Corollary V. Suppose $\text{MAM}' = o$, the line AM, AM′ will coincide with each other and form only one straight line. In this case $\cos \beta = 1$, and $\cos X \cos X' = \cos^2 X$, $\cos \text{Y} \cos \text{Y}' = \cos^2 \text{Y}$, $\cos Z \cos Z' = \cos^2 \text{Z}$, and the formula of the preceding Cor. III will be converted into the other

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tontie $$\cos^2 X + \cos^2 Y + \cos^2 Z = 1$$ nose

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which gives the relation between the angles which any straight line AM makes with the rectangular axes.

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Formulas for passing from one to another system of parallel axes.

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26. Let (fig. 19) AX, AY, AZ be any system of axes, and A″X″, A″Y″, A″Z″ a second system of axes parallel to the former. The axes of the latter system being produced so as to meet in k, r, s the planes of the first system, if $x_\circ$, $y_\circ$, $z_\circ$ represent the co-ordinates of A″ with reference to A, we shall have (23)

time

$$\text{A}''s = x_\circ,\ \text{A}''r = y_\circ,\ \text{A}''k = z_\circ$$

1 2 3 4 5 6 7 8 9 10 11 12 - 13 - 14 - 15 - 16 . 17
s m i t h s o n i a n i n s t i t

18 . 19 . 20
u t e

Let now M be any point to be referred to both systems, and let $MK''K$, $MR''R$, $MS''S$ be drawn from M parallel to the axes. Let moreover K'', R'', S'' represent the points of the planes of the second system met by the parallel lines, and K, R, S the corresponding points on the planes of the first system. If the co-ordinates of the point M with regard to the first system be called x, y, z, and those with reference to the second x'', y'', z'', we will have

$$MS = x,\ MR = y,\ MK = z$$

$$MS'' = x'',\ MR'' = y'',\ MK'' = z''$$

But on account of the parallel planes

$$MS = MS'' + A''s,\ MR = MR'' + A''r,\ MK = MK'' + A''k$$

hence

$$x = x'' + x_o,\ y = y'' + y_o,\ z = z'' + z_o$$

or

$$x'' = x - x_o,\ y'' = y - y_o,\ z'' = z - z_o$$

all of which give the relation between the co-ordinates of the two systems.

General formulas for passing from one system to any other whatever.

27. Let us come now to the most general formulas of transformation of co-ordinates, and let AX, AY, AZ (fig. 20) be any system of axes, and $A'X'$, $A'Y'$, $A'Z'$ any other system whatever. In order to have the relation between the co-ordinates of any point with reference to the first system, and the co-ordinates of the same point with reference to the second; let us draw from A' three lines $A'Z''$, $A'X''$, $A'Y''$ parallel to the axes X, Y, Z which will form a third system, and considering this system together with $A'X'$, $A'Y'$, $A'Z'$; let us first ascertain the relation between the co-ordinates of any point M with reference to these two systems of axes. To this end let us draw from M the straight line MB perpendicular to the plane $X''A'Y''$ and $A'B$ on the same plane, and let $A'H''$, $H''K''$, $K''M$ be the co-ordinates x'', y'', z'' of the point M with reference to the corresponding axes X'', Y'',

Z'', and $A'H'$, $H'K'$, $K'M$ the co-ordinates x', y', z' with reference to the axes X', Y', Z'. Let the perpendicular MB be called n. The sides MB, BA′ which lie in one plane, and $A'H'$ in another plane, with $H'K'$, $K'M$ in a third plane, constitute the polygon MBA′H′K′M. Now denoting by (nx'), (ny'), (nz') the angles which the lines $A'H'$, $H'K'$, $K'M$ form with lines parallel to MB or with MB itself, according to the given demonstration (2) we shall obtain

$$n = - K'M \cos (nz') - H'K' \cos (ny') - H'A' \cos (nx')$$

or $$n = - z' \cos (nz') - y' \cos (ny') - x' \cos (nx')$$

The angle which the last side $A'B$ of the polygon makes with the line parallel to n being a right angle, the corresponding cosine is equal to zero. If now the angle which n makes with MK'' be termed (nz'') or (since MK'' is parallel to the axis z) $(nz,)$ from the rectangular triangle MBK'' we will have

$$n = MK'' \cos (nz) = z'' \cos (nz)$$

which value being substituted in the preceding equation, we will find

$$z'' \cos (nz) = - z' \cos (nz') - y' \cos (ny') - x' \cos (nx')$$

or

$$z'' = - \frac{z' \cos (nz') + y' \cos (ny') + x' \cos (nx')}{\cos (n z)}$$

By a similar process supposing n' and n'' to be the perpendicular lines drawn from M to the planes $X''A'Z''$, $Y''A'Z''$ we may find

$$y'' = - \frac{z' \cos (n'z') + y' \cos (n'y') + x' \cos (n'x')}{\cos (n'y)}$$

$$x'' = - \frac{z' \cos (n''z') + y' \cos (n''y') + x' \cos (n''x)}{\cos (n''x)}$$

and these equations afford the relation between the co-ordinates x', y', z' and x'', y'', z'', or in the supposition of the axes X, Y, Z having common origin with X', Y', Z', between the co-ordinates x', y', z' and x, y, z.

Let us come now to the proposed case in which the origin of the systems is different. Supposing that $x_\circ$, $y_\circ$, $z_\circ$ are the co-ordinates of the origin A′ with reference to A, since $A'X''$, $A'Y''$, $A'Z''$ are parallel to AX, AY, AZ according to the relation (26) between the co-ordinates of the same point with reference to two systems of parallel axes, we will have

$$x'' = x - x_\circ,\ y'' = y - y_\circ,\ z'' = z - z_\circ$$

Therefore the first members of the preceding equations may be converted into $x - x_\circ$, $y - y_\circ$, $z - z_\circ$ and by transposing $x_\circ$, $y_\circ$, $z_\circ$ to the second members of the same equations we will obtain

$$x = x_\circ - \frac{x' \cos(n''x') + y' \cos(n''y') + z' \cos(n''z')}{\cos(n''x)}$$

$$y = y_\circ - \frac{x' \cos(n'x') + y' \cos(n'y') + z' \cos(n'z')}{\cos(n'y)}$$

$$z = z_\circ - \frac{x' \cos(nx') + y' \cos(ny') + z' \cos(nz')}{\cos(nz)}$$

which are the required general formulas for transformation.

Scholium. An observation is here to be made similar to that under n (8.) The determined value of x given by the second member of the foregoing equation is equivalent to

$$x_\circ - \frac{\cos(n''x')}{\cos(n''x)}\,x' - \frac{\cos(n''y')}{\cos(n''x)}\,y' - \frac{\cos(n''z')}{\cos(n''x)}\,z'$$

in which the coefficients of the co-ordinates x', y', z' are numerical values and the co-ordinates are linear quantities, that is, quantities of the first degree like x given by them. From which it follows that the value of x^{n} cannot be given but by powers of the same and less degree of the co-ordinates x', y', z'. The same is to be said with regard to y and z.

¶ *Corollary.* Supposing the axes X, Y, Z rectangular (fig. 21) the normals n, n', n''; that is to say, MB, MB$'$, MB$''$ shall be parallel to the same axes, and consequently the angles $(n''x)$, $(n'y)$, (nz) shall be equal to zero. Hence,

$$\cos(n''x) = 1, \cos(n'y) = 1, \cos(nz) = 1$$

Again, (fig. 22) the angle which for instance BM or n makes with K$'$M, that is, the angle K$'$ML, or (nz') being equal to 180° — K$'$MB and K$'$MB = LMe = ZAZ$'$ = (zz') will give us

$$\cos(nz') = \cos[180° - (zz')] = - \cos(zz')$$

And since the angle which K$'$ l' parallel to n makes with H$'$K$'$, that is, the angle H$'$K$'$ l', or (ny') is equal to 180° — l' K$'$ r and l' K$'$ r = ZAY$'$ = (zy'), we will have

$$\cos(ny') = \cos[180° - (zy')] = - \cos(zy')$$

The angle which AH$'$ makes with H$'l''$ parallel to n or the angle (nx') being equal to (180° — l'' H$'$X$'$) or 180° — ZAX or 180° — (zx') will give

$$\cos(nx') = \cos[180° - (zx')] = - \cos(zx')$$

In the same manner we may find

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$$\cos(n'z') = - \cos(yz')$$

$$\cos(n'y') = - \cos(yy')$$

$$\cos(n'x') = - \cos(yx')$$

$$\cos(n''z') = - \cos(xz')$$

$$\cos(n''y') = - \cos(xy')$$

$$\cos(n''x') = - \cos(xx')$$

Therefore, in the supposition that the axes X, Y, Z, are at right angles and the origin A is common to both systems, by substituting in the formulas above determined these last values, we will find

$$x = x' \cos(xx') + y' \cos(xy') + z' \cos(xz')$$

$$y = x' \cos(yx') + y' \cos(yy') + z' \cos(yz')$$

$$z = x' \cos(zx') + y' \cos(zy') + z' \cos(zz')$$

Formulas giving the relation between the co-ordinates of any point with reference to a system of rectangular axes and the co-ordinates of the same point referred to any other system having a common origin with the former.

REMARKS.

28. Every point, either of a straight or curve line, or of a plane or curve surface in space, may be referred to any system of axes. Suppose now the same relation to exist between the co-ordinates of each point of such a series; this relation is called the equation of the line or surface, and the series of points corresponding to the equation, as we said elsewhere (13), is termed locus or geometrical locus of the equation. From the same relation follow the two kinds of investigation already explained, that is, a series of points being given to derive the corresponding equation, or an equation being given to find the corresponding locus. We will now speak of the first investigation, which will constitute the following part of the present book, reserving to the third and fourth books the discussion of the second, both in plane surfaces and space, in a larger extension.

EXAMPLES.

I. Equation of the plane.

29. Let (fig. 23) the plane BCE be referred to the rectangular axes AX, AY, AZ, and let B*Cn*, E*Cm* represent the intersections of this plane with YAZ and XAZ. Let the distance AC between the origin of the co-ordinates and the point C of the axis Z met by BCE be called z_0 and the trigonometrical tangents of the angles B*n*Y, E*m*X, t and t'. Now to the straight line E*m*, if referred to the axes X, Z, will correspond (10) the equation

$$z = t'x + z_0 \quad (o)$$

Let now K be any point of the given plane, the perpendicular line KH drawn from K to the plane XAY, and the perpendicular Ha drawn from H to the axis X and Aa will be the co-ordinates z, y, x of that point. Imagine now the plane determined by KHa, which is parallel to ZAY, indefinitely produced, and let pr, pU, aV be the intersections of this new plane with BCE, with XAY, and with XAZ. From this construction it follows that not only pH is parallel to nY, but also that pr is parallel to nB, and aV is parallel to AZ ; therefore the angle VaU equal to ZAY shall be rectangular, and rpU shall be equal to BnY, and consequently $tg \,.\, rpU = t$. If now the line rp be referred to the axes aV, aU, we will have (10)

$$KH = t \,.\, aH + Ia$$

but Ia is an ordinate of mE with reference to the axes X, Y, and consequently is equal (o) to $t' \,.\, Aa + z_o$; therefore

$$KH = t \,.\, aH + t' . \, Aa + z_o$$

or
$$z = t\,y + t'x + z_o$$

The equation, or the relation between the co-ordinates of the point K : but the point K is any point of the plane ; therefore the same relation shall be verified with every other point of the same plane, and the formula thus produced is the required equation of the plane.

Corollary I. If we conceive another plane parallel to the first, since the angles formed by the intersections of this plane with XAZ, YAZ, and the axes X, Y shall be the same, and the distance between the origin A of the axes, and the point C′ of Z met by the new plane is different from z_o, so calling z_o' such a distance, the equation of this new plane shall be exhibited by

$$z = ty + t'x + z_o'$$

Corollary II. Imagine that this plane passes through a point of which x', y', z' are the co-ordinates, the preceding equation must necessarily admit of these peculiar values of the co-ordinates, and shall be

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$$z' = ty' + t'x' + z_o'$$

which subtracted from the preceding will give

$$z - z' = t\,(y - y') + t'\,(x - x')$$

the general equation of any plane passing through a given point.

Corollary III. Supposing now that the given point be the origin of the co-ordinates, then $x' = y' = z' = o$, and the last equation shall be converted into the following :

$$z = ty + t'\,x$$

general equation of any plane passing through the origin of the co-ordinates.

Scholium. Let us observe that from the equation $z = ty + t'\,x + z_o$ we may derive

$$y = \frac{1}{t}\,z - \frac{t'}{t}\,x - \frac{1}{t}\,z_o$$

$$x = \frac{1}{t'}\,z - \frac{t}{t'}\,y - \frac{1}{t'}\,z_o$$

or making $\quad \frac{1}{t} = m, -\frac{t'}{t} = n, -\frac{1}{t}\,z_o = q$

and $\quad \frac{1}{t'} = m', -\frac{t}{t'} = n', -\frac{1}{t'}\,z_o = q'$

$$y = mz + nx + q$$
$$x = m'z + n'y + q'$$

both of which are equations of the same plane, and which could be deduced directly as the first was deduced ; so m and n are the tangents of the angles made by the intersections of the plane with Z and X, and m', n' the tangents of the angles formed by the intersections with Z and Y.

II. *Equation of the surface generated by the revolution of any line about an axis.*

30. Still supposing the system of axes at right angles, to which system every point of the surface is to be referred, imagine a plane ZAV passing through the axis Z, on which is described any line *msn* (fig. 24,) and imagine the same plane moveable about Z; it is evident, first, that the line *msn* will describe a surface around Z; and if we conceive the surface already produced, it is also evident that any section of the surface made by a plane passing through Z shall reproduce the same line *msn*.

Let us now suppose, as given, a surface *mpnq* generated in the described manner by any line, and suppose *s* to be any point of the given surface, the section *msn* made on the plane ZAV passing through *s* by the surface will be the generating line. Now the perpendicular *sk* drawn from *s* to the plane XAY, the perpendicular *kh* drawn from *k* to the axis X and A*h*, are the co-ordinates z, y, x of the point *s*, the relation of which is to be found. To this end let *sr* be drawn from *s* perpendicular to Z, it shall be equal to A*k*, but $\overline{Ak}^2 = \overline{Ah}^2 + \overline{hk}^2 = x^2 + y^2$. Hence,

$$\overline{rs}^2 = x^2 + y^2$$

And if the line *msn* be referred to the axes AZ, AV, considering Z as the axis of abscissas, *rs* shall be the ordinate of the point *s*, and supposing the equation of *msn* to be expressed by

$$v = f(z) \ldots\ldots (o)$$

rs will be equal to $f(r\mathrm{A})$, but $r\mathrm{A} = sk = z$; therefore $rs^2 = [f(z)]^2$, which value, substituted in the former equation, will give

$$[f(z)]^2 = x^2 + y^2 \ldots\ldots (o_1)$$

The required relation between the co-ordinates of the point *s*. But *s* is any point of the surface; hence the relation between the co-ordinates of any point shall be that given by (o_1), and conse quently (o_1) is the general equation of the surface generated by the revolution of any line about the axis Z.

Let us now propose some applications, observing before that the second member of (o_1) is independent of $f(z)$, and whatever be the generating line shall be always the same. On the contrary, $f(z)$ is variable with the generating line, and depends upon the equation of the same line. Therefore, to obtain the equation of the surface generated by a given line, it is sufficient to determine the equation of this line with reference to the axes AX, AV; that is, it is sufficient to determine $f(z)$, and to substitute this value in the general equation (o_1).

Application I. If the generating line is a circle, having the centre in A, since (11) the equation of this line is

$$v^2 = r^2 - z^2$$

or

$$v = \sqrt{r^2 - z^2}$$

we will have (o)

$$[f(z)]^2 = r^2 - z^2$$

and consequently (o_1) the equation of the generated surface shall be $r^2 - z^2 = x^2 + y^2$, or

$$r^2 = x^2 + y^2 + z^2 \ldots\ldots (a)$$

but the surface produced by a circle turned around a fixed diameter is a sphere; therefore the formula (a) is the equation of the spherical surface, having the radius r and the centre at the origin of the axes.

Application II. If the generating line is (fig. 25) the straight line mn, making the angle nmz with the positive direction of Z, supposing $An = r$, and according to the equation of the straight line (10) we will have

$$v = [tg \,.\, nmZ]\, z + r$$

but $nmZ = 180° - nmA$; consequently $tg \,.\, nmZ = - tg \,.\, nmA$ and $- tg \,.\, nmA = - \frac{nA}{mA}$; therefore, if the altitude Am be represented by $q : tg \,.\, nmZ = - \frac{r}{q}$; consequently the equation of the describing line mn is

$$v = - \frac{r}{q} z + r$$

from which

$$v^2 = [- \frac{r}{q} z + r]^2 = \frac{r^2}{q^2} (q - z)^2$$

hence the equation of the corresponding surface is

$$\frac{r^2}{q^2} (q - z)^2 = x^2 + y^2 \quad (a_1)$$

but the surface produced by mn is that of a right cone, of which Am is the altitude and An the radius of the base; therefore the formula (a_1) is the equation of the surface of the right cone.

Application III. If the generating line is the straight line mn (fig. 26) parallel to Z, suppose $An = r$, and since the angle made by mn with Z is equal to zero and $tg\ (o) = o$, the equation of mn, with reference to the axes AZ, AV, will be

$$v = r$$

from which $$v^2 = r^2$$

and consequently the equation of the surface generated by mn is

$$r^2 = x^2 + y^2 \ldots . (a_2)$$

but the surface produced by mn is that of a right cylinder, of which the axis passes through AZ, and the radius of the base is An: therefore the formula (a_2) is the equation of the surface of the described cylinder. It is to be observed that this equation is the same as that of the circular base (11).

3

Polar co-ordinates.

31. The rectangular co-ordinates AH $= x$, HK $= y$, KM $= z$ (fig. 27) of any point M can be transformed into polar co-ordinates. Because the position of the point M with reference to A can be completely determined by the straight line AM, drawn from the origin A to the point M, by the angle MAZ, which the same line makes with Z, and by the angle that the plane ZAK, on which is AM, makes with ZAX, namely, the angle XAK. It is here to be remarked, that supposing the angle ZAM to be taken from 0° to 180°, the position of the rectilinear co-ordinate AM may be directed towards every possible point in space around A, provided the angle XAK be taken from 0° to 360°. Now, for the sake of brevity, let MA be termed ρ, and the angles ZAM, XAK, θ and ω. The values of these polar co-ordinates may be given by those of the rectangular co-ordinates corresponding to the same point. Because

$$x = \mathrm{AK} \,.\, \cos\omega$$

$$y = \mathrm{AK} \,.\, \sin\omega$$

$$z = \mathrm{MA} \,.\, \sin(90^\circ - \theta)$$

and since AK $=$ AM $\cos(90^\circ - \theta) = \rho \sin\theta$ and MA $\sin(90^\circ - \theta) = \rho\cos\theta$;

$$x = \rho \sin\theta \cos\omega$$

$$y = \rho \sin\theta \sin\omega$$

$$z = \rho \cos\theta$$

which values, if substituted instead of x, y, z in the equation of the surface or line referred to the rectangular axes, will afford the equation or relation between the polar co-ordinates of the same series of points.

Lines in space.

REMARK.

32. The object of any equation, as we observed (13), is the description or construction of the corresponding locus. For instance, from $z = tx + t'y + z_0$, the equation of the plane, we may derive the different values of z corresponding to x and y, which are given, and each one of which z marks a point of the plane. But suppose that instead of considering the whole plane we refer only to the axes, a series of points traced by a line, for example, a circle on the same plane. It is evident that the relation between the co-ordinates of every point of the plane, or the equation of the plane, shall be the equation of that peculiar series of points. And for this reason the above mentioned object of the equation cannot be obtained with regard to the circle, because no mark distinguishes the co-ordinates proper to the circle from those of the plane in general. Therefore, to have the only values proper to the line, it is necessary to modify the relation between the co-ordinates in such a manner as to exclude all of them which do not belong to the same line. This modification considered under its most general aspect will be afforded by the following discussion.

Equations of any line in space.

33. Let (fig. 28) L be any line referred to the rectangular axes X, Y, Z : and imagine two straight lines Rr, perpendicular to the plane XAY, and Rr' perpendicular to the plane XAZ running along the given L. The lines P and Q described by these two perpendiculars shall be the projections of L on the planes XAY, XAZ. Two observations are here to be made: first, that in the same manner as the perpendicular lines describe on the planes the projections P and Q, so they describe in space two cylindrical surfaces, of which L is the common intersection. Secondly, the same perpendiculars take successively the place of the co-ordi-

nates of L ; that is to say, the perpendicular Rr becomes successively the co-ordinate z of every point of L, and the perpendicular Rr' the co-ordinate y of every point of the same L. From the last observation it follows that no other co-ordinates x, y, z but those of the projections can be the co-ordinates of L. Because, after having drawn from any point R of L the perpendicular Rr to the plane XAY or the co-ordinate z, the two remaining co-ordinates of the same point are co-ordinates of a point r of the projection P ; consequently no other co-ordinates x, y belong to the line L in space, but those of its projection P on the plane XAY. Likewise we may demonstrate that no other co-ordinates x, z belong to the line L, but the co-ordinates of the projection Q. Therefore, supposing the equations of the projections P and Q to be represented by

$$x = f(y) \ldots . (e_1)$$

$$x = F(z) \ldots . (e_2)$$

To every value of x shall correspond at once y from (e_1), and z from (e_2), all co-ordinates of the same point of L. And likewise, if the given co-ordinate is either y or z, from (e_1) or (e_2) we will derive x, which substituted in the other shall give the third co-ordinate. It is now evident why the equations (e_1), (e_2) are termed equations of the line in space. It is evident, also, that the same equations must be different in the different cases, and of this we will now give some examples.

EXAMPLE 1.

Equations of the intersection of two given planes.

34. Any geometrical locus is said to be given or known when the corresponding equation is known, and the equation is known when the constant quantities are known. For example, suppose the constant quantities a, a', b + : A, A', B given or known, the equations

$$z = ax + a'y + b \,,\; z = \mathrm{A}x + \mathrm{A}'y + \mathrm{B}$$

of two planes (29) are equally known, as also the corresponding planes; because, substituting at pleasure every possible value for x and y, we are always able to derive the corresponding z. Let us suppose now these two planes inclined to each other, and let us consider the straight line produced in space by their common intersection; it is evident that the co-ordinates of this intersection shall fulfil at once the equations of the two planes, and the co-ordinates only of that intersection can fulfil at the same time both equations of the planes; therefore, every formula derived from the equations of the planes in the supposition of the common co-ordinates, is necessarily an equation between the co-ordinates of the intersection. Now in the said supposition subtract the first from the second equation of the given planes, the difference shall be

$$x\,(\mathrm{A} - a) + y\,(\mathrm{A}' - a') + \mathrm{B} - b \quad o$$

or
$$x = \frac{a' - \mathrm{A}'}{\mathrm{A} - a}\,y + \frac{b - \mathrm{B}}{\mathrm{A} - a} \; \ldots \; (e_1)$$

which being an equation between the co-ordinates x and y of a line in space, is (33) the equation of the projection of the same line on the plane of the axes X and Y. Observe now that the equations of the planes may be reduced to the following form:

$$a'y = z - ax - b \,,\; \mathrm{A}'y = z - \mathrm{A}x - \mathrm{B}$$

or

$$y = \frac{1}{a'}\,z - \frac{a}{a'}\,x - \frac{b}{a'}$$

$$y = \frac{1}{\mathrm{A}'}\,z - \frac{\mathrm{A}}{\mathrm{A}'}\,x - \frac{\mathrm{B}}{\mathrm{A}'}$$

In the same supposition of equal co-ordinates in both equations, if the first be subtracted from the second, the difference will be

$$z\left(\frac{1}{\mathrm{A}'} - \frac{1}{a'}\right) - x\left(\frac{\mathrm{A}}{\mathrm{A}'} - \frac{a}{a'}\right) - \frac{\mathrm{B}}{\mathrm{A}'} + \frac{b}{a'} = o$$

or $$z\,(a' - A') - x\,(Aa' - A'a) - Ba' + A'b = o$$

or finally

$$x = \frac{a' - A'}{Aa' - A'a}\,z + \frac{A'b - Ba'}{Aa' - A'a} \;\ldots\ldots\; (e_2)$$

which being an equation between the co-ordinates x, z of the intersection in space, necessarily must be the equation of the projection of the same line on the plane of the axes X, Z; but the equations of two projections of any line in space are the equations of the same line; consequently the last formula (e_2) with the preceding (e_1) are the equations of the intersection of the planes.

Scholium. It is evident that the equations of the intersection of any two surfaces may be determined in the same manner, provided the equation of each surface is given.

EXAMPLE II.

Equations of a straight line passing through two given points in space.

Any point is said to be given when its co-ordinates are given; consequently the points corresponding to the known or given co-ordinates

$$x_1,\, y_1,\, z_1 \;:\; x_2,\, y_2,\, z_2$$

are two given points. Now to find the equations of the line passing through these points, it is first to be remarked that the projection on a plane of any straight line situated in space is another straight line; because if (fig. 29) from any point p of the straight line ab in space we draw the perpendicular pq to the plane P, the plane determined by ab and pq is perpendicular to P, and the same which would be produced by pq if moved along ab with parallel motion; therefore, the intersection of that plane with P is the projection of ab. Now the intersection of two planes is a straight line, which was to be demonstrated, and which may be deduced, from the preceding number also. Now from this remark, and from the form (10) proper to the equation of the straight line, we may conclude that the equations of any line in space will be represented by

$$\left.\begin{array}{l} x = ay + b \\ x = a'z + b' \end{array}\right\} (o)$$

Hence, what is to be done in the present case is to ascertain the values of a, a'; b, b'. Now the co-ordinates x, y, as well as x, z of every point of the line in space, must fulfil the equations (o); but by supposition the line in space passes through the given points; hence the co-ordinates x_1, y_1, and x_2, y_2, must fulfil the first equation, and x_1, z_1 : x_2, z_2 the second; and so we will have at once

$$x_1 = ay_1 + b \;,\; x_2 = ay_2 + b$$

$$x_1 = a'z_1 + b' \;,\; x_2 = a'z_2 + b'$$

To obtain the required values of a, a', b, b', let us first subtract the second equation of each binary from the first, we will obtain

$$a = \frac{x_1 - x_2}{y_1 - y_2} \;,\; a' = \frac{x_1 - x_2}{z_1 - z_2}$$

the values of a and a' determined by the known co-ordinates. Again, since from the first equation of each binary we have

$$b = x_1 - ay_1 \,,\; b' = x_1 - a'z_1$$

and substituting the determined values of a and a'

$$b = x_1 - \frac{x_1 - x_2}{y_1 - y_2}\, y_1$$

$$b^1 = x_1 - \frac{x_1 - x_2}{z_1 - z_2}\, z_1$$

which are the values of b and b^1 given by known quantities. Substituting, now, the values of a, a^1, as well as those of b and b^1, in the equation (o), we will obtain the required equations

$$x = \frac{x_1 - x_2}{y_1 - y_2} y + [x_1 - \frac{x_1 - x_2}{y_1 - y_2} y_1]$$

$$x = \frac{x_1 - x_2}{z_1 - z_2} z + [x_1 - \frac{x_1 - x_2}{z_1 - z_2} z_1]$$

Scholium. Supposing only one point to be given of which x_1, y_1, z_1 are the co-ordinates, instead of the two preceding binaries, we will have two single equations

$$x_1 = ay_1 + b$$

$$x_1 = a'z_1 + b'$$

from which

$$b = x_1 - ay_1$$

$$b' = x_1 - a'z_1$$

and substituting these values in (o) we have

$$x = ay + [x_1 - ay_1]$$

$$x = a'z + [x_1 - a'z_1]$$

the equations of any straight line in space passing through a given point. It is here to be observed, first, that the same equations may be transformed into

$$x - x_1 = a\ (y - y_1)$$

$$x - x_1 = a'\ (z - z_1)$$

the form of which is that corresponding to the line on a plane (10) passing through a given point. Observe, also, that the constants a, a' remain undetermined in this case. And if the given point is the origin of the co-ordinates, since in this supposition $x_1 = y_1 = z_1 = o$, the last equations will be converted into

$$x = ay, \quad x = a'z$$

which are the general equations of any line in space passing through the origin of the co-ordinates.

EXAMPLE III.

Equations of the perpendicular drawn from a given point to a given plane.

36. Suppose

$$x = mz + ny + q$$

to be (29 *Sch.*) the equation of the given plane in which, consequently, (34) m, n, q are known quantities. And suppose x_1, y_1, z_1 to be the equally known co-ordinates of the given point. According to the equations of the straight line which passes through a given point, the equations of the perpendicular shall have the following form:

$$\left.\begin{aligned} x - x_1 &= a\,(y - y_1) \\ x - x_1 &= a'\,(z - z_1) \end{aligned}\right\} (o)$$

in which a and a' are to be determined. To this end imagine that BC, BD represent (fig. 30) the intersections of the given plane with XAZ and XAY, and let LM be the perpendicular line drawn from the given point to the given plane. Draw LL′ perpendicular to the plane XAZ, and LL″ to the plane XAY, from any point L of LM; and imagine the plane determined by ML and LL′, and that determined by ML and LL″, to be produced so as to intersect XAZ and XAY. These intersections L′M′ and L″M″, are (35) the projections of LM, of which (o) are the equations. Now the projection L′M′ is at right angles with BC, and L″M″ with BD; because, on account of the perpendicular LL′, the planes ZAX, LMM′L′ are perpendicular to each other, and since LM is perpendicular to the given plane, this plane CBD with the same LMM′L′ are equally perpendicular to each other. But from elementary geometry, when two planes are perpendicular to a third plane, their common intersection is also perpendicular to the same plane. Hence BC is perpendicular to the plane L′M′ML, and, consequently, to the line L′M′ on the same plane. In the same manner BD is perpendicular to $L''M''$. Now we observed (10 Cor. ii) that if the tangent of the angle formed by a straight line and an axis is equal to t, the tangent of the angle formed by the perpendicular to that line and the same axis is $-\frac{1}{t}$. But according to the equation of the plane CBD, the tangent of the angle made by BC and Z is (29 sch.) m, and that of the angle made by DB and Y is n; therefore the tangent of the angle contained by $L'M'$ and Z shall be $-\frac{1}{m}$, and that of the angle contained by $L''M''$ and Y shall be $-\frac{1}{n}$. But the coefficient a' of the second (o) is (10 Cor. i) the tangent of the angle which $M'L'$ makes with Z, and the coefficient a of the first (o) is the tangent of the angle which $L''M''$ makes with Y; hence

$$a = -\frac{1}{n}, \quad a' = -\frac{1}{m}$$

which values substituted in (o) will give

$$\left.\begin{aligned} x - x_1 &= \frac{1}{n}(y_1 - y) \\ x - x_1 &= \frac{1}{m}(z_1 - z) \end{aligned}\right\} \quad . \; . \quad (o_1)$$

The equations of LM, in which nothing more remains to be determined.

Scholium I. Suppose the co-ordinates of the point M of the plane met by the perpendicular to be sought. It is evident that the co-ordinates of M are at once co-ordinates of the plane and of the perpendicular; hence they must fulfil at once the equation of the plane and those of the perpendicular. It is also plain that the co-ordinates of that point alone are common to the equation of the plane and to those of the perpendicular. Therefore, to suppose x, y, z common to these equations, is the same as to substitute in the equations the co-ordinates of M. In order to deduce the values of such co-ordinates, let us first transform the last equations (o_1) into the following :

$$\left.\begin{aligned} y &= y_1 - nx + nx_1 \\ z &= z_1 - mx + mx_1 \end{aligned}\right\} \ldots (o_2)$$

and let us substitute these values of y and z into the equation of the plane, which shall become

$$x = m\,[z_1 - mx + mx_1] + n\,[y_1 - nx + nx_1] + q$$

hence

$$x + m^2x + n^2x = mz_1 + \underline{m^2}x_1 + ny_1 + \underline{n^2x_1} + q$$

$$= m^2x_1 + n^2x_1 + mz_1 + ny_1 + q + x_1 - x_1$$

or

$$x\,[1 + m^2 + n^2] = x_1\,[1 + m^2 + n^2] + mz_1 + ny_1 + q - x_1$$

and finally

$$x = x_1 + \frac{mz_1 + ny_1 + q - x_1}{1 + m^2 + n^2}$$

the value of the co-ordinate x given by known quantities, and corresponding to the point M. To obtain the values of the others let us modify the last equation and the preceding (o_2) in the following manner:

$$x - x_1 = \frac{mz_1 + ny_1 + q - x_1}{1 + m^2 + n^2}$$

$$y = y_1 - n\,(x - x_1)$$

$$z = z_1 - m\,(x - x_1)$$

Substituting, now, the difference $x - x_1$ given by the former in the latter equations, we will obtain

$$y = y_1 - \frac{n\,(mz_1 + ny_1 + q - x_1)}{1 + m^2 + n^2}$$

$$z = z_1 - \frac{m\,(mz_1 + ny_1 + q - x_1)}{1 + m^2 + n^2}$$

which are the required values of y and z.

Scholium II. The length of the perpendicular LM from the given point to the plane may be determined, observing that (25)

$$\sqrt{[(x - x')^2 + (y - y')^2 + (z - z')^2]}$$

gives the length of a straight line in space comprised between two points, of which x, y, z and x', y', z' are the co-ordinates. Therefore, if, in the general formula, we substitute for x, y, z the preceding values of the co-ordinates of M, and for x', y', z' the co-ordinates x_1, y_1, z_1 of the given point, that formula shall give the length of the perpendicular. But by such a substitution the first term $(x - x')^2$ of the general formula becomes

$$\frac{(mz_1 + ny_1 + q - x_1)^2}{(1 + m^2 + n^2)^2}$$

the second $(y - y')^2$

$$n^2\,\frac{(mz_1 + ny_1 + q - x_1)^2}{(1 + m^2 + n^2)^2}$$

the third $(z - z')^2$

$$m^2\,\frac{(mz_1 + ny_1 + q - x_1)^2}{(1 + m^2 + n^2)^2}$$

and consequently their sum

$$\frac{(mz_1 + ny_1 + q - x_1)^2}{1 + m^2 + n^2}$$

of which the corresponding root

$mz_1+ny_1+q-x_1$
$mz_1+ny_1+q-x_1$

$m^2y^2+m^2z_1y+mzq-mz_1x_1$
$m^2z_1z+m^2z_1^2+mz_1q+ mz_1x$
$m^2y^2+ ... nyq+q^2-qx$

$$\frac{mz_1+ny_1+q-x_1}{\sqrt{1+m^2+n^2}}$$ $\cancel{mz_1x}+qx-x_1^2$

is the required value of the length of the perpendicular.

$m^2y^2+2m^2z_1y+2mzq+m^2y^2+2my_1q+myq+q^2$

Use of the equations of the straight line and plane to derive some angular relations.

I. Value of the cosine of the angle contained by two straight lines passing through the origin of the axes.

37. We obtained (25 Cor. II) the value of the cosine of the angle MAM′ ($=\beta$) by means of the co-ordinates of the extremities M and M′ (fig. 31) of the sides, without any regard to the equations of these lines. Observe now that the equations of AM and AM′ may be exhibited (35 *Sch.*) by

$$x=ay\ ,\ x=a'z$$

$$x'=cy'\ ,\ x'=c'z'$$

from which

$$y=\frac{1}{a}x\ ,\ z=\frac{1}{a'}x$$

$$y'=\frac{1}{c}x'\ ,\ \ z'=\frac{1}{c'}x'$$

and

$$yy'=\frac{1}{ac}xx'\ \ ,\ zz'=\frac{1}{a'c'}xx'$$

where we may suppose the co-ordinates of every point of AM and AM′; hence, the co-ordinates also of the points M and M′. In this supposition, since the value of cos β already (25) determined is given by

$$\frac{xx'+yy'+zz'}{\sqrt{x^2+y^2+z^2}\sqrt{x'^2+y'^2+z'^2}}$$

in which x, y, z, x', y', z' are the co-ordinates of M and M′; we may substitute the corresponding values given by the preceding equations. Hence,

$$xx' + yy' + zz' = xx'\left(1 + \frac{1}{ac} + \frac{1}{a'c'}\right)$$

and

$$\sqrt{x^2 + y^2 + z^2} = x\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}$$

$$\sqrt{x'^2 + y'^2 + z'^2} = x'\sqrt{1 + \frac{1}{c^2} + \frac{1}{c'^2}}$$

consequently

$$\cos\beta = \frac{1 + \frac{1}{ac} + \frac{1}{a'c'}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}\sqrt{1 + \frac{1}{c^2} + \frac{1}{c'^2}}} \ldots (a)$$

the required value dependent only on the constant quantities of the equations of the lines forming the angle β.

Corollary I. If the angle β is a right angle, then $\cos\beta = o$; and in this case

$$1 + \frac{1}{ac} + \frac{1}{a'c'} = o$$

Corollary II. Let us suppose that the side AM′ of the angle β becomes coincident with the axis X, then (25 C. III) the angle β shall become X. It is further to be considered that all the co-ordinates y and z of the line coincident with X are equal to zero; therefore, in the present supposition the co-ordinates z', y' of the point M′ are equal to zero; the co-ordinate, moreover, x' of that point is equal to AM′; hence, the general formula (25)

$$\cos\beta = \frac{xx' + yy' + zz'}{\text{AM} \,.\, \text{AM}'}$$

will be converted into

$$\cos X = \frac{x \,.\, \text{AM}'}{\text{AM} \,.\, \text{AM}'} = \frac{\text{X}}{\text{AM}}$$

But (25 C. I.)

$$\text{AM} = \sqrt{x^2 + y^2 + z^2} = x\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}$$

hence,

$$\cos X = \frac{1}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

in which the sign of the second member depends on the sign given to the root; but the second member must have the positive sign when X is less than 90°, and negative when X is greater than the right angle; therefore, when the line AM makes an angle with X greater than the right angle, the radical expression

$$\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}$$

will be negative; when AM makes an angle with the axis X less than a right angle, the same radical, dependent on the constant quantities a, a', will be positive. Let us here remark, that sometimes the line, or any geometrical locus, is represented by its equation, so that it is the same to say the line AM as the line $[x = ay\ ,\ x = a'z]$

Let us now come to the second case in which we suppose the side AM′ of the angle β to coincide with the axis Y; evidently (25) the angle β will become Y. But in this second supposition the co-ordinates x', z' of M′ are equal to zero, and the co-ordinate y' equal to AM′; hence, the general formula will become

$$\cos Y = \frac{y\ .\ \text{AM}'}{\text{AM}\ .\ \text{AM}'} = \frac{y}{x\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

now from the first equation $x = ay$ of AM, we have

$$y = \frac{x}{a} = x\ .\ \frac{1}{a}\text{; hence}$$

$$\cos Y = \frac{\frac{1}{a}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

Suppose, finally, AM$'$ to coincide with Z, in this case, β becomes Z, and $x' = y' = o$ and $z' = \text{AM}'$, consequently from the general formula

$$\cos Z = \frac{z \cdot \text{AM}'}{\text{AM} \cdot \text{AM}'} = \frac{z}{x\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

but from the second equation $x = a'z$ of AM we have $z = x \cdot \frac{1}{a'}$;

hence,

$$\cos Z = \frac{\frac{1}{a'}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

It will be remarked that, by supposing the successive coincidence of AM with the axes, we might have derived the formulas

$$\cos X' = \frac{1}{\sqrt{1 + \frac{}{c^2} + \frac{1}{c'^2}}}$$

$$\cos Y' = \frac{\frac{1}{c}}{\sqrt{1 + \frac{1}{c^2} + \frac{1}{c'^2}}}$$

$$\cos Z' = \frac{\frac{1}{c'}}{\sqrt{1 + \frac{1}{c^2} + \frac{1}{c'^2}}}$$

in which the radical

$$\sqrt{1+\frac{1}{c^2}+\frac{1}{c'^2}}$$

is positive or negative according as the angle formed by the line $[x = cz\ ,\ y = c'z]$ with X is greater or less than a right angle. From the same formulas it appears, that if the equations of any line passing through the origin of the axes is given, we may always derive the angles which the same line makes with the axes; because these angles depend only on the constant quantities of the equations.

Scholium. It is evident that the projections of two parallel lines, for instance (fig. 32) aM$'$, Am', taken on the same plane XAY, are equally parallel lines; because, supposing the perpendiculars pr, qs drawn from any point p, and q of aM$'$, Am' to the plane XAY, the planes determined by aM$'$, pr, and Am', qs are of course parallel to each other, and consequently their intersections with the plane XAY are parallel to each other; but (35) such intersections are the projections of each line. Hence, the projections of the parallel line in space, taken on the same plane, are likewise parallel lines. Now let the straight lines aM, aM$'$ in space be given at any angle, and let their equations be

$$x = ay + b\ ,\ x = a'z + b'$$

$$x = cy + d\ ,\ x = c'z + d'$$

and suppose Am, Am' drawn parallel to aM, aM$'$ from the origin of the axes, evidently the angle m'Am is equal to M$'a$M, and the angles formed by Am', Am with the axes are the angles made by the directions of aM$'$, aM with the same axes. But all these angles depend on the constants a, a', c, c' of the preceding equations, as will be seen hereafter; hence, by means of these constants we may derive the values of the angle MaM$'$ and of the other mentioned before. To demonstrate it, let us remark that the projections of aM$'$ are parallel to the projections of Am', and the projections of aM are parallel to those of Am. Now (10 *Sch.*) if the equation of a line in a plane is $x = ay + b$, the equation of another line parallel to the former, and passing through the origin of the axes, is $x = ay$; hence, the equations of Am, Am' will be

$$x = ay, \quad x = a'z$$

$$x = cy, \quad x = c'z$$

But the value of the angle mAm', as well as those formed by each side with the axes, depend on the constant quantities of these equations in the explained manner; hence the angle MaM$'$, and those formed by aM, aM$'$ and the axes or parallel lines to the axes, will be given by the constant quantities a, a', c, c'.

II. Value of the cosine of the angle formed by two planes.

38. It is known from elementary geometry that the angle formed by two planes is the same angle as that formed by the lines drawn perpendicularly from any point in space to both planes. Hence, if we are able to determine the cosine of the angle made by the lines perpendicular to the planes

$$x = Az + A'y + A''$$

$$x = Bz + B'y + B''$$

we shall have the cosine of the angle formed by the planes themselves. Suppose, now, x_1, y_1, z_1 to be the co-ordinates of the point from which are drawn the perpendiculars to the planes. The equations of the perpendicular drawn to the first plane are (36 (e_1))

$$x - x_1 = \frac{1}{A'}(y_1 - y)$$

$$x - x_1 = \frac{1}{A}(z_1 - z)$$

and the equations of the perpendicular drawn to the second plane are

$$x - x_1 = \frac{1}{B'}(y_1 - y)$$

$$x - x_1 = \frac{1}{B}(z_1 - z)$$

Which equations, if the given point be on the origin of the axes, will be converted into

$$x = -\frac{1}{A'}y \qquad x = -\frac{1}{B'}y$$

$$x = -\frac{1}{A}z \qquad x = -\frac{1}{B}z$$

Now the cosine of the angle formed by these two perpendiculars is given (37) by the formula (a), substituting there, instead of a, a', c, c', the coefficients of the last equations. Hence, if we term β' the angle formed by the planes, we will have

$$\cos\beta' = \frac{1 + A'B' + AB}{\sqrt{1 + A'^2 + A^2}\sqrt{1 + B'^2 + B^2}} \cdot \cdot \cdot \cdot \cdot (a_1)$$

Observe (37 C. II) that the radicals

$$\sqrt{1 + A'^2 + A^2}, \sqrt{1 + B'^2 + B^2}$$

will be positives when the angles made by the perpendiculars with the axis X are less than 90°; will be negative when the same angles are greater than 90°.

III. Value of the cosine of the angle formed by a straight line and a plane.

39. Let us now treat of the angle contained by a given line and a plane in space, and suppose the plane represented by CB (fig. 33) and the line by pq. If from any point p of qp we draw

the line pr perpendicular to the plane and then qr on the same plane, the angle pqr is the angle which the line pq is said to make with CB, and which, from analogy with the preceding angles, we will term β''. In order to obtain the sine of this angle, imagine the plane C$'$B$'$ parallel to CB and passing through the origin of the axes. From the same origin A let us draw Ap' parallel to qp, $p'r'$ perpendicular to C$'$B$'$ and Ar'; the angle p'Ar' will be equal to β''. Suppose, now, Ap'' parallel to $r'p'$, we will have

$$p''\mathrm{A}p' = p''\mathrm{A}r' - p'\mathrm{A}r' = 90° - \beta''$$

consequently,

$$\cos\,(p''\mathrm{A}p') = \cos\,(90° - \beta'') = \sin\,\beta''$$

But the first member of the equation is the cosine of the angle formed by two lines passing through the origin of the co-ordinates, which depends on the equations of the same lines. Hence, to obtain sin β'', it is necessary first to derive from the given equations,

$$x = ay + b\,, \quad x = a'z + b'$$

of the line pq, and

$$x = \mathrm{A}z + \mathrm{A}'y + \mathrm{A}''$$

of the plane CB, the equations of the lines Ap', Ap''. Now the equations of Ap' parallel to pq are

$$x = ay\,,\; x = a'z \;.\;.\;.\;.\; (o)$$

and the equation of the plane C$'$B$'$ parallel to CB, and passing through the origin of the axes, is (29, CC. I, III)

$$x = \mathrm{A}z + \mathrm{A}'y$$

Finally, the equations of the line Ap'' perpendicular to $C'B'$ are (36 (e))

$$x = -\frac{1}{A'}y, \quad x = -\frac{1}{A}z \quad \ldots (o_1)$$

Now, by a substitution in the formula (a) [37] we deduce from (o) and (o_1) the required value of cos ($p''Ap'$) or sin β''; that is,

$$\sin\beta'' = \frac{1 - \frac{A'}{a} - \frac{A}{a'}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}\sqrt{1 + A'^2 + A^2}} \quad \ldots (a_2)$$

in which equation the usual observation with regard to the radical is to be made according as the lines Ap', Ap'' form an angle greater or less than 90°.

New form of the equations of a straight line, and of a plane passing through a given point.

40. From the angular relations already considered (37, C. II) we may derive a new and useful form of the equations of a line passing through a given point. The known equations of such a line are

$$x - x_1 = a(y - y_1), \quad x - x_1 = a'(z - z_1) \quad (e)$$

and the angular relations of the same line with the axes are given by the formulas

$$\cos X = \frac{1}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

$$\cos Y = \frac{\frac{1}{a}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

$$\cos Z = \frac{\frac{1}{a'}}{\sqrt{1 + \frac{1}{a^2} + \frac{1}{a'^2}}}$$

from which

$$\frac{\cos X}{\cos Y} = a, \quad \frac{\cos X}{\cos Z} = a$$

and substituting these values in the preceding (e)

$$x - x_1 = \frac{\cos X}{\cos Y}(y - y_1), \quad x - x_1 = \frac{\cos X}{\cos Z}(z - z_1)$$

hence

$$\frac{x - x_1}{\cos X} = \frac{y - y_1}{\cos Y}, \quad \frac{x - x_1}{\cos X} = \frac{z - z_1}{\cos Z}$$

New form of the equations of a straight line which was required.

41. Let us now consider the equation of the plane. If x_2, y_2, z_2 are the co-ordinates of the point through which the plane passes, its equation (29, C. ii) is

$$x - x_2 = \mathrm{A}\,(z - z_2) + \mathrm{A}'\,(y - y_2) \;.\;.\;.\;.\;.\; (f)$$

Imagine from any point $[x_1, y_1, z_1]$ we draw a perpendicular to the plane, the equations of this perpendicular (36 (e)) are

$$x - x_1 = -\frac{1}{\mathrm{A}'}(y - y_1), \quad x - x_1 = -\frac{1}{\mathrm{A}}(z - z_1)$$

and from the angles X, Y, Z, which this line makes with the axes, we will have, as in the preceding number,

$$\frac{\cos X}{\cos Y} = -\frac{1}{\mathrm{A}'}, \quad \frac{\cos X}{\cos Z} = -\frac{1}{\mathrm{A}}$$

hence

$$\mathrm{A}' = -\frac{\cos Y}{\cos X}, \quad \mathrm{A} = \frac{\cos Z}{\cos X}$$

which values, substituted in (f), give

$$x - x_2 = -\frac{\cos Z}{\cos X}(z - z_2) - \frac{\cos Y}{\cos X}(y - y_2)$$

from which

$$(x - x_2)\cos X + (z - z_2)\cos Z + (y - y_2)\cos Y = o$$

A new form of the equation of a plane passing through the point $[x_2, y_2, z_2]$, in which the angles X, Y, Z are those formed by the axes and any line perpendicular to the plane.

BOOK III.

LINES OF THE SECOND ORDER.

REMARKS.

42. It now remains for us to distinguish more exactly that part which belongs to the present book from that which regards the fourth.

Let us recall to mind the distinction (17) already given between algebraical and transcendental lines, and the different orders of the algebraical lines according to the different degrees of the corresponding equations. If now we suppose an algebraical equation of a certain degree so general as to comprehend all the possible cases of the equations of that degree, it appears that from the discussion of such an equation, we may derive, first, the common properties of all the lines of the same order; and secondly, by modifying the general equation, the properties of the peculiar lines corresponding to the different modifications. Now although this investigation so generally proposed constitutes the object of the present book, yet we will not extend the discussion to every order of lines, but only to those of the second order; and to proceed with the proposed method we will first establish the general equation of the second degree. Observing that since the algebraical lines described in plane surfaces may be referred to a system [X, Y] of two axes, the variable quantities of the general equation will be only x and y. We may remark, also, that the curve lines described in plane surfaces are termed lines of only one curvature; those described in curve surfaces are called lines of double curvature; and this second kind of curves must necessarily be considered in space.

General formula or equation of the second degree containing two variable quantities.

43. The degree or dimension of an equation is taken from the highest dimension of the variable quantities found in its terms, and this dimension corresponds to the sum of the exponents of the variable quantities. For instance, the term $ax^m y^n$, in which a is a constant quantity, is a term of the dimension $m+n$, and ax^r, ay^s terms of the dimensions r and s; and if $m+n$ is equal to r, the terms $ax^m y^n$, ax^r are of the same dimension. Suppose now $m+n$, r and s different whole numbers, and $m+n$ the greater, an equation of the terms $ax^m y^n$, ax^r, ay^s is termed an equation of the dimension or degree $m+n$; and if r be the greater number, the degree of the equation will be r. From these observations it follows, first, that an equation containing the variable quantities x, y will be of the second degree, if its term or terms of the highest dimension are ax^2, by^2, cxy. Secondly, the equation of the second degree, which, besides the terms of the preceding description, contains terms of the form $d.x$, $e.y$, and other terms independent of the variables x, y will be the most general equation of the second degree; therefore, collecting together all the terms similar to ax^2, all those similar to by^2, and so on, we may give to the general equation of the second degree the following form :

$$Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey = K \; . \; . \; . \; . \; . \; (i_1)$$

In which A, B, 2C, &c., are constant quantities, and the coefficients 2C, 2D, 2E are preferred to C, D, E to facilitate certain investigations. It is plain that this equation represents all the lines of second order.

A simpler form given to the general equation in order to derive the properties of the lines of second order.

DIAMETER.

44. Let (fig. 34) $n''nn'n'''$ represent any line of the second order referred to the rectangular axes X, Y; and let nn' be any chord which we will term $2c$; let us represent by $x_\circ$, $y_\circ$ the coordinates Ap, pm of the middle point m of the chord; and let us call α the angle $n'r$X formed by the same chord with AX. Draw now from n and n' the ordinates nq, $n'q'$, and from n and m, ns, mt parallel to X; the triangles mno, $n'mo'$, in which $n'mo' = mno = \alpha$, and $nm = mn' = c$, will give

$$no = mo' = c \cos \alpha \ , \ mo = n'o' = c \sin \alpha$$

but $no = qp$, $mo' = pq'$; hence, $\mathrm{A}q = \mathrm{A}p - no$, $\mathrm{A}q' = \mathrm{A}p + mo'$. Again, $nq = po = pm - mo$, $n'q' = q'o' + n'o' = pm + n'o'$; therefore,

$$\mathrm{A}q = x_\circ - c \, . \cos \alpha \ , \ qn = y_\circ - c \sin \alpha$$

$$\mathrm{A}q' = x_\circ + c \, . \cos \alpha \ , \ q'n' = y_\circ + c \sin \alpha$$

But Aq, qn, as well as Aq', $q'n'$, must fulfil the equation of the curve, which being any line of the second order is represented by the general equation (i); hence, Aq, qn, and Aq', $q'n'$ may be substituted instead of x, y in the same equation. Before this substitution is performed, observe that

$$\overline{\mathrm{A}q}^2 = x_\circ^2 - 2x_\circ c \, . \cos \alpha + c^2 \cos^2 \alpha \ , \ \overline{qn}^2 = y_\circ^2 - 2y_\circ c \, . \sin \alpha + c^2 \sin^2 \alpha$$

$$\overline{\mathrm{A}q'}^2 = x_\circ^2 + 2x_\circ c \, . \cos \alpha + c^2 \cos^2 \alpha \ , \ \overline{q'n'}^2 = y_\circ^2 + 2y_\circ c \, . \sin \alpha + c^2 \sin^2 \alpha$$

and

$$\mathrm{A}q \, . \, qn = x_\circ y_\circ - cy_\circ \cos \alpha - cx_\circ \sin \alpha + c^2 \sin \alpha \, . \cos \alpha$$

$$\mathrm{A}q' . \, q'n' = x_\circ y_\circ + cy_\circ \cos \alpha + cx_\circ \sin \alpha + c^2 \sin \alpha \, . \cos \alpha$$

Making now the partial substitutions, we will obtain

$$1\begin{cases} Ax^2 = Ax_0^2 - 2Ax_0 \,.\, c \,.\, \cos\alpha + Ac^2 \,.\, \cos^2\alpha \quad \text{or} \\ Ax^2 = Ax_0^2 + 2Ax_0 \,.\, c \,.\, \cos\alpha + Ac^2 \,.\, \cos^2\alpha \end{cases}$$

$$2\begin{cases} By^2 = By_0^2 - 2By_0 \,.\, c \,.\, \sin\alpha + Bc^2 \,.\, \sin^2\alpha \quad \text{or} \\ By^2 = By_0^2 + 2By_0 \,.\, c \,.\, \sin\alpha + Bc^2 \,.\, \sin^2\alpha \end{cases}$$

$$3\begin{cases} 2Cxy = 2Cx_0y_0 - 2C.cy_0\cos\alpha - 2C.cx_0\sin\alpha + 2C.c^2\sin\alpha\cos\alpha \\ \text{or} \\ 2Cxy = 2Cx_0y_0 + 2C.cy_0\cos\alpha + 2C.cx_0\sin\alpha + 2C.c^2\sin\alpha\cos\alpha \end{cases}$$

$$4\begin{cases} 2Dx = 2Dx_0 - 2D \,.\, c \,.\, \cos\alpha \quad \text{or} \\ 2Dx = 2Dx_0 + 2D \,.\, c \,.\, \cos\alpha \end{cases}$$

$$5\begin{cases} 2Ey = 2Ey_0 - 2Ec \,.\, \sin\alpha \quad \text{or} \\ 2Ey = 2Ey_0 + 2Ec \,.\, \sin\alpha \end{cases}$$

Observe here, that the only modification which can be made with regard to the general formula (i_1), is the suppression of some of its terms, and in this case some of the preceding binaries will be wanting; but in every case the sum of the value of Ax^2, By^2, &c., given by the first substitution, as well as that given by the second substitution, is equal to the same K, and consequently, if the first sum be subtracted from the second, the difference must be equal to zero. Before we perform this subtraction, and in order to give a compendious form to the sum equal to K, let us make

$$\left.\begin{array}{l} Ax_0^2 + By_0^2 + 2Cx_0y_0 + 2Dx_0 + 2Ey_0 = R \\ Ax_0\cos\alpha + By_0\sin\alpha + Cy_0\cos\alpha + Cx_0\sin\alpha + D\cos\alpha + E\sin\alpha \\ \text{or the corresponding} \\ (Ax_0 + Cy_0 + D)\cos\alpha + (By_0 + Cx_0 + E)\sin\alpha = Q \\ \text{and} \\ A\cos^2\alpha + B\sin^2\alpha + 2C\sin\alpha \,.\, \cos\alpha = P \end{array}\right\} (i_2)$$

By doing so the sums corresponding to the double substitution will be represented by

$$\left.\begin{array}{l} Pc^2 - 2Qc + R = K \\ Pc^2 + 2Qc + R = K \end{array}\right\} \ldots (i_3)$$

These equations may be represented by a single one

$$Pc^2 \mp 2Qc + R = K$$

which is the fundamental formula; because upon the discussion of that formula depends the whole doctrine of the lines of the second order. And first, let us derive the equation of the *diameter* by subtracting the former (i_3) from the latter. Since from this difference we deduce $4Qc = o$; hence, $Q = o$, and consequently (i_2)

$$Ax_o \cos \alpha + By_o \sin \alpha + Cy_o \cos \alpha + Cx_o \sin \alpha + D \cos \alpha + E \sin \alpha = 0$$

from which

$$y_o [C \cos \alpha + B \sin \alpha] = - [Ax_o + D] \cos \alpha - [Cx_o + E] \sin \alpha$$

and

$$y_o [C + B\ tg\ \alpha] = - [Ax_o + D] - [Cx_o + E]\ tg\ \alpha$$
$$= - x_o [A + C\ tg\ \alpha] - [D + E\ tg\ \alpha]$$

hence

$$y_o = - \frac{A + C\ tg\ \alpha}{C + B\ tg\ \alpha} x_o - \frac{D + E\ tg\ \alpha}{C + B\ tg\ \alpha} \ldots\ldots (i_4)$$

Considering, now, this equation, we may observe that the coefficient of x_o and the last term are quantities dependent on α: but the angle α is the same for every chord parallel to nn'; and since the relation between the co-ordinates x_o, y_o of the middle point of any chord is expressed by the formula (i_4), therefore the relation between the co-ordinates x_o, y_o of every middle point of a system of parallel chords will be given by the equation (i_4) in which the last term and the coefficient of x_o are constant quantities. Hence the relation between the co-ordinates of such a series is a constant one, and (i_4) represents the equation of the line passing through all the middle points of any system of parallel chords. But (i_4) is (10) the equation of a straight line; hence in every line of the second order all the middle points of any system of parallel chords are along a straight line, which line is called *diameter*. And when the diameter is perpendicular to the system of the chords it is called *axis* of the curve.

Scholium I. From the same equation (i_4) we are able to deduce the angle ω which the diameter makes with the positive axis of the abscissas, because the coefficient of x_0 is (10) the tangent of that angle, consequently

$$tg\,\omega = -\frac{A + C\,tg\,\alpha}{C + B\,tg\,\alpha} \,.\,.\,.\,.\,.\, (i_5)$$

from which we may derive ω.

Scholium II. Observe, moreover, that from this last value we may obtain a criterion to know whether the diameter is an axis. Because when two straight lines referred to the axes are perpendicular to each other, the product of the tangents of the angles formed by the two lines with the positive direction of X is equal (10, C. II) to the negative unity. Hence in this case we must have $tg\,\alpha\,.\,tg\,\omega = -1$, that is,

$$1 - \frac{A + C\,tg\,\alpha}{C + B\,tg\,\alpha}\,tg\,\alpha = o \,.\,.\,.\,.\,.\,.\, (i_6)$$

Therefore, when the system of parallel lines makes such an angle α with the axes X as to fulfil the equation (i_6), the diameter corresponding to this system is an axis of the curve.

PROPOSITION.

In every line of the second order there is a certain position of parallel chords in which they are cut perpendicularly in two equal parts by a straight line.

45. The demonstration of this proposition depends on the resolution of the equation (i_6). Because if it is possible to find a real angle α which fulfils that equation, the system of chords inclined with such an angle is a system perpendicularly bisected by the straight line. To find the real value of α let us transform the equation (i_6) into the following

$$C\,tg^2\alpha + A\,tg\,\alpha = C + B\,tg\,\alpha$$

from which

$$tg^2\alpha + \frac{A - B}{C}\,tg\,\alpha = 1$$

which, being an equation of the second degree, resolved according to the known rule, will give

$$tg\,\alpha = -\frac{A - B}{2\,C} \pm \sqrt{\left[\frac{C^2 + \left(\frac{A-B}{2}\right)^2}{C^2}\right]}$$

now the quantity under the radical sign is a positive and real quantity, consequently the second member is a real quantity, and if the angle α corresponding to the positive sign of the radical be termed α_1, and that corresponding to the negative sign α_2, the last equation may be represented by the two following

$$\left.\begin{aligned} tg\,\alpha_1 &= \frac{1}{C}\left[\frac{B - A}{2} + \sqrt{C^2 + \left(\frac{A - B}{2}\right)^2}\,\right] \\ tg\,\alpha_2 &= \frac{1}{C}\left[\frac{B - A}{2} - \sqrt{C^2 + \left(\frac{A - B}{2}\right)^2}\,\right] \end{aligned}\right\}\ (i_7)$$

from which we may derive the real values of $\alpha_1\ \alpha_2$. That is, in every line of the second order there is such a position of parallel chords as to be cut by a straight line or axis not only in two equal parts, but also at right angles.

Corollary. From this conclusion it follows that every line of the second order can be cut in two equal and symmetrical parts ; because when a system of parallel lines is cut in two equal parts by a line perpendicular to the system, every point of the extremities of each parallel from one side of the secant has the corresponding point equally distant on the opposite side ; hence the whole series of points on both sides of the secant shall be equally disposed, and the areas comprised between the same series and the secant will be equal to each other.

Scholium. On account of the use to be made of the values of the tangents lately determined, it is necessary to give to them a new form, which we will obtain by observing, first, that from (i_6) follows

$$C + B\, tg\, \alpha = [A + C\, tg\, \alpha] \frac{\sin \alpha}{\cos \alpha}$$

hence

$$C \cos \alpha + B \sin \alpha = [A \cos \alpha + C \sin \alpha] \frac{\sin \alpha}{\cos \alpha}$$

and consequently

$$\frac{C \cos \alpha + B \sin \alpha}{\sin \alpha} = \frac{A \cos \alpha + C \sin \alpha}{\cos \alpha}$$

which last is the same as the following:

$$\frac{[C \cos \alpha + B \sin \alpha] \sin \alpha}{\sin^2 \alpha} = \frac{[A \cos \alpha + C \sin \alpha] \cos \alpha}{\cos^2 \alpha}$$

But when two ratios $\frac{a}{b}$, $\frac{a'}{b'}$ are equal to each other, the sum $a + a'$ divided by $b + b'$ is equal to the same ratios. Because, supposing $\frac{a}{b} = k$, we will have, also, $\frac{a'}{b'} = k$, and, consequently,

$a = bk \; a' = b'k$, hence $a + a' = (b + b')\,k$ and $\dfrac{a + a'}{b + b'} = k = \dfrac{a}{b} = \dfrac{a'}{b'}$. Making, now, an application of this theorem to the preceding equation, we will obtain

$$\frac{[\mathrm{C} \cos \alpha + \mathrm{B} \sin \alpha] \sin \alpha}{\sin^2 \alpha} = \frac{[\mathrm{A} \cos \alpha + \mathrm{C} \sin \alpha] \cos \alpha}{\cos^2 \alpha} =$$

$$\frac{\mathrm{C} \cos \alpha \sin \alpha + \mathrm{B} \sin^2 \alpha + \mathrm{A} \cos^2 \alpha + \mathrm{C} \sin \alpha \cos \alpha}{\sin^2 \alpha + \cos^2 \alpha}$$

But the numerator of this last member is (44, i_2) compendiously termed P, and the denominator we know from trigonometry to be equal to unity ; hence

$$\frac{\mathrm{C} \cos \alpha + \mathrm{B} \sin \alpha}{\sin \alpha} = \frac{\mathrm{A} \cos \alpha + \mathrm{C} \sin \alpha}{\cos \alpha} = \mathrm{P}$$

The first and third member of which equation may be modified in the following manner:

$$\frac{\mathrm{C} \cos \alpha + \mathrm{B} \sin \alpha}{\sin \alpha} \;\; \frac{\sin \alpha}{\cos \alpha} = \mathrm{P}\, tg\, \alpha$$

hence

$$\left.\begin{aligned} \mathrm{C} + \mathrm{B}\, tg\, \alpha &= \mathrm{P}\, tg\, \alpha \\ \mathrm{A} + \mathrm{C}\, tg\, \alpha &= \mathrm{P} \end{aligned}\right\} \; (i_8)$$

from the first of which

$$tg\, \alpha = \frac{\mathrm{C}}{\mathrm{P} - \mathrm{B}}$$

from the second

$$tg\, \alpha = \frac{\mathrm{P} - \mathrm{A}}{\mathrm{C}}$$

Observe that P (i_2) is a quantity dependent on α, therefore if we term P_1, P_2 the values of P corresponding to α_1 and α_2 (i_7), from the last equations we will derive

$$\left.\begin{aligned} tg\,\alpha_1 &= \frac{C}{P_1 - B} = \frac{P_1 - A}{C} \\ tg\,\alpha_2 &= \frac{C}{P_2 - B} = \frac{P_2 - A}{C} \end{aligned}\right\} (i_9)$$

the new form of the values of $tg\,\alpha_1$, $tg\,\alpha_2$ which was required; but P_1 and P_2 are still to be determined by means of known quantities, which we may obtain by substituting in the last equations the values of $tg\,\alpha_1$ and $tg\,\alpha_2$ given by (i_7); because from this substitution

$$\frac{1}{C}\left[\frac{B-A}{2} + \sqrt{C^2 + \left(\frac{A-B}{2}\right)^2}\right] = \frac{1}{C}\left(P_1 - A\right)$$

$$\frac{1}{C}\left[\frac{B-A}{2} - \sqrt{C^2 + \left(\frac{A-B}{2}\right)^2}\right] = \frac{1}{C}\left(P_2 - A\right)$$

and, observing that $\dfrac{B-A}{2} = \dfrac{B+A}{2} - A$,

$$\left.\begin{aligned} P_1 &= \frac{B+A}{2} + \sqrt{\left[C^2 + \left(\frac{A-B}{2}\right)^2\right]} \\ P_2 &= \frac{B+A}{2} - \sqrt{\left[C^2 + \left(\frac{A-B}{2}\right)^2\right]} \end{aligned}\right\} (i_{10})$$

the second members of which equations contain only the known coefficients of (i_1).

Equations of the axes of the curve.

46. The equation of any axis of the curves of second order is a peculiar case of the general formula or equation of the diameter [44 (i_4)]; hence, to deduce from that formula the equation of any axis, let us remark, that in the peculiar case of the axes the angle α becomes either α_1 or α_2, and from (i_8) we have

$$A + C\,tg\,\alpha_1 = P_1$$

$$A + C\,tg\,\alpha_2 = P_2$$

and
$$C + B\, tg\, \alpha_1 = P_1\, tg\, \alpha_1$$
$$C + B\, tg\, \alpha_2 = P_2\, tg\, \alpha_2$$

therefore, the coefficient of $x_\circ$ in (i_4) will become

$$-\frac{1}{tg\, \alpha_1} \quad \text{or} \quad -\frac{1}{tg\, \alpha_2}$$

and the last term of the same equation

$$-\frac{D + E\, tg\, \alpha_1}{P_1\, tg\, \alpha_1} \quad \text{or} \quad -\frac{D + E\, tg\, \alpha_2}{P_2\, tg\, \alpha_2}$$

and consequently the equations of the axes are

$$\left.\begin{aligned} y_\circ &= -\frac{1}{tg\, \alpha_1}\, x_\circ - \frac{D + E\, tg\, \alpha_1}{P_1\, tg\, \alpha_1} \\ y_\circ &= -\frac{1}{tg\, \alpha_2}\, x_\circ - \frac{D + E\, tg\, \alpha_2}{P_2\, tg\, \alpha_2} \end{aligned}\right\} (i_{11})$$

PROPOSITION.

The curves of the second order cannot all have the same number of axes.

47. Different hypotheses may be made with regard to the values of P_1 and P_2; we may first suppose both values equal to zero. Secondly, we may suppose only one to be equal to zero, or next neither the first nor the second; and in this last supposition it may happen that both values of P_1 and P_2 are equal to each other or not. Let us consider every hypothesis.

Suppose, first, that of the two P_1, P_2, the second is equal to zero, in which case the second (i_{10}) gives

$$\frac{B + A}{2} - \sqrt{\left[C^2 + \left(\frac{A - B}{2}\right)^2\right]} = o$$

or

$$\sqrt{\left[C^2 + \left(\frac{A - B}{2}\right)^2\right]} = \frac{B + A}{2}$$

which value, if substituted in the first (10), will give

$$P_1 = \frac{B+A}{2} + \frac{B+A}{2} = B + A$$

and consequently (i_9)

$$tg\,\alpha_1 = \frac{B}{C} = \frac{C}{A}$$

which equations determine the angle α_1 of inclination of a system of chords bisected at right angles by the axis. The second (i_9) would give

$$tg\,\alpha_2 = -\frac{A}{C} = -\frac{C}{B}$$

but there is only one position in which a system of parallel chords can be bisected by the axis; because, with $P_2 = o$, the second (i_{11}) becomes

$$y_\circ = -\frac{1}{tg\,\alpha_2}\,x_\circ - \infty$$

or

$$y_\circ = \frac{C}{A}\,x_\circ - \infty = \frac{B}{C}\,x_\circ - \infty$$

From which it appears that to every abscissa corresponds an infinite ordinate, and, consequently, the position of the axis corresponding to the last equation cannot be assigned.

Let us now examine the second hypothesis, in which we put $P_1 = P_2$ and not equal to zero; it will be $P_1 - P_2 = o$; and consequently (i_{10})

$$2\sqrt{\left[C^2 + \left(\frac{A-B}{2}\right)^2\right]} = o$$

or

$$\sqrt{\left[C^2 + \left(\frac{A-B}{2}\right)^2\right]} = o$$

But C^2, as well as $\left(\frac{A-B}{2}\right)^2$ are either equal to zero, or necessarily are positive quantities; the second supposition cannot be admitted, because then $\sqrt{\left[C^2 + \left(\frac{A-B}{2}\right)^2\right]}$ would never be equal to zero; hence,

$$C^2 = o \, , \left(\frac{A-B}{2}\right)^2 = o$$

or

$$C = o \, , \frac{A-B}{2} = o$$

consequently, with $P_1 = P_2$ will be also

$$C = o \, , \; A - B = o \, , \; A = B$$

Therefore, in the present supposition, every diameter is an axis; because, the general equation (i_4) of the diameter on account of the last values becomes

$$y_o = -\frac{1}{tg\,\alpha} x_o - \frac{D + E\,tg\,\alpha}{B\,tg\,\alpha}$$

in which α still represents the angle formed by the corresponding system of chords with X, and $-\frac{1}{tg\,\alpha}$ the tangent of the angle formed by the diameter with the same axis X; but $-\frac{1}{tg\,\alpha}$ is (10 C. II) the tangent of the angle formed with X by any perpendicular to the chords of which the preceding equation represents the diameter; therefore, in the present supposition every diameter is an axis.

In the third supposition, in which neither P_1, P_2 are equal to each other, nor equal to zero, the formulas (i_{10}), as well as (i_{11}) will remain unvaried, and the systems of chords inclined to the axis X with the angles α_1, α_2 are alone bisected by the axes.

Let us finally suppose $P_1 = P_2 = o$, we will have at once

$$P_1 + P_2 = o \, , \; P_1 - P_2 = o$$

and consequently (i_{10})

$$\frac{B+A}{2}+\frac{B+A}{2}=o \quad \text{or} \quad B+A=o$$

and

$$2\sqrt{\left[C^2+\left(\frac{A-B}{2}\right)^2\right]}=o$$

or

$$\sqrt{\left[C^2+\left(\frac{A-B}{2}\right)^2\right]}=o$$

From this last equation, as we observed in the second hypothesis, follows $C=o$, $A-B=o$; hence, we have at once

$$A+B=o \ , \ A-B=o$$

and consequently

$$A=o \ , \ B=o$$

Therefore, we cannot suppose P_1 and P_2 at once equal to zero, without supposing at the same time $A=B=C=o$; that is, without supposing the general formula (i_1) converted into the following:

$$2Dx+2Ey=K$$

which being an equation of the first degree cannot represent any line of the second order; therefore, as long as we suppose a line of the second order, P_1 and P_2 can never be at once equal to zero.

Discussion of the curves of the second order with reference to their axes.

PROPOSITION I.

The curve of the second order in which all the diameters are axes is a circle.

49. In the preceding proposition we remarked, that in the supposition of $P_1=P_2$ every diameter is converted in an axis, and in the same supposition we have $C=o$, and $A=B$; consequently, the general formula (i_1) becomes in this case

$$Ax^2+Ay^2+2Dx+2Ey=K$$

from which $$x^2+y^2+\frac{2D}{A}x+\frac{2E}{A}y=\frac{K}{A}$$
and also
$$x^2+y^2+2\frac{D}{A}x+2\frac{E}{A}y+\left(\frac{D}{A}\right)^2+\left(\frac{E}{A}\right)^2=\left(\frac{D}{A}\right)^2+\left(\frac{E}{A}\right)^2+\frac{K}{A}$$

now $$x^2+2\frac{D}{A}x+\left(\frac{D}{A}\right)^2=\left(x+\frac{D}{A}\right)^2$$
$$y^2+2\frac{E}{A}y+\left(\frac{E}{A}\right)^2=\left(y+\frac{E}{A}\right)^2$$
$$\frac{K}{A}+\left(\frac{D}{A}\right)^2+\left(\frac{E}{A}\right)^2=\frac{KA+D^2+E^2}{A^2}$$
hence, the same equation may be changed into
$$\left(x+\frac{D}{A}\right)^2+\left(y+\frac{E}{A}\right)^2=\frac{KA+D^2+E^2}{A^2}\ldots(o)$$
but this is the equation of the circle, because, supposing (fig. 35) the origin A of the rectangular axes at the centre of the circle, the equation of this curve is (11) $x^2+y^2=r^2$; but if the origin of the axes be transposed to A and the axes X′, Y′ are parallel X, Y, the co-ordinates x, y may be given by those of the second through the formulas (8), in which some modification is yet to be made, because in the present case the angle (xx') is equal to zero and $(y'x)=(yx)=90°$; hence $\sin(xx')=0$, $\cos(xx')=1$, $\sin(y'x)=1$, $\cos(y'x)=0$: moreover $x_\circ=\mathrm{A}a$, $y_\circ=a\mathrm{A}'$; consequently the values of the co-ordinates with reference to the first system will be given by the formulas
$$y=a\mathrm{A}'+y',\quad x=\mathrm{A}a+x'$$
which values substituted in $x^2+y^2=r^2$, give
$$(x'+\mathrm{A}a)^2+(y'+\mathrm{A}'a)^2+r^2$$
which, compared with the preceding (o) manifests that if $P_1=P_2$ the line is a circle whose radius is

$$\sqrt{\frac{KA + D^2 + E^2}{A^2}}$$

But since the properties of the circle are commonly treated of in elementary geometry, any further discussion of this curve will be omitted in the present treatise.

PROPOSITION II.

In the curve having only one axis, the diameters are all parallel to this axis.

50. The curve of the second order, which admits only one axis, corresponds (47) to the case of $P_2 = o$ and

$$tg\,\alpha_1 = \frac{B}{C} = \frac{C}{A}$$

Now since the tangent of the angle which any diameter makes with the positive X is given by the co-efficient $-\frac{A + C\,tg\,\alpha}{C + B\,tg\,\alpha}$ of x_o (i_4) and the tangent of the angle which, in the present case, forms the axis with the same X is equal to $-\frac{1}{tg\,\alpha_1}$ (i_{11}); if these two tangents be the same, the corresponding angles also shall be the same, and all the diameters will be parallel to the axis of the curve. Now, from the above mentioned equation we have

$$\frac{A}{C} = \frac{1}{tg\,\alpha_1}$$

and since the coefficient $-\frac{A + C\,tg\,\alpha}{C + B\,tg\,\alpha}$ may be transformed into $-\frac{\frac{A}{C} + tg\,\alpha}{1 + \frac{B}{C}\,tg\,\alpha}$, which, on account of $\frac{B}{C} = \frac{C}{A}$, is equal to the following:

$$-\frac{\frac{A}{C}+tg\,\alpha}{1+\frac{C}{A}\,tg\,\alpha},\ \text{and}\ 1+\frac{C}{A}\,tg\,\alpha=\frac{C}{A}\left(\frac{A}{C}+tg\,\alpha\right)$$

therefore

$$-\frac{A+C\,tg\,\alpha}{C+B\,tg\,\alpha}=\frac{A}{C}=\frac{1}{tg\,\alpha_1}$$

that is to say, the tangent of the angle formed by every diameter with the positive X is equal to that formed by the axis of the curve with the same X. And in the curve of the second order, having only one axis, the diameters are all parallel to this axis.

PROPOSITION III.

The curve having only one axis has no centre.

51. The centre of curves is said to be that point in which the chords drawn in different directions are equally bisected. Now, such a point is impossible in the case of a single axis. Because let (fig. 36) O represent any point within the curve. From this we can conceive OA, a line passing parallel to the axis of the curve, which, of course, will be a diameter. Imagine *ab* to represent the direction of the system of chords bisected by the diameter AO. Any other line *a'b'* will belong to another system of chords bisected by a diameter parallel to AO; but any line parallel to AO never can pass through any point O of AO; consequently the chord represented by *a'b'* shall be bisected in a point different from O; therefore the only one *ab* is divided in two equal parts in O. Now, the same demonstration may be applied to every other point, consequently the curve of the second order, having only one axis, is without centre.

PROPOSITION IV.

When the curve admits two axes, these axes are perpendicular to each other.

52. When the curve of the second order has two axes, the equations of these axes are represented by (i_{11}). Now the line represented by the first of those equations is perpendicular to the second, when (10, C. II) the coefficients $-\frac{1}{tg\,\alpha_1}, -\frac{1}{tg\,\alpha_2}$ multiplied with each other give a product equal to -1. But this is the case at present, because from the equations (i_7) we may readily derive

$$tg\,\alpha_1\, tg\,\alpha_2 = \frac{1}{C^2}\left[\left(\frac{B-A}{2}\right)^2 - \left(C^2 + \left(\frac{A-B}{2}\right)^2\right)\right]$$

hence $$tg\,\alpha_1\;\; tg\,\alpha_2 = -1$$

but $$-\frac{1}{tg\,\alpha_1} \cdot -\frac{1}{tg\,\alpha_2} = \frac{1}{tg\,\alpha_1\;\; tg\,\alpha_2}$$

therefore $$-\frac{1}{tg\,\alpha_1} \cdot -\frac{1}{tg\,\alpha_2} = -1$$

and the two axes are at right angles.

PROPOSITION V.

In the same curves all the diameters pass through the point of intersection of the axes.

53. If the co-ordinates of the point of intersection between the two axes fulfil in every case the equation of the diameter, every diameter shall pass through that point, because the co-ordinates of the points of the line to which an equation belongs can alone fulfil the same equation. Therefore, to demonstrate the present proposition it is sufficient to determine the value of the co-ordinates of the point of intersection, and then observe if, in every

case, these values fulfil the equation of the diameter. And first observe that the co-ordinates of the point of intersection, and only these co-ordinates, belong at once to both of the axes; consequently, if we suppose in (i_{11}) the same co-ordinates $x_\circ$, $y_\circ$, the values of $x_\circ$, $y_\circ$ derived from the formulas (i_{11}) in this supposition shall be those of the co-ordinates of the point of intersection. Subtracting, now, the first (i_{11}) from the second, and reducing their difference to the same denominator, we will obtain

$$-x_\circ(tg\,\alpha_2 - tg\,\alpha_1) = \frac{(D + E\,tg\,\alpha_1)P_2\,tg\,\alpha_2 - (D + E\,tg\,\alpha_2)P_1\,tg\,\alpha_1}{P_1\,P_2}$$

but (52) $tg\,\alpha_1 \cdot tg\,\alpha_2 = -1$; hence

$$-x_\circ(tg\,\alpha_2 - tg\,\alpha_1) = \frac{DP_2\,tg\,\alpha_2 - EP_2 - DP_1\,tg\,\alpha_1 + EP_1}{P_1\,P_2} \ldots (o)$$

again, (i_7)

$$tg\,\alpha_2 - tg\,\alpha_1 = -\frac{2}{C}\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} \text{ and } (i_7), (i_{10})$$

$$P_2\,tg\,\alpha_2 = \frac{1}{C}\left[\frac{B(B-A)}{2} + C^2 - B\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}\right]$$

$$P_1\,tg\,\alpha_1 = \frac{1}{C}\left[\frac{B(B-A)}{2} + C^2 + B\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}\right]$$

hence $DP_2\,tg\,\alpha_2 - DP_1\,tg\,\alpha_1 = -2\,\dfrac{DB}{C}\sqrt{\left(\dfrac{A-B}{2}\right)^2 + C^2}$

moreover (i_{10}) $P_1 - P_2 = 2\sqrt{\left(\dfrac{A-B}{2}\right)^2 + C^2}$; hence

$$EP_1 - EP_2 = 2E\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}$$

finally, from the same (i_{10})

$$P_1\,P_2 = AB - C^2$$

Substituting all these values in the preceding equation (o) we will have

$$x_\circ \cdot \frac{2}{C}\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} = \frac{-2\frac{DB}{C}\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} + 2E\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}}{AB - C^2}$$

hence
$$\frac{x_\circ}{C} = \frac{-\frac{DB}{C} + E}{AB - C^2}$$

and
$$x_\circ = \frac{EC - DB}{AB - C^2}$$

one of the values required ; to have the second let us first reduce the equations (i_{11}) to the form

$$x_\circ = -tg\,\alpha_1\, y_\circ - \frac{D + E\,tg\,\alpha_1}{P_1}$$

$$x_\circ = -tg\,\alpha_2\, y_\circ - \frac{D + E\,tg\,\alpha_2}{P_2}$$

and subtracting the first from the second, we will have

$$y_\circ\,(tg\,\alpha_2 - tg\,\alpha_1) = \frac{D\,(P_2 - P_1) + E\,(P_2\,tg\,\alpha_1 - P_1\,tg\,\alpha_2)}{P_1\,P_2}$$

From the preceding values of $P_1 - P_2$ we derive

$$P_2 - P_1 = -2\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}$$

and from (i_7), (i_{10})

$$P_2\,tg\,\alpha_1 = \frac{1}{C}\left[A\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} - \frac{1}{2}A^2 + \frac{1}{2}AB - C^2\right]$$

$$P_1\,tg\,\alpha_2 = \frac{1}{C}\left[-A\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} - \frac{1}{2}A^2 + \frac{1}{2}AB - C^2\right]$$

hence,

$$P_2\,tg\,\alpha_1 - P_1\,tg\,\alpha_2 = 2\,\frac{A}{C}\sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}$$

So, by a substitution similar to that of the former case, we will find

$$-y_\circ \cdot \frac{2}{C}\sqrt{\left(\frac{A-B}{2}\right)^2+C^2} = \frac{-2D\sqrt{\left(\frac{A-B}{2}\right)^2+C^2}+\frac{2AE}{C}\sqrt{\left(\frac{A-B}{2}\right)^2+C^2}}{AB-C^2}$$

or

$$-\frac{y_\circ}{C} = \frac{-D+\frac{AE}{C}}{AB-C^2}$$

hence,

which with

$$\left.\begin{aligned} y_\circ &= \frac{CD-AE}{AB-C^2} \\ x_\circ &= \frac{CE-BD}{AB-C^2} \end{aligned}\right\} \ldots (i_{12})$$

will determine the point of intersection of the two axes.

Let us now come to the second part, that is, to the substitution of these co-ordinates in the general equation (i_4) of any diameter. It is plain that only the co-ordinates of any point of the line represented by an equation substituted in the same equation, shall make the first member equal to the second. Hence, if from the substitution of the values of $x_\circ$, $y_\circ$ we will derive the first member of (i_4) equal to the second, every diameter will pass through the point of which the co-ordinates are $x_\circ$, $y_\circ$; that is, through the point of intersection of the axes of the curve. But, substituting in the formula (i_4) the values (i_{12}), that formula becomes

$$\frac{CD-AE}{AB-C^2} = -\frac{A+C\,tg\,\alpha}{C+B\,tg\,\alpha}\cdot\frac{CE-BD}{AD-C^2} - \frac{D+E\,tg\,\alpha}{C+B\,tg\,\alpha}$$

or

$$\frac{CD-AE}{AB-C^2} = \frac{-(A+C\,tg\,\alpha)(CE-BD)-(D+E\,tg\,\alpha)(AB-C^2)}{(C+B\,tg\,\alpha)(AB-C^2)}$$

But

$$\begin{aligned} -(A+C\,tg\,\alpha)(CE-BD) &= -ACE+BAD-EC^2\,tg\,\alpha \\ &\qquad +BCD\,tg\,\alpha \\ -(D+E\,tg\,\alpha)(AB-C^2) &= -ABD+DC^2-ABE\,tg\,\alpha \\ &\qquad +EC^2\,tg\,\alpha \end{aligned}$$

hence,

$$\frac{CD - AE}{AB - C^2} = \frac{DC^2 - ACE + BDC\ tg\ \alpha - ABE\ tg\ \alpha}{(C + B\ tg\ \alpha)\ (AB - C^2)}$$

Again,

$$DC^2 - ACE + BDC\ tg\ \alpha - ABE\ tg\ \alpha = CD\,(C + B\ tg\ \alpha) - AE\,(C + B\ tg\ \alpha)$$
$$= (CD - AE)\ (C + B\ tg\ \alpha)$$

therefore, substituting,

$$\frac{CD - AE}{AB - C^2} = \frac{CD - AE}{AB - C^2}$$

that is, the co-ordinates $(i_{1\,2})$ substituted in the formula (i_4), make the first member of that formula equal to the second ; and all the diameters of any curve of the second order having two axes, pass through the point of intersection of these two axes.

PROPOSITION VI.

The point of intersection of the two axes is the centre of the curve.

54. This proposition is a corollary of the preceding ; for imagine any chord passing through the point of intersection of the two axes : this chord, as well as the system of its parallels, is cut in two equal parts by the corresponding diameter, but the diameter passes through the same intersection ; hence, the chord is cut in two equal parts in the point of intersection, but (51) the point in which the different chords are equally bisected is the centre ; therefore, the point of intersection of the axes is the centre of the curve. It is yet to be remarked, that all the straight lines passing through the centre of the curve do not reach the curve, as we shall see in the following discussion.

Discussion of the chords passing through the centre of the curve.

GENERAL FORMULA.

55. The general formula of any chord whatever may be derived from the sum of the equations (i_3), because from this we have

$$2Pc^2 = 2K - 2R$$

and
$$c^2 = \frac{K - R}{P}$$

which is the square of any semichord. Now, in order to have the value of those chords only which pass through the centre of the curve, it is to be observed that among the elements by which the preceding value c^2 is given, K is (44) a constant one, P depends on the angle formed by the chords with the axis of the abscissas, and R depends on the co-ordinates of the middle point of the chord. Hence, to have the expression of the chords bisected at the centre of the curve it is first necessary to determine the peculiar value of R by substituting the co-ordinates (i_{12}) of the centre. The general value of R is (i_2) $Ax_o^2 + By_o^2 + 2Cx_oy_o + 2Dx_o + 2Ey_o$ and substituting,

$$Ax_o^2 = \frac{AC^2E^2 - 2ACEBD + AB^2D^2}{(AB - C^2)^2}$$

$$By_o^2 = \frac{BC^2D^2 - 2ACEBD + BA^2E^2}{(AB - C^2)^2}$$

$$2Cx_oy_o = \frac{2C^3ED - 2C^2D^2B - 2C^2E^2A + 2ACEBD}{(AB - C^2)^2}$$

before making the substitution, in the two last terms, of the value of R, let us take the sum of the three preceding, which is

$$Ax_\circ^2 + By_\circ^2 + 2Cx_\circ y_\circ =$$

$$\frac{-AC^2E^2 - BC^2D^2 + 2C^3ED + BA^2E^2 + AB^2D^2 - 2ACEBD}{(AB - C^2)^2}$$

$$= \frac{-C^2(AE^2 + BD^2 - 2CED) + AB(AE^2 + BD^2 - 2CED)}{(AB - C^2)^2}$$

$$= \frac{(AE^2 + BD^2 - 2CED)(AB - C^2)}{(AB - C^2)^2}$$

$$= \frac{(AE^2 + BD^2 - 2CED)}{(AB - C^2)}$$

Substituting, now, the same values in the remaining terms, we will have

$$2Dx_\circ + 2Ey_\circ = \frac{4CDE - 2BD^2 - 2AE^2}{AB - C^2}$$

hence the whole value of R, which, in the present case (and to distinguish this peculiar form from the most general value) we shall call r, will be given by the formula

$$r = \frac{2CDE - AE^2 - BD^2}{AB - C^2} \; \ldots\ldots \; (i_{13})$$

and the square of this semichord corresponding to the same value of R, and passing through the centre of the curve, will be given by

$$c^2 = \frac{K - r}{P} \; \ldots\ldots \; (i_{11})$$

Different cases with regard to the lines passing through the centre of the curve.

56. We have already observed that the value of P depends upon the angle α which the system of parallel chords makes with the axis of the abscissas; hence, although the curve admits of a

centre, and consequently, in this case, the peculiar values P_1, P_2 of P corresponding to the angles, α_1 and α_2 (45) are different from zero; yet it can happen that the value of P, corresponding to another angle α, different from α_1 and α_2, becomes equal to zero. Hence we may distinguish three different cases with regard to the last formula (i_{14}). Either the value of P is equal to zero, or, if not equal to zero, the ratio $\frac{K - r}{P}$ is positive or negative. The first case will be considered afterwards; let us now examine the two latter. And first, as far as the second member of the formula (i_{14}) is a positive quantity, there are two real corresponding values for c; that is to say, as far as P is different from zero, and such as to effect the ratio $\frac{K - r}{P}$, a positive quantity, the semichord c shall be a real one, and the lines passing through the centre of the curve forming, with the axis of the abscissas, the angles α corresponding to such values of P are chords. On the contrary, if the value of P is such as to make the same ratio $\frac{K - r}{P}$ a negative quantity, no real value of c corresponds to this ratio, and all the lines passing through the centre of the curve, and forming the angles α corresponding to such values of P, are not chords; because the distances c between the centre and the point of the curve met by these lines are imaginary quantities. Therefore, the condition to be fulfilled by the lines passing through the centre of the curve, in order to be chords, is that the angles formed by these lines with the axis of abscissas be such as to make the ratio $\frac{K - r}{P}$ a positive quantity.

Corollary. From these observations follows an important corollary with regard to the axes; because axes are diameters which bisect the system of parallel chords at right angles. But we demonstrated (52) that the axes are perpendicular to each other, consequently, each axis is at once one of the parallel lines cut by the other, and it is that which passes through the centre of the curve. Therefore, when the axes meet the curve, these are at once axes and chords, and since, in the case of the axes, the value P becomes either P_1 or P_2, if the *semi-axes* be termed c_1, c_2 from the general formula (i_{14}), we may derive

$$c_1^2 = \frac{K - r}{P_1}, \quad c_2^2 = \frac{K - r}{P_2} \; \ldots\ldots \; (i_{15})$$

Vice versa from these formulas we may perceive if the axes meet the curve according to the positive or negative value of the second members.

Peculiar case of the infinite chords.

57. Let us now come to the case in which the value P is equal to zero. In this case evidently the ratio $\frac{K - r}{P}$ becomes an infinite quantity, and, consequently, the corresponding square c^2 also, that is, that chord passing through the centre and forming an angle α with the axis of the abscissas, which makes $P = o$. Let us determine the values of these angles. Since (44) (i_2) $P = A \cos^2\alpha + B \sin^2\alpha + 2C \sin\alpha \cos\alpha$, we will have, also,

$$A \cos^2\alpha + B \sin^2\alpha + 2C \sin\alpha \cos\alpha = o$$

Hence those values of α which fulfil this equation are the values corresponding to $P = o$, and if no real value of α may fulfil the same equation, P will never be equal to zero. In order to resolve the last equation, let us transform it into the following:

$$A + B\, tg^2\alpha + 2C\, tg\,\alpha = o$$

from which $$tg^2\alpha + 2\frac{C}{B}\, tg\,\alpha = -\frac{A}{B}$$

and consequently, by resolving the equation according to the known rule,

$$tg\,\alpha = -\frac{C}{B} \pm \frac{1}{B}\sqrt{C^2 - AB} \; \ldots \; (i_{16})$$

from which equation it is plain that two real values of the angle α (α', α'') will fulfil the equation when $C^2 - AB > 0$, and when $C^2 - AB < 0$ no real value of α can fulfil the same equation. And in the former case only two chords passing through the centre of the curve, making the angles α', α'' with the axis of the abscissas will be infinite chords.

Functions of the curves.

Equations of the tangent and normal.

58. Let $AA'CB'R$ (fig. 37) be any line of the second order, and let TT' be a tangent drawn to any point C of that line. Suppose a system of chords $A'B'$, AB, ... to be parallel to the tangent TT', and let $m'm$D be the diameter corresponding to this system. Such a diameter must pass through C, the point of contact of the tangent TT'. Because, imagine the chord, for instance AB, to be moved in a parallel manner towards TT', the points A, B must be always equally distant from m, and always approaching nearer to the same point of the diameter, so as to become only one point when the chord will have reached the extremity C of the same diameter. But then the chord becomes tangent, therefore the diameter passes through C, the point of contact. Hence the co-ordinates x, y of the point C are at once co-ordinates of the curve and of the diameter CD, and they must fulfil the equation of the curve as well as the equation of that diameter. Now, from the general equation (i_4) of the diameter, we have

$$tg\,\alpha = -\frac{Cy_0 + Ax_0 + D}{By_0 + Cx_0 + E}$$

an equation which is fulfilled by the co-ordinates x_0, y_0 of every point of the diameter corresponding to the chords which form an angle α with the axis of abscissas. Let us now substitute in the preceding formula the co-ordinates x, y of the point common to the diameter and to the curve; we will have

$$tg\,\alpha = -\frac{Cy + Ax + D}{By + Cx + E}$$

But the co-ordinates x, y are those of the point of contact of the tangent forming an angle α with the axis of abscissas; and if, to prevent any confusion, we term v the abscissas, and u the ordinates of the same tangent, the equation of that line is (10)

$$u = tg\,\alpha\, v + u_\circ$$

or, since that tangent passes through a point of which x, y are the co-ordinates, (10 C. I), the equation of the same tangent is also

$$u - y = tg\,\alpha\,(v - x)$$

and substituting the value of $tg\,\alpha$ given by the preceding equation

$$u - y = -\frac{Ax + Cy + D}{Cx + By + E}(v - x)$$

which is the required equation of the tangent, and in which x, y are the co-ordinates of the point of contact.

The normal corresponding to the tangent TT′ is (21) the perpendicular CP drawn to the tangent from the point of contact; and if the abscissas of this new line be termed v' and the ordinates u', the equation (10 C. II) will be

$$u' = -\frac{1}{tg\,\alpha}\,v' + u_\circ'$$

But CP passes through the same point C; consequently, the equation of the same line is also given by

$$u' - y = -\frac{1}{tg\,\alpha}(v' - x)$$

and substituting the value of $tg\,\alpha$

$$u' - y = \frac{Cx + By + E}{Ax + Cy + D}(v' - x)$$

Tangent, normal, subtangent, and subnormal of any point.

59. We remarked already (21) that the portion CT of the tangent contained between the point of contact and the axis of the abscissas, is called tangent of the point C; and CP the normal, and the segments TH and HP, in which the hypothenuse of CTP is divided by the perpendicular CH, are termed subtangent and subnormal of the same point C. Now it is plain that, supposing u and u', in the preceding equation, to be equal to zero, the corresponding v and v', in the same equations, must be the abscissas AT and AP of the points of the axis X, met by the tangent and normal; therefore, if those abscissas or distances of the tangent and normal from the origin of the axes be termed Δ and Δ', we will have

$$\Delta = x + \frac{Cx + By + E}{Ax + Cy + D}y$$

$$\Delta' = x - \frac{Ax + Cy + D}{Cx + By + E}y$$

But $TH = AH - AT = x - \Delta$; hence, the absolute value of the subtangent TH:

$$\frac{Cx + By + E}{Ax + Cy + D}y$$

Again, $HP = AP - AH = \Delta' - x$; hence, the absolute value of the subnormal

$$\frac{Ax + Cy + D}{Cx + By + E}y$$

And, since $CT = \sqrt{[CH^2 + TH^2]}$, $CP = \sqrt{[CH^2 + HP^2]}$ we will obtain the values of the tangent CT ($= t$), and of the normal CP ($= n$) by the following equations:

$$t = \sqrt{[y^2 + (x - \Delta)^2]}$$

$$n = \sqrt{[y^2 + (\Delta' - x)^2]}$$

Different species of the lines of the second order.

Modification of the general equation.

60. From the discussion of the axes, and of the chords passing through the centre of the curve, it is plain that there are different species of lines of the second order. But a further investigation of the subject is necessary to separate entirely each kind from the others. To this end let us first begin by modifying the general formula (i) or equation of these lines, which shall be performed—supposing any diameter of the curve to become the axis X of the abscissas, and any straight line parallel to the chords bisected by that diameter the axis Y of the ordinates; because in this supposition the general formula must necessarily be

$$A'x^2 + B'y^2 + 2Dx = K \ldots (o)$$

For let AX (fig. 39) be the diameter, and YY' the line parallel to the system pq , $p'q'$, ... of chords bisected by AX. Now, to every abscissa Ah, Ah', &c., correspond two ordinates hp, hq; $h'p'$, $h'q'$, &c., the one positive, the other negative, and equal to each other. But if in (o) we suppose to be added any other term, we will no longer have two equal ordinates for every abscissa; because, the terms which may be added are (43) $2C'xy$, $2E'y$, that is, $2(C'x + E')y$, and introducing this new term in (o), we will obtain

$$B'y^2 + 2(C'x + E)y = K' - A'x^2 - 2D'x$$

an equation of the second degree, which resolved will give

$$y = -\frac{C'x + E}{B'} \pm \sqrt{\left[\frac{K' - A'x^2 - 2D'x}{B'} + \left(\frac{C'x + E'}{B'}\right)^2\right]}$$

Now, to each value of x correspond two values of y different from each other, and this difference depends on the term $-\frac{C'x + E}{B'}$ that is, it depends on the introduction of the term $2(C'x + E')y$.

Hence, in (o), where C' and E' are equal to zero, to each value of x correspond two equal values of y affected by contrary signs, and given by the formula

$$y = \pm\sqrt{\left[\frac{K' - A'x^2 - 2D'x}{B'}\right]}$$

Different cases contained in the general equation.

61. Let us now consider how many partial forms may be given to the preceding general equation. There may happen five cases; we may suppose either the general formula

$$A'x^2 + B'y^2 + 2D'x = K'$$

as it is modified in the preceding number, or the same formula wanting some terms, first, the term K' so as to have

$$A'x^2 + B'y^2 + 2D'x = o$$

Secondly, the last term of the first member; and, consequently,

$$A'x^2 + B'y^2 = K'$$

Thirdly, the first term, so that

$$B'y^2 + 2D'x = K'$$

And finally, the first term and the term K, so that we have

$$B'y^2 + 2D'x = o$$

These are the only possible cases. We cannot suppose, for instance, the general equation without the first and second terms at once; because, in this case, the equation would become an equation of the first degree, and it would preserve, moreover, only one co-ordinate. The same would happen supposing the general equation deprived of the first and third terms; that is, it would preserve only one co-ordinate; and the same is to be said if the formula is deprived of the second and third terms, and of the second alone, or of the second and last, K'. We can no more suppose the same general equation wanting at once the last term of the first member, and the term K′ of the second ; because, in that case, the equation might be converted into the following :

$$x = \pm \sqrt{\frac{B'}{A'}}$$

which being an equation of the first degree cannot represent any line of the second order ; therefore, only the five cases first considered are those which can be admitted. But, as we shall presently see, the curves corresponding to the first three equations may be represented by a single equation ; and the curves corresponding to the two remaining equations may be likewise represented by a single equation. Therefore, the five cases may be reduced to two only.

The lines corresponding to the preceding cases represented by two principal formulas.

62. Suppose the curve $p'pqq'$ (fig. 39) to correspond to the general equation $A'x^2 + B'y^2 + 2D'x = K'$, and let AX be the diameter taken for the axis of the abscissas, and AY the axis of the ordinates parallel to the chords pq, $p'q'$, &c., bisected by AX ; if the origin A of the axes be transposed to A′, so that the length AA′ be equal to $\frac{D'}{A'}$, the ordinates ph, $p'h'$, &c., corresponding to the abscissas Ah, Ah' in the first supposition shall correspond to the abscissas $A'A + Ah$, $A'A + Ah'$, &c., in the second supposition ; and if the new abscissas $A'h$, $A'h'$, &c., be represented by x', we will have

$$x' = \frac{D'}{A'} + x \text{ or } x = x' - \frac{D'}{A'}$$

and introducing this value of x in the general equation, it will become

$$A'\left(x' - \frac{D'}{A'}\right)^2 + B'y^2 + 2D'\left(x' - \frac{D'}{A'}\right) = K'$$

or $$A'x'^2 + By^2 = K' - \frac{D'^2}{A'}$$

in which the second member is a constant quantity; therefore, this formula corresponds exactly to the third case considered in the preceding number; and consequently, all the lines which may be represented by the general formula of the first case, may be also represented by that of the third, and the difference of the form of the equations is to be referred only to the different origin of the axis. In the same manner we may demonstrate, that all the lines corresponding to the second equation of the preceding number may be represented by the same third formula; therefore, all the lines of the second order, corresponding to the first three cases, may be represented by only one formula, that corresponding to the third case. Let us come to the last two cases, and suppose again the line $p'pVqq'$ corresponding to the equation $B'y^2 + 2D'x = K'$; if the origin A be transferred in A'', so as to have $AA'' = \frac{K'}{2D'}$, and if the abscissas taken from this new origin be termed x', we will obtain the value of every abscissa x taken from the origin A, equal to $x' + \frac{K'}{2D'}$. Substituting now this value in the preceding equation, we shall obtain

$$B'y^2 + 2D'\left(x' + \frac{K'}{2D'}\right) = K'$$

or $$B'y^2 + 2D'x' = o$$

Which equation exactly corresponds to that of the fifth case. Hence all the lines represented by the equation of the fourth case may also be represented by that corresponding to the fifth, and the difference of the forms depends on the different origin of the axes. Therefore all the lines corresponding to the different cases of the preceding number are represented by two principal formulas,

$$A'x^2 + B'y^2 = K'',$$

$$B'y^2 + 2D'x = o$$

Further modification of the two principal formulas.

63. From the conclusion lately deduced it is plain that the discussion of every possible case is limited to that of the two principal formulas, and from such a discussion we may be able to derive the different species, as well as the properties of each species of the lines of the second order. However, before entering upon this subject, it is necessary to modify the same formulas, not only to facilitate our investigations, but also to give to the equations the form usually adopted. To this end let us first transform the principal formulas into the following:

$$\frac{A'}{K''}x^2 + \frac{B'}{K''}y^2 = 1$$

$$y^2 = -\frac{2D'}{B'}x$$

and then let us determine the values of some positive quantities a, b, p, so as to have

$$\pm a^2 = \frac{K''}{A'} \quad \text{or} \quad \pm\frac{1}{a^2} = \frac{A'}{K''}$$

$$\pm b^2 = \frac{K''}{B'} \quad \text{or} \quad \pm\frac{1}{b^2} = \frac{B'}{K''}$$

$$\pm p = \frac{D'}{B'}$$

And since this determination is always possible, we may consequently substitute these values in the preceding equations, which shall become

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

$$y^2 = \pm 2px$$

Which equations, although apparently two, are really six, because in the first either the signs may be supposed both positive or both negative, or the first positive and the second negative, or, next, the first negative and the second positive. Likewise the second formula corresponds to two on account of the double sign. But some of these suppositions are evidently to be excluded, because we never can have either $-\frac{x^2}{a^2}-\frac{y^2}{b^2}=+1$, or $y^2=-2px$ when $x>o$, and if $x<o$, then $y^2=-2p\,.-x$ is the same as $y^2=+2p\,.\,x$. Hence the only equations to be discussed are

$$\left.\begin{array}{c} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\[2ex] \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \\[2ex] \frac{y^2}{b^2}-\frac{x^2}{a^2}=1 \\[2ex] y^2=2px \end{array}\right\}(g)$$

With regard to the corresponding curves, that represented by the last equation has (14) two indefinite branches, and it is called *parabola*. That represented by the first is circumscribed within certain limits (15) and is called *ellipse*. The curves corresponding to the second and third equations are evidently of the same kind; and if, for instance, in the third the axis of the abscissas be changed in that of the ordinates, and vice versa, the third equation shall take the very same form of the second. Hence we may consider the first of the two equations alone, the corresponding curve of which is (16) a curve having four indefinite branches, and is called *hyperbola*.

Observe that since the equations (g) are derived from that general formula in which a diameter is supposed to be taken for the axis of the abscissas, and since any axis of the curve is nothing but a peculiar diameter, so we may suppose, and generally we shall suppose, an axis of the curve to be taken for axis of the abscissas, and consequently the co-ordinates to be rectangular.

Criterion applied to the curves represented by the last formulas to detect the number of the axes.

64. Before coming to the discussion of each curve in particular, it is convenient to make a comparison between the last formulas (g) and the most general (i), in order to apply a criterion, deduced from the discussion of that equation. And comparing, first, the equation $y^2 - 2px = o$ of the parabola with the general $Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey = K$, we derive $A = o$, $B = 1$, $C = o$, $D = -p$, $E = K = o$; consequently [45 (10)]

$$P_1 = 1, \quad P_2 = 0$$

But when one of the values P_1, P_2 is equal to zero, the curve (47) admits of only one axis; therefore in the parabola only one axis is to be found, and since in the case of a single axis the diameters of the curve are (50) all parallel to that axis; therefore, supposing (fig. 40) AC to be the axis of the parabola tVt'; the lines ac, $a'c'$, &c., parallel to AC, shall be all diameters of the curve; the chords, moreover, bisected by the axis constitute the only system perpendicularly bisected.

Let us, secondly, compare the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of the ellipse with the same (i) we shall find

$$A = \frac{1}{a^2}, B = \frac{1}{b^2}, C = D = E = 0, K = 1$$

hence $\quad A + B = \frac{b^2 + a^2}{a^2 b^2}, \; A - B = \frac{b^2 - a^2}{a^2 b^2}$

and [45 (i_{10})]

$$P_1 = \frac{1}{a^2}, \; P_2 = \frac{1}{b^2}$$

Therefore the ellipse admits of two axes (fig. 41) AC, BD which (52) are perpendicular to each other, and the point O of intersection is (54) the centre of the curve, and all the diameters *db*, *d'b'*, &c., must pass (53) through that point.

The comparison between the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of the hyperbola, and the general equation (i) gives

$$A = \frac{1}{a^2}, \quad B = -\frac{1}{b^2}, C = D = E = 0, K = 1$$

hence $$A + B = \frac{b^2 - a^2}{a^2 b^2}, \quad A - B = \frac{b^2 + a^2}{a^2 b^2}$$

and [45 (i_{10})]

$$P_1 = \frac{1}{a^2}, \; P_2 = -\frac{1}{b^2}$$

Therefore the hyperbola also admits of two axes AC, BD (fig. 42) perpendicular to each other, and all the diameters *bd*, *b'd'*, &c., will pass through O, the centre of the curve. The other properties derived from the general formula will be more appropriately examined in the discussion of the different curves.

PARABOLA.

Discussion of each peculiar equation; and first of that of the parabola.

DEFINITIONS.

65. In the equation of the parabola

$$y^2 = 2px$$

the constant quantity $2p$ is called *parameter;* and since the equation may be resolved into the proportion $x : y :: y : 2p$, therefore the parameter is the third proportional term to any abscissa and the corresponding ordinate. The point V of the parabola met by the axis AC is called *vertex*; the point F (fig. 43) of the same axis within the curve, and so far from the vertex as to have $VF = \frac{1}{2}p$, is termed *focus*. The straight line QB perpendicular to the axis and passing through a point E distant from V as far as F is the *directrix*. Observe, that since the origin of the co-ordinates (14) is to be taken in V, the abscissa corresponding to the double ordinate drawn from the focus must be equal to $VF = \frac{1}{2}p$; but then from the equation of the curve we have $\overline{Fs}^2 = 2p \cdot \frac{1}{2}p = p^2$, and, consequently, $Fs = p$ and $ss' = 2p$; that is, the double ordinate passing through the focus is equal to the parameter.

PROPOSITION I.

Every point of the parabola is equally distant from the focus and the directrix.

66. Let M be any point of the curve; the distance of M from QB is given by Mt $(= \rho')$ perpendicular to QB, FM $(= \rho)$ is the distance of the same point from the focus. Now, supposing Mn perpendicular to VC, we will have $Vn = x$, $Mn = y$; consequently $\rho' = tM = En = EV + Vn = \frac{1}{2}p + x$, but $\rho^2 = \overline{MF}^2 = \overline{Mn}^2 + \overline{Fn}^2 = y^2 + (x - \frac{1}{2}p)^2 = y^2 + x^2 - px + \frac{p^2}{4}$;

and since $y^2 = 2px$, $\rho^2 = x^2 + px + \frac{p^2}{4} = (x + \frac{1}{2}p)^2$, hence $\rho = x + \frac{1}{2}p$, and

$$\rho' = \rho$$

That is, every point of the parabola is equally distant from the focus and the directrix.

Corollary. From this property of the parabola it follows that if we suppose a straight line BD perpendicular to AC (fig. 44), every curve having all its points equally distant from the line BD and from any point F of AC is a parabola. Whence we deduce the manner of describing this line; because suppose (fig. 45) AB the edge of a rule, R to represent the directrix, and let CD be perpendicular to AB, that is the edge of the square S perpendicular to that of the rule. Suppose, again, on the same plane a thread F*t*D equal in length to CD, having one end fixed in the extremity D of the square, and the other end in some point F of the plane, and the part *t*D of the thread to be kept close to the edge CD by the stile *tq* while the square slides along CB. *t*F will evidently be always equal to C*t*; therefore the path described by *t*, the extremity of the stile, is a parabola.

PROPOSITION II.

The subtangent corresponding to any point of the curve is double of the abscissa of the same point.

67. The general formula by which any subtangent is represented is (59) $\frac{Cx + By + E}{Ax + Cy + D} y$, but in the present case (64) $C = A = E = 0$, $B = 1$, $D = -p$; hence the subtangent of any point (x, y) is equal to $-\frac{y^2}{p}$; and since $y^2 = 2px$, the value of the same subtangent will be given by $-2x$; and having no regard to the sign, the subtangent of any point of the parabola is equal to the double abscissa of the same point.

PROPOSITION III.

The subnormal of any point of the parabola preserves a constant value equal to the half of the parameter.

68. The general formula of the subnormal is (59)

$$-\frac{Ax + Cy + D}{Cx + By + E} y$$

consequently, in the discussion of the present curve, the subnormal of the point (x, y) will be $\frac{py}{y}=p$, but $2p$ is the parameter; hence, the subnormal of any point of the parabola is a constant quantity, and equal to the half of the parameter.

PROPOSITION IV.

The tangent of any point of the parabola is the mean proportional between the distance of that point from the focus, and the quadruple of the abscissa of the same point.

69. The square of any tangent MT (fig. 46) is equal to

$$\overline{Mn}^2 + \overline{Tn}^2$$

the square of the ordinate of the point of contact plus the square of the corresponding subtangent. Now (67) the subtangent is double the corresponding abscissa; hence,

$$\overline{MT}^2 = y^2 + 4x^2$$

and, since $y^2 = 2px$

$$\overline{MT}^2 = 2px + 4x^2 = 4x\,(\tfrac{1}{2}p + x)$$

but (66) $\frac{1}{2}p + x = \rho$, the distance of the point M from the focus; hence,

$$\overline{MT}^2 = 4x \,.\, \rho$$

or

$$4x : MT : : MT : \rho$$

Scholium. The angle which any tangent MT makes with the axis AX, may be derived from the general formula (58)

$$tg\,\alpha = -\frac{Cy + Ax + D}{By + Cx + E}$$

and, since $C = A = E = 0$, $D = -p$, $B = 1$, we will have

$$tg\ \alpha \text{ or } tg\ \mathrm{MTX} = \frac{p}{y}$$

namely, the trigonometrical tangent of the angle which the tangent of the curve forms with the axis X, is equal to the half of the parameter divided by the ordinate of the point of contact.

PROPOSITION V.

The normal corresponding to any point of the parabola is the mean proportional between the parameter and the distance of that point from the focus.

70. The square of any normal MR is equal to $\overline{\mathrm{M}n}^2 + \overline{n\mathrm{R}}^2$, that is, to the square of the ordinate of the point M of contact plus the square of the subnormal; but $\overline{\mathrm{M}n}^2$, or $y^2 = 2px$, and (68) $\overline{n\mathrm{R}}^2 = p^2$; hence,

$$\overline{\mathrm{MR}}^2 = 2px + p^2 = 2p\,(x + \tfrac{1}{2}p)$$

and since (66) $\frac{1}{2}p + x = \rho$; consequently,

$$\overline{\mathrm{MR}}^2 = 2p\,.\,\rho$$

or

$$2p : \mathrm{MR} :: \mathrm{MR} : \rho$$

PROPOSITION VI.

The focus of the parabola is equidistant from the tangent and the normal, the distances being taken on the axis.

71. FT and FR are the distances of the focus from the tangent and normal reckoned on the axis AX. Now

$$\mathrm{TF} = \mathrm{AT} + \mathrm{AF}$$

but $\mathrm{AT} = \mathrm{T}n - \mathrm{A}n$, and $\mathrm{T}n$ is the subtangent, $\mathrm{A}n$ the abscissa of M; that is, (67) $\mathrm{T}n = 2x$, and $\mathrm{A}n = x$. Again, AF is equal (65) to $\frac{1}{2}p$; hence,

$$TF = x + \tfrac{1}{2}p = \rho$$

Moreover,

$$FR = AR - AF$$

but $AR = An + nR$, and An is the abscissa, nR the subnormal of M; hence, $An = x$, and (68) $= nR = p$. Again, $AF = \frac{1}{2}p$; therefore,

$$FR = p + x - \tfrac{1}{2}p = x + \tfrac{1}{2}p = \rho$$

and consequently

$$TF = FR$$

The distance of the focus from the tangent equal to the distance of the same focus from the normal, and both equal to the radius ρ or FM drawn from the focus to the point M, to which the tangent and normal belong, so that the triangle RFM, as well as MFT, are isosceles; it follows, also, that

$$TR = 2TF = 2\rho$$

PROPOSITION VII.

The normal of any point of the parabola is double the perpendicular drawn from the focus to the corresponding tangent.

72. Let FN be (fig. 47) the perpendicular drawn to the tangent Tt. This perpendicular shall be parallel to the normal MR, and consequently the triangles TFN TRM similar to each other; hence from elementary geometry

$$\frac{TR}{TF} = \frac{RM}{FN}$$

and since (71) $TR = 2TF$, hence $\frac{RM}{FN} = 2$, and

$$RM = 2 \, . \, FN$$

the normal is twice the perpendicular FN. Now, since (70)

$\overline{RM}^2 = 2p \cdot \rho$, and consequently $RM = \frac{2p \cdot \rho}{RM}$, the value of the normal will be also given by the formula

$$RM = \frac{p \cdot \rho}{FN}$$

PROPOSITION VIII.

Two straight lines drawn from any point of the parabola, the one to the focus the other parallel to the axis, form the same angle with the tangent of that point.

73. Let the radius FM be drawn from any point M of the curve, and from the same point let Mi be drawn parallel to the axis. The angle iMt which Mi makes with the tangent is equal to the angle which the same tangent makes with AX. But (71) in the triangle TFM the side TF is equal to MF; consequently the angle FTM = FMT; but FTM = iMt; hence

$$iMt = FMT.$$

Equation of the parabola with reference to the polar co-ordinates.

74. Let β be any angle AFM (fig. 45) which the radius ρ or FM makes with the axis AX, and let An, nM be the co-ordinates x, y of the point M of the curve to which the radius is driven. In every supposition we will have

$$x = \tfrac{1}{2}p - \rho \cos \beta$$

The angle β may be equal, less, or greater than a right angle; in the first case $\cos \beta = o$, and consequently the preceding equation becomes $x = \frac{1}{2}p$, the value of the abscissa corresponding to m, to which, in this case, is drawn the radius. In this second case the perpendicular drawn from m' to the axis must fall between A and F, and consequently An' or $x = AF - Fn'$; now $AF = \frac{1}{2}p$ and $Fn' = + \rho \cos \beta$; hence $x = \frac{1}{2}p - \rho \cos \beta$. In the third case the perpendicular drawn from M must fall at some point n beyond F; hence An or $x = AF + Fn$; but again, $AF = \frac{1}{2}p$ and $Fn = -\rho \cos \beta$; therefore $x = \frac{1}{2}p - \rho \cos \beta$. Now the value of the radius ρ in every case (66) is given by $x + \frac{1}{2}p$, x being the abscissa of that point to which the radius is drawn; hence substituting, instead of x, the value given by the preceding equation, we will have $\rho = \frac{1}{2}p - \rho \cos \beta + \frac{1}{2}p$; that is

$$\rho = \frac{p}{1 + \cos \beta}$$

which equation, containing only the variable quantities ρ and β, is the required equation.

Equation of the parabola with reference to any diameter considered as the axis of the abscissas, and the tangent parallel to the system of chords bisected by that diameter, considered as the axis of the ordinates.

75. Let Mi be the diameter on which are to be taken the abscissas. Mt the tangent parallel (58) to the system of chords bisected by Mi on which are to be taken the ordinates. Let the co-ordinates of the curve with reference to the new system be termed x', y', since (6) the angle $(xx') = o$ and $(y'x) = \alpha$, that is, the inclination of the tangent on the axis X, we will have (7 C. I)

$$y = y_{\circ} + y' \sin \alpha$$

$$x = x_{\circ} + x' + y' \cos \alpha$$

in which $x_{\circ}$, $y_{\circ}$ are the co-ordinates of the new origin M with reference to the former system, and consequently we will have $y_{\circ}^2 = 2px_{\circ}$, and $x_{\circ} = \frac{y_{\circ}^2}{2p}$. Again, (69 *Sch.*) $tg\ \alpha = \frac{p}{y_{\circ}}$, from which $y_{\circ} = \frac{p}{tg\ \alpha} = \frac{p \cos \alpha}{\sin \alpha}$; hence

$$x_{\circ} = \frac{p^2 \cos^2 \alpha}{2p \cdot \sin^2 \alpha} = \frac{p \cos^2 \alpha}{2 \sin^2 \alpha}$$

which value, with that of $y_\circ$, substituted in the preceding formulas, they shall become

$$y = \frac{p \cos \alpha}{\sin \alpha} + y' \sin \alpha$$

$$x = \frac{p \cos^2 \alpha}{2 \sin^2 \alpha} + x' + y' \cos \alpha$$

Let us now resume the equation $y^2 = 2px$, and substituting in it the last determined values of x and y, we will have

$$\left(\frac{p \cos \alpha}{\sin \alpha} + y' \sin \alpha\right)^2 = 2p \cdot \frac{p \cos^2 \alpha}{2 \sin^2 \alpha} + 2p \cdot x' + 2p \cdot y' \cos \alpha$$

hence

$$\frac{p^2 \cos^2 \alpha}{\sin^2 \alpha} + 2py' \cos \alpha + y'^2 \sin^2 \alpha = \frac{p^2 \cos^2 \alpha}{\sin^2 \alpha} + 2px' + 2py' \cos \alpha$$

and consequently $y'^2 \sin^2 \alpha = 2px'$, from which

$$y'^2 = \frac{2p}{\sin^2 \alpha} x'$$

the required equation.

ELLIPSE.

PROPOSITION I.

The axes of the ellipse meet the curve.

76. We observed (56 Cor.) that when the second members of the equations $c_1{}^2 = \frac{K - r}{P_1}$, $c_2{}^2 = \frac{K - r}{P_2}$ are positive quantities the axis $2c_1$, $2c_2$, having then a real and determined value, meet the curve. Now from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that here taken in consideration, and compared with the most general (i) we have (64) $P_1 = \frac{1}{a^2}$, $P_2 = \frac{1}{b^2}$, $K = 1$ and $A = \frac{1}{a^2}$,

$B = \frac{1}{b^2}$, $C = D = E = O$; hence (55 (i_{13})) $r = o$; therefore

$$c_1{}^2 = a^2 \ , \ c_2{}^2 = b^2$$

that is, the axes of the ellipse meet the curve at a distance from the centre, the one equal to b, the other equal to c.

Supposing $a > b$ the axis $2a$ is termed *transverse*, and the axis $2b$ *conjugate*. Let us here remark that, supposing $a = b$ the equation of the ellipse is converted (11) in that of the circle, and of course the circle may be regarded as a peculiar case of the ellipse.

PROPOSITION II.

The equation of the Ellipse considered in the present discussion necessarily supposes the origin of the co-ordinates in the centre.

77. Supposing the centre or point of intersection between the two axes to be different from the point of origin of the co-ordinates, the co-ordinates (53) of that point will be given by the formulas (i_{12})

$$y_\circ = \frac{CD - AE}{AB - C^2} \ , \ x_\circ = \frac{CE - BD}{AB - C^2} \ ; \text{ but } C = D = E = O$$

hence $$x_\circ = o \ , \ y_\circ = o$$

but the point of which both co-ordinates are equal to o is the origin of the axes; consequently, where is the point of intersection between the axes of the curve, there is the origin of the co-ordinates.

PROPOSITION III.

If any chord passing through the centre of the ellipse is the diameter of another chord passing through the same centre, the latter is by turns diameter of the former.

78. Let (fig. 49) aa' be any line passing through C the centre of the ellipse; since this line meets the curve with both extre-

mities, it may be considered like a chord; and since it passes through the centre it may be considered (53) as the diameter bisecting the system of chords parallel to the tangent (58) in a or a'. Hence, if from the centre we draw a line bb' parallel to that tangent, this line shall be one of the chords of the system; but it is a diameter also, and such a diameter as to bisect the system of chords parallel to aa'. Because, supposing α to be the angle formed by bb' with the positive X, the angle ω formed by aa' with the same X, that of the diameter corresponding to the system of chords parallel to bb', and consequently given by the formula (i_5), which in the present case becomes $tg\,\omega = -\frac{b^2}{a^2\,tg\,\alpha}$. Now, if bb' is the diameter of the system of chords parallel to aa', we must have $tg\,\alpha = -\frac{b^2}{a^2\,tg\,\omega}$; which will be easily demonstrated; because, let us substitute instead of $tg\,\omega$ the value given by the preceding formula, we will have $tg\,\alpha = tg\,\alpha$. Such lines considered together are termed *conjugate diameters.*

PROPOSITION IV.

The sum of the squares of the conjugate diameters is constantly equal to the sum of the squares of the axes.

79. Every diameter, as we observed before, may be regarded as a chord passing through the centre of the figure; and the square of any semichord passing through the centre is (55) $\frac{K-r}{P}$, in which P depends on the angle which the chord makes with the positive X. Now, since in the present investigation $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = o$, so the value of P [44 (i_2)] will become $\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2}$. Hence, let aa', bb' be any two conjugate dia-

meters, and the angle which bb' forms with CX be termed α, that which aa' makes with the same axis ω, between which angles there is the relation remarked before. Now, with regard to the former bb', the value of P is that mentioned above, and which may be modified as follows:

$$P = \frac{b^2 \cos^2\alpha + a^2 \sin^2\alpha}{a^2 b^2}$$

with regard to the latter aa', we will have

$$P = \frac{b^2 \cos^2\omega + a^2 \sin^2\omega}{a^2 b^2}$$

Observe, now, that (64, 76) $K = 1$, $r = 0$; therefore, substituting the preceding values of P in the ratio $\frac{K - r}{P}$, we shall obtain the values of the semi-diameters by means of the following formulas:

$$\left.\begin{aligned} \overline{Cb}^2 &= \frac{a^2 b^2}{b^2 \cos^2\alpha + a^2 \sin^2\alpha} \\ \overline{Ca}^2 &= \frac{a^2 b^2}{b^2 \cos^2\omega + a^2 \sin^2\omega} \end{aligned}\right\} \ldots (o)$$

but we observed in the preceding number that $tg\,\omega = -\frac{b^2}{a^2\,tg\,\alpha}$ from which follows

$$tg^2\omega = \frac{b^4}{a^4\,tg^2\alpha}$$

or

$$\frac{\sin^2\omega}{\cos^2\omega} = \frac{b^4 \cos^2\alpha}{a^4 \sin^2\alpha}$$

and

$$\frac{\cos^2\omega}{\sin^2\omega} = \frac{a^4 \sin^2\alpha}{b^4 \cos^2\alpha}$$

by adding unity to each member of both equations, and reducing to the same denominator. If we observe that $\sin^2\omega + \cos^2\omega = 1$, we will obtain

$$\frac{1}{\cos^2\omega} = \frac{b^4 \cos^2\alpha + a^4 \sin^2\alpha}{a^4 \sin^2\alpha}$$

$$\frac{1}{\sin^2\omega} = \frac{a^4 \sin^2\alpha + b^4 \cos^2\alpha}{b^4 \cos^2\alpha}$$

hence,

$$b^2 \cos^2\omega = \frac{b^2 a^4 \sin^2\alpha}{b^4 \cos^2\alpha + a^4 \sin^2\alpha}$$

$$a^2 \sin^2\omega = \frac{a^2 b^4 \cos^2\alpha}{a^4 \sin^2\alpha + b^4 \cos^2\alpha}$$

and consequently,

$$b^2 \cos^2\omega + a^2 \sin^2\omega = \frac{a^2 b^2 (a^2 \sin^2\alpha + b^2 \cos^2\alpha}{a^4 \sin^2\alpha + b^4 \sin^2\alpha}$$

which value substituted in the second (o) shall give

$$\overline{\mathrm{C}a}^2 = \frac{a^4 \sin^2\alpha + b^4 \cos^2\alpha}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}$$

or since

$$\begin{aligned} a^4 \sin^2\alpha + b^4 \cos^2\alpha &= a^2 b^2 + a^4 \sin^2\alpha + b^4 \cos^2\alpha - a^2 b^2 \\ &= a^2 b^2(\sin^2\alpha + \cos^2\alpha) + a^4 \sin^2\alpha + b^4 \cos^2\alpha - a^2 b^2 \\ &= a^2 \sin^2\alpha\,(b^2 + a^2) + b^2 \cos^2\alpha\,(b^2 + a^2) - a^2 b^2 \\ &= (b^2 + a^2)\,[a^2 \sin^2\alpha + b^2 \cos^2\alpha] - a^2 b^2 \end{aligned}$$

and consequently,

$$\frac{a^4 \sin^2\alpha + b^4 \cos^2\alpha}{a^2 \sin^2\alpha + b^2 \cos^2\alpha} = b^2 + a^2 - \frac{a^2 b^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}$$

we have also

$$\overline{\mathrm{C}a}^2 = b^2 + a^2 - \frac{a^2 b^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}$$

but the last term is the value of $\overline{\mathrm{C}b}^2$ given by the first (o); hence,

$$\overline{Ca}^2 + \overline{Cb}^2 = a^2 + b^2$$

and

$$4.\overline{Ca}^2 + 4.\overline{Cb}^2 = 4.a^2 + 4.b^2$$

that is, the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.

PROPOSITION V.

The parallelogram on the conjugates is equal to the rectangle on the axes.

80. If from the extremities a, a' , b, b' (fig. 50) of the diameters we draw the tangents mn, $m'n'$, nn', mm', the first two shall be parallel to the diameter bb', the other to aa'; hence, $mn = m'n' = bb'$, and $nn' = mm' = aa'$, and the parallelogram $mnm'n'$ is the parallelogram on the diameters aa' , bb'. In the same way, if rs, $r's'$, rr', ss' are tangents drawn from the extremities of the axes, the rectangle $rr's's$ is the rectangle on the axes. Now, on account of the parallel lines the diameters aa' , bb' divide in four equal parts the parallelogram $mm'n'n$, and the axes AA' , BB' divide in four equal parts the rectangle $rss'r'$, so that the areas may be given by the equations

$$\left.\begin{aligned} rr's's &= 4\ \mathrm{ACB}'r' = 4\ a\,.\,b \\ mm'n'n &= 4\ ma\,\mathrm{C}\,b' = 4\ a\,\mathrm{C}\,.\,b't \end{aligned}\right\}\ \ldots\ (a)$$

supposing $b't$ perpendicular to aa'. Now, $b't = b'\mathrm{C} \sin b'\mathrm{C}a'$; and preserving still the same denominations of ω and α of the angles $a'\mathrm{CA}'$, $b'\mathrm{CA}'$, we will have $\sin b'\mathrm{C}a' = \sin(\alpha - \omega)$; hence,

$$b't = b'\mathrm{C}\,.\,\sin(\alpha - \omega)\ \ldots\ (a_1)$$

but from trigonometry

$$\begin{aligned} \sin(\alpha - \omega) &= \sin\alpha\cos\omega - \cos\alpha\sin\omega \\ &= \cos\omega\,(\sin\alpha - \cos\alpha\ tg\ \omega) \end{aligned}$$

hence,

$$\sin^2(\alpha - \omega) = \cos^2\omega\,(\sin\alpha - \cos\alpha\, tg\,\omega)^2$$

but (78) $tg\,\omega = -\dfrac{b^2}{a^2\, tg\,\alpha}$; consequently,

$$\sin^2(\alpha - \omega) = \frac{\cos^2\omega}{a^4 \sin^2\alpha}\left[a^2 \sin^2\alpha + b^2 \cos^2\alpha\right]^2$$

substituting now instead of $\cos^2\omega$ the corresponding value given in the preceding number, we will derive

$$\sin^2(\alpha - \omega) = \frac{[a^2 \sin^2\alpha + b^2 \cos^2\alpha]^2}{a^4 \sin^2\alpha + b^4 \cos^2\alpha}$$

but from the same preceding number

$$\frac{a^2 \sin^2\alpha + b^2 \cos^2\alpha}{a^4 \sin^2\alpha + b^4 \cos^2\alpha} = \frac{1}{\overline{\mathrm{C}a}^2}$$

hence,

$$\sin^2(\alpha - \omega) = \frac{a^2 \sin^2\alpha + b^2 \cos^2\alpha}{\overline{\mathrm{C}a}^2}$$

again, from the first (o)

$$a^2 \sin^2\alpha + b^2 \cos^2\alpha = \frac{a^2\, b^2}{\overline{\mathrm{C}b'}^2}$$

hence,

$$\sin^2(\alpha - \omega) = \frac{a^2\, b^2}{\overline{\mathrm{C}a}^2 . \overline{\mathrm{C}b'}^2}$$

or

$$\sin(\alpha - \omega) = \frac{a\, b}{\mathrm{C}a . \mathrm{C}b'}$$

let us now substitute this value in (a_1), it will become

$$b't = b'\mathrm{C}\,\frac{a . b}{\mathrm{C}a . \mathrm{C}b'} = \frac{a . b}{\mathrm{C}a}$$

from which finally substituted in the second (a), we deduce

$$mm'n'n = 4 . a . b$$

but the first (a) gives $rr's's = 4 \,.\, a \,.\, b$; therefore,

$$mm'n'n = rr's's$$

that is, the parallelogram on the conjugates is equal to the rectangle on the axes.

PROPOSITION VI.

In every ellipse there are two conjugate diameters equal to each other.

81. Let aa' be any diameter, and let us draw from a' $a'n$ perpendicular to CA$'$ the axis of the abscissas. Cn and na' will be the co-ordinates x and y of the point a', and the square $\overline{Ca'}^2$ of the semi-diameter is equal to $x^2 + y^2$, or

$$Ca' = \sqrt{x^2 + y^2}$$

Now, from the equation considered in the present discussion of the ellipse we derive

$$y^2 = b^2 - \frac{b^2}{a^2} x^2$$

which value substituted in the preceding equation gives

$$Ca' = \sqrt{x^2 + b^2 - \frac{b^2}{a^2} x^2}$$

or
$$Ca' = \sqrt{\frac{x^2}{a^2}(a^2 - b^2) + b^2}$$

From which equation we know, first, that all the diameters in the quadrant BCA$'$ are different among themselves, because the second member of the equation depends on the variable x, which is different for every point of the curve corresponding to that quadrant. Secondly, since $a > b$ and consequently $a^2 - b^2 > o$, the diameters are increasing with x, but x is to be taken from o

to a; and to $x = o$ corresponds $Ca' = b$; to $x = a$ corresponds $Ca' = a$; therefore all the linear values comprised between the semi-axes a and b are the values of the semi-diameters drawn from every point of the curve contained between B′ and A′. Now, on account of $a > b$

$$\sqrt{\frac{a^2 + b^2}{2}} < \sqrt{\frac{a^2 + a^2}{2}} \text{ and } \sqrt{\frac{a^2 + b^2}{2}} > \sqrt{\frac{b^2 + b^2}{2}}$$

but $\sqrt{\frac{a^2 + a^2}{2}} = a$, $\sqrt{\frac{b^2 + b^2}{2}} = b$; hence $\sqrt{\frac{a^2 + b^2}{2}}$ is a value

comprised between a and b, and consequently the value of one of the semi-diameters drawn from some point a' of the branch B′A′. But (78) to every diameter corresponds another conjugate, and the sum of the squares of any two conjugates is (79) equal to $a^2 + b^2$. Therefore if χ' represents the conjugate semi-diameter of $\sqrt{\frac{a^2 + b^2}{2}} = \chi$; the value of χ' is to be derived from the equation

$$\frac{a^2 + b^2}{2} + \chi'^2 = a^2 + b^2$$

which gives $\chi'^2 = \frac{a^2 + b^2}{2}$ and $\chi' = \sqrt{\frac{a^2 + b^2}{2}}$

therefore $\chi = \chi'$ and

$$2\chi = 2\chi'$$

that is, in every ellipse there are two conjugate diameters equal to each other. It is here to be remarked that only one binary of such diameters can be found in the ellipse, because since the sum of the squares of the semi-diameters must always be the same, if we take any diameter greater than 2χ its conjugate must, of course, be less than $2\chi'$, and vice versa.

PROPOSITION VII.

The sine of the angle formed by any two conjugate diameters cannot be less than that of the angle contained by the equal conjugates.

82. The sine of the angle $(\alpha - \omega)$ formed by any two conjugate diameters is given (80) by the general formula

$$\sin(\alpha - \omega) = \frac{a \,.\, b}{Ca \,.\, Cb}$$

hence if $Ca = Cb$, that is, (81) if Ca and Cb are equal to $\sqrt{\frac{a^2 + b^2}{2}}$; the sine of the angle formed by these diameters shall be given by

$$\sin(\alpha - \omega) = \frac{2\, a \,.\, b}{a^2 + b^2}$$

If we now suppose the sine of the former equation to be less than the sine of the latter, we must suppose, also,

$$\frac{a \,.\, b}{Ca \,.\, Cb} < \frac{2 \,.\, a \,.\, b}{a^2 \,.\, b^2}$$

and $$a \,.\, b < \frac{2 \,.\, Ca \,.\, Cb}{a^2 + b^2}\, a \,.\, b$$

from which it follows

$$2\, Ca \,.\, Cb > a^2 + b^2$$

But since $(Ca - Cb)^2 = \overline{Ca}^2 + \overline{Cb}^2 - 2\, Ca \,.\, Cb$, and the first member of this equation is essentially positive, hence

$$\overline{Ca}^2 + \overline{Cb}^2 > 2\, Ca + Cb$$

and, of course, if $2\, Ca \,.\, Cb > a^2 + b^2$

$$\overline{Ca}^2 + \overline{Cb}^2 > a^2 + b^2$$

but (79) we demonstrated that the sum of the squares of the semi-diameters is constantly equal to the sum of the squares of the semi-axes; hence we cannot suppose the sine of the angle formed by any two conjugate diameters less than $\frac{2a \cdot b}{a^2 + b^2}$.

PROPOSITION VIII.

The sum of the distances of any point of the ellipse from the FOCI *is constantly equal to the transverse axis.*

83. Let (fig. 51) two points, F and F$'$, be taken on the transverse axis AA$'$ equally distant from the centre, and let the distance be equal to $\sqrt{a^2 - b^2}$; such points are called *foci* of the ellipse. The difference $a^2 - b^2$ is a positive quantity less than a^2, making, consequently,

$$a^2 - b^2 = \varepsilon^2 a^2$$

we must suppose $\varepsilon < 1$. Such a fraction is termed the *eccentricity*. Observe, now, that the equation of the ellipse reduced to the form

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \ . \ . \ . \ (o)$$

may be transformed, also, into

$$y^2 = (1 - \varepsilon^2)(a^2 - x^2) \ . \ . \ . \ (o).$$

Let us now draw from any point M of the curve the lines MF, MF$'$ to the foci, which lines are termed *radii* or *radius-vectors*, and let MF be represented by ρ, MF$'$ by ρ'. Mn the perpendicular drawn from M to the axis X the ordinate y of that point, so that, considering the triangles MFn, MF$'n$, we may derive the equations

$$\left.\begin{aligned} \rho^2 &= y^2 + \overline{nF}^2 \\ \rho'^2 &= y^2 + \overline{nF'}^2 \end{aligned}\right\} \ . \ . \ . \ (o_1)$$

but $\quad nF = CF - Cn = \sqrt{a^2 - b^2} - x = \varepsilon a - x$

$$nF' = CF' + Cn = \sqrt{a^2 - b^2} + x = \varepsilon a + x$$

therefore, substituting in (o_1) these values and that of y^2, given by the preceding formula (o), we shall have

$$\rho^2 = (1 - \varepsilon^2)(a^2 - x^2) + (\varepsilon a - x)^2$$

$$\rho'^2 = (1 - \varepsilon^2)(a^2 - x^2) + (\varepsilon a + x)^2$$

And since

$$(1 - \varepsilon^2)(a^2 - x^2) + (\varepsilon a - x)^2 = a^2 + \varepsilon^2 x^2 - 2\varepsilon ax = (a - \varepsilon x)^2$$

$$(1 - \varepsilon^2)(a^2 - x^2) + (\varepsilon a + x)^2 = a^2 + \varepsilon^2 x^2 + 2\varepsilon ax = (a + \varepsilon x)^2$$

so $\quad \rho^2 = (a - \varepsilon x)^2, \quad \rho'^2 = (a + \varepsilon x)^2$

therefore $\quad \rho = a - \varepsilon x, \quad \rho' = a + \varepsilon x$

and $\quad \rho + \rho' = 2a$

That is, the sum of the radius-vectors of any point M of the ellipse is constantly equal to the transverse axis.

Observe, that since $(a - \varepsilon x)^2 = (\varepsilon x - a)^2$, so $\rho^2 = (a - \varepsilon x)^2 = (\varepsilon x - a)^2$, and consequently $\rho = \varepsilon x - a$, in this case we would have $\rho' - \rho = 2a$. But considering, for instance, the extremity A of the transverse axis, of which the radius-vectors are AF, AF′, according to the last equation it would be AF — AF′ $= 2a =$ AA′, which is absurd; consequently the value $\varepsilon x - a$ of ρ cannot be admitted. Moreover, the required value of ρ must be positive. Now, since $\varepsilon < 1$ and x can never be greater than a, hence $\varepsilon x - a$ necessarily is negative, and of course the value of ρ, to be excluded.

On the property of the ellipse above discovered depends a mechanical construction of the same curve. Because if the ends of a thread of the exact length of the transverse axis AA′ be fixed by pins in the foci F and F′ (fig. 52), then moving a stile or pencil P within the thread FMF′ so as to keep it always stretched, it will describe the ellipse AB′A′B.

PROPOSITION IX.

The distance from the tangent of any point of the ellipse to the centre, if reckoned on the transverse axis, is third proportional to the abscissa of that point and the transverse semi-axis.

84. Let (fig. 53) the tangent tT be drawn from any point M of the ellipse. Since the origin of the co-ordinates is still supposed in C, if Mn be the perpendicular drawn from M to the transverse axis; Cn shall be the abscissa of that point, and CT (59) the distance Δ of the origin of the co-ordinates from the tangent, reckoned on the axis X. Such a distance is third proportional to Cn or x and the semi-axis CA$'$ or a: because the value of this distance (59) is given by the general formula

$$\Delta = x + \frac{Cx + By + E}{Ax + Cy + D}\, y$$

and consequently on account of $C = D = E = o$, $B = \frac{1}{b^2}$, $A = \frac{1}{a^2}$; by the formula

$$\Delta = x + \frac{a^2 \ y^2}{b^2 \ x}$$

but (83) $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ hence,

$$\Delta = x + \frac{a^2 - x^2}{x} = \frac{a^2}{x}$$

therefore, $$x : a :: a : \Delta$$

the indicated relation. Observe that the coefficient $\frac{Cx + By + E}{Ax + Cy + D}$ is (58) equal to $-\frac{1}{tg\,\alpha}$ being α, the angle formed by Tt with the positive X, hence the tangent of the angle tTX will be given by $\frac{Ax + Cy + D}{Cx + By + E}$ or substituting by $-\frac{b^2\,x}{a^2\,y}$

PROPOSITION X.

The normal corresponding to any point of the ellipse divides in two equal parts the angle formed by the radius-vectors of the same point.

85. Let (fig. 54) tT be the tangent of any point M of the ellipse. The line MR perpendicular to tT will be the normal of M. If from this point we draw the radii $MF = \rho$, $MF' = \rho'$, the distances RF and RF' of the normal from these radii, reckoned on the axis AA', have the same proportion as the radii themselves. Because

$$\left.\begin{aligned} FR &= FC - CR \\ F'R &= F'C + CR \end{aligned}\right\} \ldots (o)$$

and (83) $FC = F'C = \varepsilon a$, CR is (59) the distance Δ' reckoned on the axis X from the normal to the origin of the co-ordinates; which value given by the general formula

$$\Delta' = x - \frac{Ax + Cy + D}{Cx + By + E}\, y$$

becomes in the present case

$$\Delta' = x - \frac{b^2}{a^2}\, x = x\left(\frac{a^2 - b^2}{a^2}\right)$$

and since (83) $a^2 - b^2 = \varepsilon^2 a^2$

$$\Delta' = \varepsilon^2 x$$

therefore $CR = \varepsilon^2 x$. which value with that of FC and F'C substituted in the preceding (o) will give

$$FR = \varepsilon\,(a - \varepsilon x)$$

$$F'R = \varepsilon\,(a + \varepsilon x)$$

but (83) $a - \varepsilon x = \rho$, $a + \varepsilon x = \rho'$; hence

$$FR = \varepsilon\rho\,,\quad F'R = \varepsilon\rho'$$

and $$\frac{FR}{F'R} = \frac{\rho}{\rho'} \left(= \left(\frac{MF}{MF'}\right)\right.$$

Therefore, the side FF′ of the triangle FF′M is divided by MR in two parts, which constitute a proportion with the other sides of the triangle ; but it necessarily supposes the angle FMF′ divided in two equal parts by MR. Because let F′M be produced so far as to have $MF = Mf$, and let the points f, F be joined by fF, the triangles F′MR, $F'fF$ are similar to each other, for FR : F′R : : FM : F′M, and consequently FR : F′R : : fM : F′M; hence the angle $F'fF = F'MR$, but on account of the equal sides FM and fM, the angle MfF is equal to the angle fMF, and consequently $FfM + fFM = 2\,MfF$. Again, the angle $F'MF = FfM + fFM$ or $F'MF = 2MfF$, but MfF or $F'fF = F'MR$; hence

$$F'MF = 2F'MR$$

that is to say, the normal MR divides in two equal parts the angle formed by the radii MF, MF′.

PROPOSITION XI.

The normal of any point of the ellipse is a fourth proportional to the perpendicular drawn from any one of the two foci to the corresponding tangent, the radius drawn from the same focus to that point, and the half of the parameter.

86. The double of the ratio $\frac{b^2}{a}$ existing between the square of the conjugate semi-axis, and the transverse semi-axis is termed *parameter*, and if it be compendiously termed $2p$, we will have $2p = 2\,\frac{b^2}{a}\cdot$ Now, before speaking of the proposed subject, let us determine the values of the tangent and normal of any point M, the second of which is to be used in the following demonstration :

Since the most general values of the tangent and normal of any point (x, y) are given (59) by the formulas $t = \sqrt{y^2 + (x - \Delta)^2}$, $n = \sqrt{y^2 + (\Delta' - x)^2}$, and being (84, 85) with regard to the ellipse,

$$x - \Delta = \frac{x^2 - a^2}{x}, \ \Delta' - x = -\frac{b^2}{a^2}\,x$$

so we will have

$$t^2 = y^2 + \left(\frac{x^2 - a^2}{x}\right)^2$$

$$n^2 = y^2 + \frac{b^4}{a^4}\,x^2$$

but $\frac{b^4}{a^4} = \frac{b^2}{a^2} \cdot \frac{b^2}{a^2}$ and (83) from the equation $a^2 - b^2 = \varepsilon^2 a^2$ we have $\frac{b^2}{a^2} = 1 - \varepsilon^2$; again, since (83) $y^2 = \frac{b^2}{a^2}\,(a^2 x^2)$ the value of the square n^2 may be converted into the following

$$n^2 = \frac{b^2}{a^2}\,(a^2 - x^2) + \frac{b^2}{a^2}\,(1 - \varepsilon^2)\,x^2$$

$$= \frac{b^2}{a^2}\,[a^2 - \varepsilon^2 x^2]$$

but $a^2 - \varepsilon^2 x^2 = (a + \varepsilon x)(a - \varepsilon x)$ and (83) $a + \varepsilon x = \rho'$, $a - \varepsilon x = \rho$

hence $$n^2 = \frac{b^2}{a^2}\,\rho\rho'$$

Let us now come to the proposed demonstration, and first observe that the distance FT of the focus F from the tangent reckoned on the axis X corresponds to CT — CF, and the distance F′T of the other focus from the same point T of the tangent corresponds to CT + CF′, but CT or Δ is (84), equal to $\frac{a^2}{x}$ and (83) CF = CF′ = εa; hence

$$FT = \frac{a^2}{x} - \varepsilon a, \ F'T = \frac{a^2}{x} + \varepsilon a;$$

but $$\frac{a^2}{x} \mp \varepsilon a = \frac{a}{x}(a \mp \varepsilon x)$$

and $$a - \varepsilon x = \rho\,,\ a + \varepsilon x = \rho'$$

therefore $$\mathrm{FT} = \frac{a}{x}\rho\,,\ \mathrm{F'T} = \frac{a}{x}\rho'$$

Again, the distance TR of the normal MR from the tangent is equal to the difference CT — CR and (84, 85) equal also to

$$\Delta - \Delta' = \frac{a^2}{x} - \varepsilon^2 x = \frac{a^2}{x} - \frac{\varepsilon^2 x^2}{x}\text{; hence}$$

$$\mathrm{TR} = \frac{a^2 - \varepsilon^2 x^2}{x}$$

but $$a^2 - \varepsilon^2 x^2 = (a - \varepsilon x)(a + \varepsilon x) = \rho \cdot \rho'\text{; therefore}$$

$$\mathrm{TR} = \frac{\rho\rho'}{x}$$

Let us now draw (fig. 55) $\mathrm{F}m = q$ and $\mathrm{F}'m' = q'$ perpendicular to the tangent MT; the triangles RMT, $\mathrm{F}m\mathrm{T}$, $\mathrm{F}'m'\mathrm{T}$ are of course similar to each other; hence the proportions

$$\mathrm{TR} : \mathrm{FT} :: \mathrm{MR} : \mathrm{F}m$$

$$\mathrm{TR} : \mathrm{F'T} :: \mathrm{MR} : \mathrm{F}'m'$$

and substituting the preceding values

$$\frac{\rho\rho'}{x} : \frac{a\rho}{x} :: n : q$$

$$\frac{\rho\rho'}{x} : \frac{a\rho'}{x} :: n : q'$$

or $$\rho' : a :: n : q$$

$$\rho : a :: n : q'$$

from which $$q = \frac{an}{\rho'}\,,\ q' = \frac{an}{\rho}$$

and consequently $\quad qn = \frac{an^2}{\rho'}$, $q'n = \frac{an^2}{\rho}$

but the square n^2 of the normal determined before is equal to $\frac{b^2}{a^2}\rho\rho'$, hence

$$qn = \frac{b^2}{a}\rho \ , \ q'n = \frac{b^2}{a}\rho'$$

and since $\frac{b^2}{a} = p$ the semi-parameter, so

$$qn = p\rho \ , \ q'n = p\rho'$$

and consequently,

$$q : \rho :: p : n \quad , \quad q' : \rho' :: p : n$$

the required relations, and

$$n = \frac{p\rho}{q} \ , = \frac{p\rho'}{q'}$$

Scholium I. Let $Cn = q''$ be the perpendicular drawn from the centre C to the tangent; on account of the similarity of the triangles MRT , nCT, we have

$$TR : CT :: RM : Cn$$

and substituting the corresponding values

$$\frac{\rho\rho'}{x} : \frac{a^2}{x} :: n : q''$$

hence,

$$q'' = \frac{a^2 n}{\rho\rho'} \quad \text{and} \quad q''n = \frac{a^2 n^2}{\rho\rho'}$$

and substituting the above determined value of n^2 ,

$$q''n = b^2$$

or

$$n : b :: b : q''$$

that is, the conjugate semi-axis is a mean proportional between any normal and the perpendicular drawn from the centre to the corresponding tangent.

Scholium II. If in $y^2 = (1 - \varepsilon^2)(a^2 - x^2)$ the equation (83) of the ellipse, we substitute $\pm \varepsilon a$ instead of x; that is to say, the abscissa equal to the distance of the foci from the centre, the corresponding square of the ordinate y becomes

$$(1 - \varepsilon^2)(a^2 - \varepsilon^2 a^2) = a^2 (1 - \varepsilon^2)^2$$

but (83) $\qquad (1 - \varepsilon^2) = \frac{b^2}{a^2}$; hence,

$$y^2 = \frac{b^4}{a^2} = \left(\frac{b^2}{a}\right)^2$$

and

$$y = \frac{b^2}{a} = p$$

the semi-parameter; therefore, the double ordinates passing through the foci are equal to the parameter.

Equation of the ellipse with reference to the polar co-ordinates.

87. Let β be the angle formed by any radius $\rho = Fm'$, or $=$ FM, $= \ldots$ (fig. 56) and the positive axis X. Such an angle may be equal or less or greater than a right angle. In the first case $\cos\beta = o$; in the second $\cos\beta > o$; in the third $\cos\beta < o$. Suppose the three different cases to be represented by $m'FX <$ 90°, MFX $=$ 90°, mFX $>$ 90°; the first of the three perpendiculars $m'r'$, MF, mr drawn from the three different points must fall between F and A; the second shall fall in the focus F; and the third between the focus and the other extremity A of the axis; and Cr', CF, Cr will be the abscissas of each point. But $Cr' = CF + Fr'$, $Cr = CF - Fr$. Again, $CF = \varepsilon a$, and $Fr' = Fm' . \cos m'FX = \rho \cos\beta$, $- Fr = Fm \cos mFX =$ $\rho \cos\beta$; hence,

$$Cr' = \varepsilon a + \rho \cos \beta \ , \ Cr = \varepsilon a + \rho \cos \beta$$

and since CF $= \varepsilon a = \varepsilon a + \rho \cos 90^\circ = \varepsilon a + \rho \cos \text{MFX} = \varepsilon a + \rho \cos \beta$; so generally

$$x = \varepsilon a + \rho \cos \beta$$

But observe that (83) $\rho = a - \varepsilon x$; hence, substituting in this equation the last value of x, we will obtain

$$\rho = a - \varepsilon (\varepsilon a + \rho \cos \beta)$$

and consequently $\rho (1 + \varepsilon \cos \beta) = a (1 - \varepsilon^2)$

or
$$\rho = \frac{a (1 - \varepsilon^2)}{1 + \varepsilon \cos \beta}$$

which is the required equation.

Equation of the ellipse with reference to the conjugate diameters.

88. Let us now pass from the rectangular axes AA$'$, BB$'$ to any other system determined (fig. 57) by the conjugate diameters bb' , aa' ; the general formulas (8. C. I) to pass from one to another system of axes must be converted into the following :

$$\left.\begin{array}{l} y = x' \sin \alpha + y' \sin \omega \\ x = x' \cos \alpha + y' \cos \omega \end{array}\right\} \ldots (o)$$

because, since all the diameters must pass through the centre of the ellipse, the origin of the new system is of course common with that of the former ; hence, $x_\circ = y_\circ = o$. Again, the angle formed by one of the diameters, for instance bb', with X being termed α, that formed by the corresponding conjugate aa' will be accordingly termed ω ; and supposing the former to be taken for axis of the abscissas, we shall have $(xx') = \alpha$, $(xy') = \omega$. Substituting now the value (o) in the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we will have

$$\frac{(x' \cos \alpha + y' \cos \omega)^2}{a^2} + \frac{(x' \sin \alpha + y' \sin \omega)^2}{b^2} = 1$$

or

$$\frac{x'^2 \cos^2 \alpha + y'^2 \cos^2 \omega + 2x'y' \cos \alpha \cos \omega}{a^2} +$$

$$\frac{x'^2 \sin^2 \alpha + y'^2 \sin^2 \omega + 2x'y' \sin \alpha \sin \omega}{b^2} = 1$$

and consequently,

$$x'^2 \left[\frac{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha}{a^2 b^2}\right] + y'^2 \left[\frac{b^2 \cos^2 \omega + a^2 \sin^2 \omega}{a^2 b^2}\right]$$

$$+ 2x'y' \left[\frac{a^2 \sin \omega \sin \alpha + b^2 \cos \omega \cos \alpha}{a^2 b^2}\right] = 1$$

but since (78) $tg\ \omega = -\frac{b^2}{a^2\ tg\ \alpha}$; and consequently,

$$a^2 \sin \omega \sin \alpha = - b^2 \cos \omega \cos \alpha$$

or

$$a^2 \sin \omega \sin \alpha + b^2 \cos \omega \cos \alpha = o$$

therefore,

$$x'^2 \left[\frac{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha}{a^2 b^2}\right] + y'^2 \left[\frac{b^2 \cos^2 \omega + a^2 \sin^2 \omega}{a^2 b^2}\right] = 1$$

but (79 (o))

$$\frac{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha}{a^2 b^2} = \frac{1}{\overline{Cb}^2}$$

$$\frac{b^2 \cos^2 \omega + a^2 \sin^2 \omega}{a^2 b^2} = \frac{1}{\overline{Ca}^2}$$

therefore,

$$\frac{x'^2}{\overline{Cb}^2} + \frac{y'^2}{\overline{Ca}^2} = 1$$

the same equation as that of the ellipse having Cb and Ca for semi-axes and referred to them. Supposing, moreover, the diameters equal to each other, the equation of the ellipse does not differ from that of the circle referred to the rectangular diameters and having the radius equal to Ca.

It must be remarked, that from the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of the equation of the ellipse, we cannot conclude that the corresponding curve is an ellipse whose axes are $2a$ and $2b$; because, although such an ellipse also is represented by that equation, yet to the same equation may correspond any other ellipse referred to the conjugate diameters $2a$, $2b$. Observe, moreover, that since the equations of the diameters and axes, and every function of the lines of the second order derived from the most general formula (i), are deduced in the supposition of the curve referred to the rectangular axes, so in comparing any other with that general equation, we must suppose the curve referred to the rectangular axes. Hence, in comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the general equation (i), we must necessarily suppose $2a$ and $2b$ to be the axes, since in this case the curve is referred to the rectangular axes, and among the conjugate diameters the axes alone are perpendicular to each other.

HYPERBOLA.

PROPOSITION I.

Only one axis of the hyperbola meets the curve.

89. Let us now discuss

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the equation [62. (g)] of the hyperbola; which, compared with the most general (i) affords (64) $P_1 = \frac{1}{a^2}$, $P_2 = -\frac{1}{b^2}$, $K = 1$ and on account of $C = D = E = o$, $(i_{13})\ r = o$. Hence, $c_1{}^2$, $c_2{}^2$, the square values of the semi-axes (i_{15}) shall become

$$c_1{}^2 = a^2\ ,\ c_2{}^2 = -\,b^2$$

of which equations the first only being a real one, it follows that only one axis can meet the hyperbola and at a distance equal to a from both sides of the centre. Yet, to preserve the analogy between the ellipse and the hyperbola, the axis $2a$, or AA′ (fig. 58) is termed *transverse axis*, and cutting from both sides of the centre at a distance CB, CB′ equal to b, the unlimited axis aa', the portion BB′ of this axis is termed *conjugate axis*.

PROPOSITION II.

The equation of the hyperbola considered in the present discussion supposes the origin of the co-ordinates at the centre of the curve.

90. The values of the co-ordinates x_o, y_o of the centre given by the formulas (i_{10}) on account of $C = D = E = o$, both become, in the present case, equal to zero. But the values $C = D = E = o$ depend, as it was observed (64), on the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of our equation, and both the co-ordinates equal to zero are the co-ordinates of the origin of the axes; hence the equation of the hyperbola considered in the present discussion supposes the origin of the co-ordinates in the centre of the curve, or in the point C of intersection of the two axes of the same curve.

PROPOSITION III.

Every branch of the hyperbola has its proper asymptote.

91. If two lines rs, $r's'$, passing through the centre of the hyperbola (fig. 59), are continually approaching, the first to the branches Al, $A'm'$, the second to the branches Al', $A'm$, without touching them, but at infinite distance, according to the definition given (19) of the asymptote, each branch of the hyperbola has its proper asymptote. To demonstrate, now, that this is the case with regard to the hyperbola, let us first recollect the condition or criterion to be fulfilled when the chords passing through the

centre of the curve do not touch it, except at an infinite distance (57). It is proved that when the difference $C^2 - AB$ is positive, there are two such chords making the angles α', α'' with the positive axis X, to be determined by the equation

$$tg\,\alpha = -\frac{C}{B} \pm \frac{1}{B}\sqrt{C^2 - AB}$$

but in the present discussion (64) $A = \frac{1}{a^2}$, $B = -\frac{1}{b^2}$, $C = o$;

hence $$C^2 - AB = +\frac{1}{a^2\,b^2}$$

and, consequently,

$$tg\,\alpha' = -\frac{b}{a}, \quad tg\,\alpha'' = +\frac{b}{a}$$

therefore, in the hyperbola, two straight lines rs, $r's'$ passing through the centre, and making, with the axis CX, the angles α', α'', determined by the last equations, are two straight lines, which do not touch the curve but at an infinite distance from C. Therefore, if the same two lines are, moreover, continually approaching to the four branches, each branch has its proper asymptote. Now, the equations of the two infinite chords are

$$y' = \frac{b}{a}x', \quad y' = -\frac{b}{a}x'$$

from which $$y'^2 = \frac{b^2}{a^2}x'^2, \quad y'^2 = \frac{b^2}{a^2}x'^2$$

Again, the equation of the hyperbola may be transformed into

$$y^2 = \frac{b^2}{a^2}x^2 - b^2$$

hence, if we take the same abscissa in the latter and in the former equations, it follows evidently, first,

$$y < y'$$

secondly, $$y'^2 - y^2 = b^2$$

and since $$y'^2 - y^2 = (y' + y)(y' - y)$$

$$y' - y = \frac{b^2}{y' + y}$$

but y and y' are increasing with x, therefore the fraction $\frac{b^2}{y' + y}$, and of course the difference $y' - y$, becomes smaller and smaller by the increasing value of x; and consequently the lines rs, $r's'$ continually approach the curve.

Corollary. From this property it follows that all the branches of the curve are contained within the angles $r'Cr$, $s'Cs$ bisected by the transverse axis; hence any straight line pp' passing through the centre of the hyperbola, and contained within the angles $r'Cs$, $s'Cr$ bisected by the conjugated axis, can never reach any point of the curve; nay, shall continually diverge from the curve. On the contrary, any line qq' passing through the centre of the curve, and contained within the angles $r'Cr$, $s'Cs$ must necessarily meet the curve; because, while the curve is continually approaching the asymptote, that line is continually diverging from it. Now, every line passing through the centre of the curve and reaching the curve is a determined chord bisected at the centre.

PROPOSITION IV.

The diameter corresponding to any determined chord passing through the centre of the hyperbola is contained within the angle bisected by the conjugate axis.

92. We observe (56), with regard to the general formula (i_{14}) of the chord passing through the centre of the curve, that the second member of that equation must necessarily be a positive quantity. But in the present discussion, since (64) $K = 1$,

and (on account of $C = D = F = o$) $r = o$ and [44 (i_2)] $P = \frac{1}{a^2} \cos {}^2\alpha - \frac{1}{b^2} \sin {}^2\alpha = \frac{b^2 \cos {}^2\alpha - a^2 \sin {}^2\alpha}{a^2 b^2}$ the formula (i_{14}) becomes

$$c^2 = \frac{a^2 b^2}{b^2 \cos {}^2\alpha - a^2 \sin {}^2\alpha}$$

of which the second member cannot be a positive quantity except by supposing

$$a^2 \sin {}^2\alpha < b^2 \cos {}^2\alpha$$

or

$$tg {}^2\alpha < \frac{b^2}{a^2}$$

and in the present case it must really be so, because the angle formed by each asymptote and the transverse axis must be greater than that formed by the chord and the same axis. Now, the tangent of the former angle is (91) $+ \frac{b}{a}$ or $- \frac{b}{a}$, and of course the square of the same tangent $\frac{b^2}{a^2}$; therefore, the square of the tangent of any angle less than the angle made by the asymptotes with the transverse axis, must be less than $\frac{b^2}{a^2}$. Hence, if the angle formed by the diameter corresponding to the chord and the transverse axis be such as to give the square of its tangent greater than $\frac{b^2}{a^2}$, this diameter is evidently contained within the angle bisected by the conjugate axis. Now the angle ω, made by the diameter and the transverse axis, is to be deduced from (i_5), which in the present discussion is converted into $tg\,\omega = \frac{b^2}{a^2\, tg\,\alpha}$ from which

$$tg {}^2\omega = \frac{b^4}{a^4} \cdot \frac{1}{tg {}^2\alpha}$$

but since in the present case

$$tg^2\alpha < \frac{b^2}{a^2}$$

consequently,

$$\frac{1}{tg^2\alpha} > \frac{a^2}{b^2}$$

and if instead of $\frac{1}{tg^2\alpha}$ we substitute $\frac{a^2}{b^2}$ in the preceding equation, we shall obtain

$$tg^2\omega > \frac{b^2}{a^2}$$

that is to say, the diameter corresponding to any determined chord passing through the centre of the hyperbola is contained within the angle bisected by the conjugate axis, and consequently can never reach the curve.

PROPOSITION V.

If any line passing through the centre of the hyperbola is the diameter corresponding to a determined chord passing through the same point, this chord by turns shall be the diameter of the system of chords parallel to that line.

93. Let (fig. 60) bb' be the diameter corresponding to the system of chords parallel to aa'. Since aa' is a chord of determined length; according to the demonstration of the preceding member, bb' must be contained within the angle of the asymptotes bisected by the axis BB′, and will never reach the curve; yet the lines nn', . . . mm', . . . parallel to bb', are chords of determined length, and of which aa' produced is the diameter. Because, calling ω the angle formed by bb' and the system of parallel chords with AA′, the equation of the diameter corresponding to this system will be derived from (i_4) by changing α into ω, which will consequently become

$$y_0 = \frac{b^2}{a^2\ tg\ \omega}\ x_0$$

But by supposition bb' is diameter of aa', which forms the angle α with AA', consequently (i_5)

$$tg\ \omega = \frac{b^2}{a^2\ tg\ \alpha}$$

which value substituted in the preceding formula gives

$$y_o = tg\ \alpha\ .\ x_o$$

the equation of a straight line passing through the origin of the co-ordinates, and making with AA' the angle α. But such a line is aa', therefore aa' is the diameter of the system of chords parallel to bb'. To follow the analogy of the ellipse, observe that from the formula $tg\ ^2\omega > \frac{b^2}{a^2}$. corresponding to the angle made by bb' with the conjugate axis, we

deduce $$\frac{\sin\ ^2\omega}{\cos\ ^2\omega} - \frac{b^2}{a^2} > o$$

and $$a^2 \sin\ ^2\omega - b^2 \cos\ ^2\omega > o$$

consequently, the ratio

$$\frac{a^2\ b^2}{a^2 \sin\ ^2\omega - b^2 \cos\ ^2\omega}$$

has a determined value; and taking from both sides of C on the diameter bb' the portions Cd, Cd' equal to the square root of the late ratio, dd' shall be termed *conjugate* diameter of aa' and vice-versa.

PROPOSITION VI.

The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.

94. Let aa', dd' be any two conjugate diameters. Since

$$aa' = 2\,Ca\,,\ dd' = 2\,Cd\,;$$

consequently $\overline{aa'}^2 = \overline{4\,Ca}^2,\ \overline{dd'}^2 = \overline{4\,Cd}^2$

and (92, 93)

$$\overline{Ca}^2 = \frac{a^2\,b^2}{b^2\cos^2\alpha - a^2\sin^2\alpha}\,,\ Cd^2 = \frac{a^2\,b^2}{a^2\sin^2\omega - b^2\cos^2\omega}$$

hence

$$\overline{aa'}^2 - \overline{dd'}^2 = 4\left[\frac{a^2\,b^2}{b^2\cos^2\alpha - a^2\sin^2\alpha} - \frac{a^2\,b^2}{a^2\sin^2\omega - b^2\cos^2\omega}\right] \ldots (o)$$

Now from (i_5) we have

$$\frac{\sin\omega}{\cos\omega} = \frac{b^2\cos\alpha}{a^2\sin\alpha}$$

and consequently,

$$\frac{\sin^2\omega}{\cos^2\omega} = \frac{b^4\cos^2\alpha}{a^4\sin^2\alpha}$$

From which we may derive the following in the same manner as we derived similar equations for the ellipse :

$$\left.\begin{aligned} b^2\cos^2\omega &= \frac{b^2\,a^4\sin^2\alpha}{a^4\sin^2\alpha + b^4\cos^2\alpha} \\ a^2\sin^2\omega &= \frac{a^2\,b^4\cos^2\alpha}{a^4\sin^2\alpha + b^4\cos^2\alpha} \end{aligned}\right\} \ldots (a)$$

and

$$a^2\sin^2\omega - b^2\cos^2\omega = \frac{a^2\,b^2\,(b^2\cos^2\alpha - a^2\sin^2\alpha)}{a^4\sin^2\alpha + b^4\cos^2\alpha}$$

hence

$$\frac{a^2\ b^2}{a^2 \sin{}^2\omega - b^2 \cos{}^2\omega} = \frac{a^4 \sin{}^2\alpha + b^4 \cos{}^2\alpha}{b^2 \cos{}^2\alpha - a^2 \sin{}^2\alpha} \ldots (a_1)$$

but the numerator of the second member

$a^4 \sin{}^2\alpha + b^4 \cos{}^2\alpha =$

$$a^4 \sin{}^2\alpha + b^4 \cos{}^2\alpha + a^2\,b^2 - a^2\,b^2$$

$$= a^4 \sin{}^2\alpha + b^4 \cos{}^2\alpha + a^2\,b^2 - a^2\,b^2\,(\sin{}^2\alpha + \cos{}^2\alpha)$$

$$= b^2 \cos{}^2\alpha\,(b^2 - a^2) - a^2 \sin{}^2\alpha\,(b^2 - a^2) + a^2\,b^2$$

$$= (b^2 - a^2)\,(b^2 \cos{}^2\alpha - a^2 \sin{}^2\alpha) + a^2\,b^2$$

consequently,

$$\frac{a^2\ b^2}{a^2 \sin{}^2\omega - b^2 \cos{}^2\omega} = b^2 - a^2 + \frac{a^2\ b^2}{b^2 \cos{}^2\alpha - a^2 \sin{}^2\alpha}$$

which value substituted in the formula (o), will give

$$\overline{aa'}^2 - \overline{dd'}^2 = 4\,(a^2 - b^2)$$

or $$\overline{aa'}^2 - \overline{dd'}^2 = (2a)^2 - (2b)^2$$

that is to say, the difference of the squares of any two diameters aa', dd' is equal to the difference of the squares of the axes $2a$, $2b$.

Corollary. Supposing $2a = 2b$. (In which case the hyperbola is called *equilateral*,) the conjugate diameters also are equal to each other, because in this case $(2a)^2 - (2b)^2 = o$; hence $\overline{aa'}^2 - \overline{dd'}^2 = o$, and

$$aa' = dd'$$

Observe that in the same hypothesis the asymptotes are at right angles. Because the angle formed by each asýmptote is (91) to be derived from $tg\ \alpha = \pm \frac{b}{a}$; hence, with regard to the equilateral hyperbola, from $tg\ \alpha = \pm 1$. That is, the angle formed by each asymptote with the tranverse axis is equal to 45°; hence that contained by the asymptotes themselves is equal to 90°.

PROPOSITION VII.

The parallelogram on the conjugates is equal to the rectangle on the axes.

95. Let us draw (fig. 61) from the extremities a, a' of the conjugate diameter aa', the tangents mm', nn' which (50, 78) must be parallel to each other, and to the conjugate dd'. Let us draw also $m'\,n'$, mn parallel to aa' from the extremities d and d' of dd', we evidently have

$$mnn'm' = 4\ \mathrm{C}a'n'd' \ldots . \ (o)$$

and in the same manner the rectangle on the axes

$$rss'r' = 4\ \mathrm{CA}s'\mathrm{B}' \ldots . \ (o_1).$$

Again, let $d'\,t'$ be drawn perpendicular to aa'. For the area of the parallelogram $\mathrm{C}a'n'd'$ we may substitute the product $\mathrm{C}a' \,.\, d't'$ but $d't' = \mathrm{C}d' \,.\, \sin d'\mathrm{C}a' = \mathrm{C}d' \,.\, \sin (d'\mathrm{CA}' - \mathrm{A}'\mathrm{C}a') = \mathrm{C}d' \,.$ $\sin (\omega - \alpha)$; hence

$$\mathrm{C}a'n'd' = \mathrm{C}a' \,.\, \mathrm{C}d' \sin (\omega - \alpha) \ldots . \ (o_2).$$

Now from trigonometry

$$\sin (\omega - \alpha) = \sin \omega \cos \alpha - \cos \omega \sin \alpha$$

$$= \cos \omega \ (tg\ \omega \cos \alpha - \sin \alpha)$$

hence $\quad \sin^2 (\omega - \alpha) = \cos^2 \omega \ (tg\ \omega \cos \alpha - \sin \alpha)^2$

but (92) $\quad tg\ \omega = \frac{b^2}{a^2\ tg\ \alpha} = \frac{b^2 \cos \alpha}{a^2 \sin \alpha}$

consequently

$$\sin^2(\omega - \alpha) = \cos^2\omega \left[\frac{b^2 \cos^2\alpha}{a^2 \sin\alpha} - \sin\alpha\right]^2$$

$$= \frac{\cos^2\omega}{a^4 \sin^2\alpha}\left[b^2 \cos^2\alpha - a^2 \sin^2\alpha\right]^2$$

but from the former (a) of the preceding number, we may derive the value of $\cos^2\omega$, which substituted in our last equation will give

$$\sin^2(\omega - \alpha) = \frac{[b^2 \cos^2\alpha - a^2 \sin^2\alpha]^2}{a^4 \sin^2\alpha + b^4 \cos^2\alpha}$$

$$= \frac{b^2 \cos^2\alpha - a^2 \sin^2\alpha}{a^4 \sin^2\alpha + b^4 \cos^2\alpha}(b^2 \cos^2\alpha - a^2 \sin^2\alpha)$$

but from (a_1) of the same number, we derive

$$\frac{b^2 \cos^2\alpha - a^2 \sin^2\alpha}{a^4 \sin^2\alpha + b^4 \sin^2\alpha} = \frac{a^2 \sin^2\omega - b^2 \cos^2\omega}{a^2\, b^2}$$

and the value of $\overline{Ca}^2$ gives

$$b^2 \cos^2\alpha - a^2 \sin^2\alpha = \frac{a^2\, b^2}{\overline{Ca'}^2}$$

hence

$$\sin^2(\omega - \alpha) = \frac{a^2 \sin^2\omega - b^2 \cos^2\omega}{\overline{Ca'}^2}$$

again, from the value of $\overline{Cd}^2$ we deduce

$$a^2 \sin^2\omega - b^2 \cos^2\omega = \frac{a^2\, b^2}{\overline{Cd'}^2}$$

consequently

$$\sin^2(\omega - \alpha) = \frac{a^2\, b^2}{\overline{Ca'}^2 . \overline{Cd'}^2}$$

and

$$\sin(\omega - \alpha) = \frac{a \,.\, b}{Ca' . Cd'}$$

therefore

$$Ca' \cdot Cd' \cdot \sin(\omega - \alpha) = a \cdot b$$

which value substituted in (o_2) gives

$$Ca'n'd' = a \cdot b$$

but $a \cdot b = CAs'B$, hence

$$Ca'n'd' = CAs'B$$

therefore the second members of the equations (o) (o_1) are equal to each other ; consequently

$$mnn'm = rss'r'$$

that is, the area of the rectangle on the axes is equal to the area of the parallelogram on any two conjugate diameters.

PROPOSITION VIII.

The difference of the distance of any point of the hyperbola from the foci *is equal to the transverse axis.*

96. Join (fig. 62) the extremities B′ and A′ of the semi-axes CB′ , CA′ and take on the transverse axis two points F , F′ at a distance from the centre C, equal to the hypothenuse B′A′. Since $B'A' = \sqrt{a^2 + b^2}$ $CF = CF' = \sqrt{a^2 + b^2}$, the points taken at such a distance from the centre are called *foci* of the hyperbola. Let us make now

$$a^2 + b^2 = \varepsilon^2 a^2$$

ε (which is termed the *eccentricity* of the hyperbola) must evidently be greater than unity. Now from this equation we may derive

$$\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ and } -\frac{b^2}{a^2} = 1 - \varepsilon^2$$

and since (63) the second (g) which is the equation considered in the present discussion, may be transformed into

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2) = -\frac{b^2}{a^2}(x^2 - a^2)$$

so by substituting the value of $-\frac{b^2}{a^2}$ the same equation will become

$$y^2 = (1 - \varepsilon^2)(a^2 - x^2) \;.\;.\;.\;.\; (o)$$

Let ρ and ρ' be the straight lines MF , MF′ drawn from any point M of the curve to the foci, which lines are termed radii or *radius-vectors.* Again, let y be the ordinate Mn of the point M. From the triangles MnF′ , MnF we shall have

$$\left.\begin{aligned}\rho'^2 &= y^2 + \overline{n\mathrm{F}'}^2\\ \rho^2 &= y^2 + \overline{n\,\mathrm{F}}^2\end{aligned}\right\} \;.\;.\;.\;.\; (o_1)$$

But $\quad n\mathrm{F}' = \mathrm{C}n - \mathrm{CF}' = x - \sqrt{a^2 + b^2} = x - \varepsilon a$

$$n\mathrm{F} = \mathrm{C}n + \mathrm{CF} = x + \sqrt{a^2 + b^2} = x + \varepsilon a$$

Therefore, substituting these values in (o_1) and the values of y given by (o) we shall obtain

$$\rho'^2 = (1 - \varepsilon^2)(a^2 - x^2) + (x - \varepsilon a)^2$$

$$\rho^2 = (1 - \varepsilon^2)(a^2 - x^2) + (x + \varepsilon a)^2$$

now

$$(1 - \varepsilon^2)(a^2 - x^2) + (x - \varepsilon a)^2 = a^2 + x^2\varepsilon^2 - 2x\,\varepsilon a = (\varepsilon x - a)^2$$

$$(1 - \varepsilon^2)(a^2 - x^2) + (x + \varepsilon a)^2 = a^2 + x^2\varepsilon^2 + 2x\,\varepsilon a = (\varepsilon x + a)^2$$

hence

$$\rho'^2 = (\varepsilon x - a)^2 \;,\; \rho^2 = (\varepsilon x + a)^2$$

and $\quad \rho' = (\varepsilon x - a) \;,\; \rho = \varepsilon x + a$

therefore

$$\rho - \rho' = 2a$$

that is to say, the difference of the radius-vectors of any point M of the hyperbola is equal to the transverse axis. We can here make the same observation which we made (59) with regard to the ellipse, that is, since $(\varepsilon x - a)^2 = (a - \varepsilon x)^2$ so we could take the last form of the square of ρ'^2. But then $\rho' = a - \varepsilon x$ and consequently $\rho + \rho' = 2a$, which would be an equation to be verified with every point of the hyperbola. But the radii of the extremity A of the transverse axis are AF , AF′ and their sum is FF′. Hence, supposing $\rho' = a - \varepsilon x$ we would have also AA′ = FF′, which being absurd, that supposition is to be excluded. Again, since the values of the radius-vectors considered here are positive, it is plain that $\rho' = a - \varepsilon x$ is to be excluded, because x cannot be less than a and ε is > 1.

From this property of the hyperbola is derived a mechanical method of constructing this curve. Because let (fig. 63) the ends of two threads F Mm , F′Mm' be fixed in the points F and F of the straight line FF′, and suppose these two threads to pass through a small ring M. Now, if by means of a stile t we make the ring glide in such a manner as to cause the same length of each thread to pass by the ring, the difference between the stretched part of the threads will be constantly the same ; for instance, AA′, and of course the path AM marked by the point of the stile, must be a hyperbola.

PROPOSITION IX.

The distance from the tangent of any point of the hyperbola to the centre, reckoned on the transverse axis, is a third proportional to the abscissa of that point and the transverse axis.

97. Let M (fig. 64) be any point of the hyperbola, and tT the correspondent tangent ; the distance CT or Δ from the centre to the point T being, according to the general equation (59), $x + \dfrac{Cx + By + E}{Ax + Cy + D}\, y$ in the present case will become

$$\Delta = x - \frac{a^2}{b^2}\frac{y^2}{x}$$

but (96) $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$, hence

$$\Delta = x - \frac{x^2 \ a^2}{x} = \frac{a^2}{x}$$

from which $x : a :: a : \Delta$

that is, the distance CT from the centre to the tangent is a third proportional to the abscissa Cn and the semi-axis CA′.

Observe, that since $\frac{Cx + By + E}{Ax + Cy + D} = -\frac{a^2 \ y}{b^2 \ x}$ and since (58) $\frac{Cx + By + E}{Ax + Cy + B} = -\frac{1}{tg\,\alpha}$ being α the angle tTX, we will also have

$$tg\,\alpha = \frac{b^2 \ x}{a^2 \ y}$$

PROPOSITION X.

The tangent corresponding to any point of the hyperbola bisects into two equal parts the angle formed by the radius-vectors of the same point.

98. Let MT be the tangent and MF , MF′ the radius-vectors of any point M. In the discussion of the analogous property of the ellipse we observed (85) that the angle M of the triangle FMF′ is equally bisected by a line MT when the segments FT and TF′ are proportional with the corresponding sides FM , F′M ; therefore, to demonstrate that the angle FMF′ is divided into two equal parts by the tangent MT, it is sufficient to prove that

$$\frac{FT}{F'T} = \frac{FM}{FM'} = \frac{\rho}{\rho'} \cdot \quad \text{Now}$$

$$\left.\begin{array}{l} TF = CF + CT \\ TF' = CF' - CT \end{array}\right\} \cdots (o)$$

But CT or Δ , as we observed before, is equal to $\frac{a^2}{x}$ and (96) $CF = CF' = \varepsilon a$; hence

$$TF = \varepsilon a + \frac{a^2}{x} = \frac{a}{x}(\varepsilon x + a)$$

$$TF' = \varepsilon a - \frac{a^2}{x} = \frac{a}{x}(\varepsilon x - a)$$

again, $\varepsilon x + a = \rho$, $\varepsilon x - a = \rho'$; consequently

$$TF = \frac{a}{x}\rho \,,\; TF' = \frac{a}{x}\rho'$$

and $$\frac{TF}{TF'} = \frac{\rho}{\rho'}$$

Corollary. Let us produce FM towards g, the angle FMT shall be equal to gMt , but FMT = F′MT , hence $gMt = F'MT$; let now MR be drawn perpendicular to the tangent, the angles RMT , RMt are of course equal to each other. But $RMT = RMF' + F'MT$, and $RMt = RMg + gMt$, hence

$$RMF' + F'MT = RMg + gMt$$

and on account of $$F'MT = gMt$$

$$RMF' = RMg$$

that is to say, the other angle F′Mg , formed by the same radius-vectors, is equally bisected by the normal.

PROPOSITION XI.

The normal of any point of the hyperbola is a fourth proportional to the perpendicular drawn from any one of the two foci to the correspondent tangent, the radius drawn from that focus to the same point and the half of the parameter.

99. The double ratio $2\dfrac{b^2}{a}$ existing between the square of the conjugate semi-axis is termed the *parameter* of the hyperbola, and is compendiously represented by $2p$. Now, to follow the analogy with the ellipse, let us first ascertain the values of the tangent and normal corresponding to any point of the hyperbola. The values of which functions given by the general formulas (59) $t=\sqrt{y^2+(x-\Delta)^2}$, $n=\sqrt{y^2+(\Delta'-x)^2}$ in the present case may be modified by the substitution of the values of Δ and Δ' corresponding to the hyperbola, the first of which is already (97) determined and found equal to $\dfrac{a^2}{x}$: the second, that is (59) the distance from the centre to the normal, being, according to the general formula, $x-\dfrac{Ax+Cy+D}{Cx+By+E}\,y$ is consequently in the present case equal to $x+\dfrac{b^2}{a^2}\,x$; therefore

$$x-\Delta=\frac{x^2-a^2}{x},\ \Delta'-x=\frac{b^2}{a^2}\ x$$

$$t=\sqrt{y^2+\left(\frac{x^2-a^2}{x}\right)^2},\ n=\sqrt{y^2+\frac{b^4}{a^4}x^2}$$

but (96) $\quad y^2=\dfrac{b^2}{a^2}(x^2-a^2)$ and $\dfrac{b^2}{a^2}=\varepsilon^2-1$

hence, taking the square of the normal

$$n^2=\frac{b^2}{a^2}(x^2-a^2)+\frac{b^2}{a^2}(\varepsilon^2-1)\,x^2=\frac{b^2}{a^2}(\varepsilon^2\,x^2-a^2)$$

Again, $(\varepsilon^2 x^2 - a^2) = (\varepsilon x + a)(\varepsilon x - a)$ and (96) $\varepsilon x + a = \rho$, $\varepsilon x - a = \rho'$; therefore, $(\varepsilon^2 x^2 - a^2) = \rho . \rho'$, and

$$n^2 = \frac{b^2}{a^2} \rho \rho'$$

or

$$n = \frac{b}{a} \sqrt{\rho \rho'}$$

Now, let M (fig. 65) be any point of the hyperbola, and MR the corresponding normal n, MT the corresponding tangent t. Again, let $\mathrm{MF} = q$, $m'\mathrm{F}' = q'$ be the perpendicular lines drawn from the foci to the tangent, we will manifestly have

$$\mathrm{TR} : \mathrm{F'T} :: \mathrm{MR} : \mathrm{F'}m'$$

$$\mathrm{TR} : \mathrm{FT} :: \mathrm{MR} : \mathrm{F}m$$

or

$$\left.\begin{array}{l} \mathrm{TR} : \mathrm{F'T} :: n : q' \\ \mathrm{TR} : \mathrm{FT} :: n : q \end{array}\right\} \ldots (o)$$

But (97, 99) $\mathrm{TR} = \mathrm{CR} - \mathrm{CT} = \Delta' - \Delta$, and $\Delta' = x + \frac{b^2}{a^2} x$ $= x + (\varepsilon^2 - 1) x = \varepsilon^2 x$, $\Delta = \frac{a^2}{x}$; hence, $\mathrm{TR} = \varepsilon^2 x - \frac{a^2}{x}$ $= \frac{\varepsilon^2 x^2 - a^2}{x}$; again, $(\varepsilon^2 x^2 - a^2) = \rho \rho'$; therefore,

$$\mathrm{TR} = \frac{\rho \rho'}{x}$$

Observe, moreover, that $\mathrm{F'T} = \mathrm{CF'} - \mathrm{CT} = \mathrm{CF'} - \Delta$, and $\mathrm{FT} = \mathrm{FC} + \mathrm{CT} = \mathrm{FC} + \Delta$; but $\mathrm{CF} = \mathrm{CF'} = \varepsilon a$; hence, $\mathrm{F'T} = \varepsilon a - \frac{a^2}{x}$, $\mathrm{FT} = \varepsilon a + \frac{a^2}{x}$; and since

$$\varepsilon a - \frac{a^2}{x} = \frac{a}{x} (\varepsilon x - a); \ \varepsilon a + \frac{a^2}{x} = \frac{a}{x} (\varepsilon x + a)$$

and $\varepsilon x - a = \rho'$, $\varepsilon x + a = \rho$; so

$$\mathrm{F'T} = \frac{a\rho'}{x}, \ \mathrm{FT} = \frac{a\rho}{x}$$

which with the preceding value of TR substituted in (o), will give

$$\frac{\rho\rho'}{x} : \frac{a\rho'}{x} : : n : q'$$

$$\frac{\rho\rho'}{x} : \frac{a\rho}{x} : : n : q$$

from which

$$\rho : a : : n : q' \qquad \rho' : a : : n : q$$

and

$$q' = \frac{an}{\rho}, \ q = \frac{an}{\rho'}$$

or

$$q'n = \frac{an^2}{\rho}, \ qn = \frac{an^2}{\rho'}$$

but we observed above that $n^2 = \frac{b^2}{a^2}\rho\rho'$; hence,

$$q'n = \frac{b^2}{a}\rho', \ qn = \frac{b^2}{a}\rho$$

but $\frac{b^2}{a}$ is equal to p the half of the parameter ; hence,

$$q'n = p\rho', \ qn = p\rho$$

and finally,

$$q' : \rho' : : p : n \qquad q : \rho : : p : n$$

the required proportions, and

$$n = \frac{p\rho'}{q'} = \frac{p\rho}{q}$$

Scholium I. Let $Cn = q''$ be the perpendicular drawn from the centre to the tangent, the similar triangles CnT, TMR will afford the proportion·

$$TR : CT :: MR : Cn$$

or, substituting the preceding values

$$\frac{\rho\rho'}{x} : \Delta\left(= \frac{a^2}{x}\right) :: n : q''$$

hence,

$$q'' = \frac{a^2 n}{\rho\rho'}$$

and

$$q''n = \frac{a^2}{\rho\rho'} n^2$$

but $n^2 = \frac{b^2}{a^2}\rho\rho'$; hence,

$$q''n = b^2$$

and

$$n : b :: b : q''$$

That is to say, the conjugate semi-axis b is a mean proportional between the normal of any point and the perpendicular drawn from the centre to the corresponding tangent.

Scholium II. If in the equation of the hyperbola (96) $y^2 = (1 - \varepsilon^2)(a^2 - x^2)$ we substitute $\pm \varepsilon a$ instead of x; that is, if we give to the abscissa the value equal to the distance from the centre to the foci, the corresponding value of y^2 will become

$$y^2 = (1 - \varepsilon^2)(a^2 - \varepsilon^2 a^2) = a^2 (1 - \varepsilon^2)^2$$

but (96) $(1 - \varepsilon) = \frac{b^2}{a^2}$; hence, $y^2 = \frac{b^4}{a^2}$, and

$$y = \pm \frac{b^2}{a} (= p)$$

Therefore, the double ordinate $2y$ passing through the foci are equal to $2p$, that is, to the parameter.

Equation of the hyperbola with reference to the polar co-ordinates.

100. Let β (fig. 66) be the angle formed by any radius-vector with the positive axis of the abscissas, we shall have either $\cos\beta = o$, or $< o$, or $> o$, according as the angle is equal, greater, or less than a right angle. Suppose, now, the three different cases to be represented by $m\text{F}'\text{X} < 90°$, $\text{MFX} = 90°$, $m\text{F}'\text{X} > 90°$, and let us draw mr and mr' perpendicular to the axis X. We will have $\text{F}'r = \text{F}'m \cos m\text{FX} = \rho \cos\beta$, $\text{F}'r' = \text{F}'m \cos m'\text{F}'\text{X} = \rho \cos\beta$, and $o = \text{F}'\text{M} \cos \text{MFX} = \rho' \cos\beta$. Now $\text{C}r$, CF', $\text{C}r'$ are the abscissas x of the three different points, and $\text{C}r = \text{CF}' + \text{F}'r$, $\text{CF}' = \text{CF}' + o$ $\text{C}r' = \text{CF}' - \text{F}'r'$; consequently, since (96) $\text{CF}' = \varepsilon a$, by substituting we shall obtain in every case

$$x = \varepsilon a + \rho' \cos\beta$$

but $\rho' = \varepsilon x - a$; hence, substituting in this formula the preceding value of x, we will have

$$\rho' = \varepsilon(\varepsilon a + \rho' \cos\beta) - a$$

and

$$\rho'(1 - \varepsilon\cos\beta) = a(\varepsilon^2 - 1)$$

or

$$\rho' = \frac{a(\varepsilon^2 - 1)}{1 - \varepsilon\cos\beta}$$

the required equation.

Equation of the hyperbola referred to the conjugate diameters.

101. Supposing α and ω to be the angles formed by the diameters aa', dd' (fig. 67) with CX. Since the origin of the co-ordinates in the present transformation is in the centre, the general formulas (7) giving the values of the former by the new co-ordinates must be transformed into

$$y = x' \sin\alpha + y' \sin\omega$$

$$x = x' \cos\alpha + y' \cos\omega$$

which values substituted in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of the hyperbola referred to the axes will give

$$\frac{(x' \cos \alpha + y' \cos \omega)^2}{a^2} - \frac{(x' \sin \alpha + y' \sin \omega)^2}{b^2} = 1$$

or (88)

$$\frac{b^2 [x'^2 \cos^2 \alpha + y'^2 \cos^2 \omega] - a^2 [x'^2 \sin^2 \alpha + y'^2 \sin^2 \omega]}{a^2 b^2} +$$

$$\frac{2x'y' (b^2 \cos \alpha \cos \omega - a^2 \sin \alpha \sin \omega)}{a^2 b^2} = 1$$

but (92) $tg\, \omega = \frac{b^2}{a^2\, tg\, \alpha}$. Hence

$$a^2 \sin \omega \sin \alpha = b^2 \cos \omega \cos \alpha$$

and consequently

$$b^2 \cos \omega \cos \alpha - a^2 \sin \omega \sin \alpha = o$$

therefore

$$\frac{b^2 [x'^2 \cos^2 \alpha + y'^2 \cos^2 \omega] - a^2 [x'^2 \sin^2 \alpha + y'^2 \sin^2 \omega]}{a^2 b^2} = 1$$

or

$$x'^2 \left[\frac{b^2 \cos^2 \alpha - a^2 \sin^2 \alpha}{a^2 b^2}\right] - y'^2 \left[\frac{a^2 \sin^2 \omega - b^2 \cos^2 \omega}{a^2 b^2}\right] = 1$$

but (92, 94)

$$\frac{b^2 \cos^2 \alpha - a^2 \sin^2 \alpha}{a^2 b^2} = \frac{1}{\overline{Ca}^2}, \quad \frac{a^2 \sin^2 \omega - b^2 \cos^2 \omega}{a^2 b^2} = \frac{1}{\overline{Cd}^2}$$

hence

$$\frac{x'^2}{\overline{Ca}^2} - \frac{y'^2}{\overline{Cd}^2} = 1$$

the equation of the hyperbola referred to the conjugate diameters, to which we may extend the same observations made with regard to the ellipse (88).

Equation of the hyperbola referred to the asymptote.

102. Suppose (fig. 68) the asymptote rs to be taken for axis of the abscissas, and $r's'$ for axis of the ordinates, the angle α', formed by the former asymptote with CX, as well as $-\alpha'$ formed by the second with the same axis, are to be derived (91) from the formula $tg\ ^2\alpha' = \frac{b^2}{a^2}$, or $\frac{\sin\ ^2\alpha'}{\cos\ ^2\alpha'} = \frac{b^2}{a^2}$; but from this last equation we have

$$\frac{\sin\ ^2\alpha'}{\cos\ ^2\alpha'} + 1 = \frac{b^2}{a^2} + 1$$

$$\frac{\cos\ ^2\alpha'}{\sin\ ^2\alpha'} + 1 = \frac{a^2}{b^2} + 1$$

or

$$\frac{\sin\ ^2\alpha' + \cos\ ^2\alpha'}{\cos\ ^2\alpha'} = \frac{b^2 + a^2}{a^2}$$

$$\frac{\cos\ ^2\alpha' + \sin\ ^2\alpha'}{\sin\ ^2\alpha'} = \frac{a^2 + b^2}{b^2}$$

and, since $\sin\ ^2\alpha' + \cos\ ^2\alpha' = 1$

$$\cos\ ^2\alpha' = \frac{a^2}{b^2 + a^2}\ ,\quad \sin\ ^2\alpha' = \frac{b^2}{a^2 + b^2}$$

or

$$\cos\ \alpha' = \frac{a}{\sqrt{b^2 + a^2}}\ ,\quad \sin\ \alpha' = \frac{b}{\sqrt{a^2 + b^2}}$$

Considering, now, that the equations (7) to pass from a system of rectangular axes to any other become, in the present case,

$$x = x_1 \cos\ \alpha' + y_1 \cos\ \alpha'$$

$$y = x_1 \sin\ \alpha' + y_1 \sin\ \alpha'$$

and substituting the preceding values of $\cos \alpha'$ and $\sin \alpha'$ we will have

$$x = x_1 \frac{a}{\sqrt{b^2 + a^2}} + y_1 \frac{a}{\sqrt{b^2 + a^2}}$$

$$y = x_1 \frac{b}{\sqrt{b^2 + a^2}} - y_1 \frac{b}{\sqrt{b^2 + a^2}}$$

from which

$$x = \frac{a\,(x_1 + y_1)}{\sqrt{b^2 + a^2}}\,,\ y = \frac{b\,(x_1 - y_1)}{\sqrt{b^2 + a^2}}$$

Substituting, finally, these values in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of the hyperbola referred to the axes we will obtain the equation between the co-ordinates x_1, y_1 of the same hyperbola referred to the asymptotes ; that is

$$\frac{(x_1 + y_1)^2}{b^2 + a^2} - \frac{(x_1 - y_1)^2}{b^2 + a^2} = 1$$

from which

$$\frac{4\,x_1\,y_1}{b^2 + a^2} = 1$$

and

$$x_1\,y_1 = \frac{b^2 + a^2}{4}$$

that is the product $x_1 \,.\, y_1$ of the co-ordinates of the hyperbola referred to the asymptotes is a constant quantity. Observe, again, that since in the supposition of the equilateral hyperbola $a = b$;

$$x_1\,y_1 = \frac{a^2}{2}$$

Equation of the hyperbola referred to an asymptote taken for axis of abscissas, and to a diameter taken for axis of ordinates.

103. Let (fig. 69) the asymptote rCs and the diameter dCd' be the new system of axes to which the hyperbola is to be referred, and let x'', y'' be the co-ordinates corresponding to the new system. Since the origin of the same system is still at the centre of the curve, the above mentioned formulas (7) will become

$$\left.\begin{aligned} x &= x'' \cos (x''x) + y'' \cos (y''x) \\ y &= x'' \sin (x''x) + y'' \sin (y''x) \end{aligned}\right\} \ldots (o)$$

Now $(x''x) = s\mathrm{CX} = \alpha'$ and (102) $\sin \alpha' = \dfrac{b}{\sqrt{b^2 + a^2}}$, $\cos \alpha' = \dfrac{a}{\sqrt{b^2 + a^2}}$; hence $\sin (x''x) = \dfrac{b}{\sqrt{b^2 + a^2}}$, $\cos (x''x) = \dfrac{a}{\sqrt{b^2 + a^2}}$
Again, being ω the angle $d'\mathrm{CX}$ formed by the diameter with the axis X, since we consider the positive direction of the new axis of ordinates from C to b; the angle formed by $\mathrm{C}d$ with CX, that is, $d\mathrm{CA} + \mathrm{AC}d' + d'\mathrm{CA}'$, will be equal to $180° + \omega$; and consequently $y''x = (180° + \omega)$; hence $\sin (y''x) = -\sin \omega$, $\cos (y''x) = -\cos \omega$; which value and the preceding, being substituted in the formulas (o), will give

$$x = x''.\frac{a}{\sqrt{b^2 + a^2}} - y'' \cos \omega$$

$$y = x'' \frac{b}{\sqrt{b^2 + a^2}} - y'' \sin \omega$$

but (96) $b^2 + a^2 = \varepsilon^2 a^2$; hence

$$x = \frac{1}{\varepsilon}\, x'' - y'' \cos \omega = \frac{x'' - \varepsilon y'' \cos \omega}{\varepsilon}$$

$$y = \frac{b}{\varepsilon a}\, x'' - y'' \sin \omega = \frac{bx'' - \varepsilon a y'' \sin \omega}{\varepsilon a}$$

Let us now substitute these values in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of the hyperbola referred to the axes, we will obtain

$$\frac{(x'' - \varepsilon y'' \cos \omega)^2}{\varepsilon^2 a^2} - \frac{(bx'' - \varepsilon ay'' \sin \omega)^2}{\varepsilon^2 a^2 b^2} = 1$$

or

$$\frac{b^2 (x'' - \varepsilon y'' \cos \omega)^2 - (bx'' - \varepsilon ay'' \sin \omega)^2}{\varepsilon^2 a^2 b^2} = 1$$

from which, since

$$b^2 (x'' - \varepsilon y'' \cos \omega)^2 = b^2 x''^2 - 2b^2 \varepsilon x'' y'' \cos \omega + b^2 \varepsilon^2 y''^2 \cos^2 \omega$$

$$(bx'' - \varepsilon ay'' \sin \omega)^2 = b^2 x''^2 - 2\varepsilon bax'' y'' \sin \omega + \varepsilon^2 a^2 y''^2 \sin^2 \omega$$

we derive

$$\frac{\varepsilon^2 (b^2 \cos^2 \omega - a^2 \sin^2 a) y''^2 + 2b \varepsilon (a \sin \omega - b \cos \omega) x'' y''}{\varepsilon^2 a^2 b^2} = 1$$

and

$$\frac{b^2 \cos^2 \omega - a^2 \sin^2 \omega}{a^2 b^2} y''^2 + \frac{2 (a \sin \omega - b \cos \omega)}{\varepsilon a^2 b} x'' y'' = 1$$

but (93) $$\frac{b^2 \cos^2 \omega - a^2 \sin^2 \omega}{a^2 b^2} = -\frac{1}{\overline{Cd}^2}$$

hence

$$y''^2 - 2 \frac{Cd (a \sin \omega - b \cos \omega)}{\varepsilon a^2 b} y'' x'' = - \overline{Cd}^2$$

equation of the second degree, which resolved, will give

$$y'' = \frac{\overline{Cd}^2 (a \sin \omega - b \cos \omega)}{\varepsilon a^2 b} x'' \pm$$

$$\sqrt{\frac{\overline{Cd}^4 (a \sin \omega - b \cos \omega)^2}{\varepsilon^2 a^2 b^2} x''^2 - \overline{Cd}^2} \quad \ldots . (e)$$

the required equation of the hyperbola referred to the above mentioned system of axes.

Corollary. From this equation we may derive another property of the hyperbola; because let aa' be the conjugate diameter of dd'; the tangent a'T drawn from the extremity a' of the former diameter is (58, 93) parallel to the latter dd'. Now, considering the equation lately determined, it is to be observed first, that when the quantity under the radical sign is positive, to every value of x'' correspond two values of y'' different from each other; secondly, if the value of x'' is such as to make the quantity under the same sign equal to zero, to this peculiar value of x'' will correspond only one value of y''; and finally, when the value of x'' becomes such as to cause the quantity under the radical sign to be negative, no real value corresponds to y''. Now, since the tangent a'T drawn from a' is parallel to the new axis Cd of the ordinates, it is the ordinate of that point a' of the curve. But the tangent cannot meet the curve but in that point; hence to the abscissa CT corresponds only one real value of the ordinate, that is to say, the abscissa CT is that peculiar value of x'' by which the radical quantity becomes zero. But the radical quantity cannot become equal to zero, except in the case of

$$\frac{\overline{Cd}^4\ (a \sin \omega - b \cos \omega)^2}{\varepsilon^2\ a^4\ b^2}\ x''^2 = Cd^2$$

or, what is the same, except in the case of

$$\frac{\overline{Cd}^2\ (a \sin \omega - b \cos \omega)^2}{\varepsilon^2\ a^4\ b^2}\ x''^2 = 1$$

from which equation

$$x'' = \frac{1}{Cd}\ \frac{\varepsilon\ a^2\ b}{a \sin \omega - b \cos \omega}$$

but in the present case $x'' = \text{CT}$, hence

$$\text{CT} = \frac{1}{Cd}\ \frac{\varepsilon\ a^2\ b}{a \sin \omega - b \cos \omega}$$

and from (e) we have

$$y'' = \frac{\overline{Cd}^2 (a \sin \omega - b \cos \omega)}{\varepsilon\, a^2\, b}\, CT$$

hence $$y'' = Cd$$

or $$a'T = Cd$$

that is, the tangent drawn from any point of the curve, and terminated to the asymptote, is equal to the semi-diameter to which it is parallel.

Scholium. If we suppose that the asymptote $r'\, s'$ is referred to the system of axes Cs , Cb ; since the equation of $r'\, s'$ referred to the rectangular axes CB′, CA′ is $y = -\, tg\, \alpha'\, x$, or (91) $y = -\frac{b}{a}\, x$, it will be sufficient to substitute instead of x an y, the values given by the preceding formulas (o), to have the equation of the asymptote referred to the new system. And to obviate confusion between the co-ordinates x'' , y'' of the curve and those of the asymptote, we will term x''' , y''' the co-ordinates of the latter, and by substituting, we will obtain

$$x''' \sin (x''x) + y''' \sin (y''x) = -\frac{b}{a} (x''' \cos (x''x) + y''' \cos (y'x))$$

but we observed that

$$\sin (x''x) = \frac{b}{\sqrt{b^2 + a^2}}\, ,\;\; \cos (x''x) = \frac{a}{\sqrt{b^2 + a^2}}$$

$$\sin (y''x) = -\sin \omega\, ,\; \cos (y''x) = -\cos \omega$$

hence

$$\frac{|b}{\sqrt{b^2 + a^2}} - y''' \sin \omega = -x''' \frac{b}{\sqrt{b^2 + a^2}} + y''' \frac{b}{a} \cos \omega$$

consequently

$$y''' \frac{(b \cos \omega + a \sin \omega)}{a} = 2\, x''' \frac{b}{\sqrt{b^2 + a^2}} = \frac{2b\, x''}{\varepsilon\, a}$$

and
$$y''' = \frac{2b\,x'''}{\varepsilon\,(b \cos \omega + a \sin \omega)}$$

but supposing $x''' = \text{CT}$, that is $x''' = \frac{1}{\text{C}d}\,\frac{\varepsilon\, a^2\, b}{a \sin \omega - b \cos \omega}$;

$$y''' = \frac{1}{\text{C}d}\,\frac{2a^2\, b^2}{a^2 \sin^2 \omega - b^2 \cos^2 \omega}$$

hence (93)

$$y''' = \frac{1}{\text{C}d}\,.\,2\,\overline{\text{C}d}^{\,2} = 2\,.\,\text{C}d$$

that is to say, the ordinate TT$'$ corresponding to the abscissa CT is equal to the conjugate diameter dd', but Ca' = Cd = $\frac{1}{2}$ dd'. Hence the tangent TT$'$ contained between the asymptotes is bisected in two equal parts at a', the point of contact.

Equations of the surfaces generated by the lines of the second order revolved about their axes.

104. We observed (30) that when the equation of a curve described on the plane ZAV (fig. 70) moveable about AZ is represented by

$$v = f(z)\;;\;.\;.\;.\;.\;(o)$$

the equation of the surface generated by that line is

$$[f(z)]^2 = x^2 + y^2\;.\;.\;.\;.\;(o_1)$$

Now, let the lines of the second order be referred to the axes AZ, AV of the moveable plane ZAV, and first the parabola, supposing AZ to be the axis of the abscissas, and AV the axis of the ordinates, the equation (o) in this case (63 . (g)) will become

$$v = \sqrt{2\,px}$$

hence (o_1) $\qquad 2pz = y^2 + x^2$

the equation of the surface generated by the revolution of the parabola about its proper axis, and referred to the rectangular system (X , Y , Z.)

Secondly, let the ellipse be referred to the same axes AZ , AV , and let us first suppose that the transverse axis of the curve be taken on AV, the equation will then be (63 . (g)) $\frac{v^2}{a^2}+\frac{z^2}{b^2}=1$ and consequently $v^2=a^2-a^2\frac{z^2}{b^2}$, therefore the equation (o) will become

$$v=\sqrt{a^2-\frac{a^2}{b^2}z^2}=\frac{a}{b}\sqrt{b^2-z^2}$$

hence (o_1)

$$\frac{a^2}{b^2}(b^2-z^2)=x^2+y^2 \; .\; .\; .\; . \; (b)$$

the equation of the surface generated by the ellipse revolved about its conjugate axis.

If we suppose the transverse axis to be taken on AZ, then the equation of the ellipse is $\frac{z^2}{a^2}+\frac{v^2}{b^2}=1$; hence $v^2=b^2-\frac{b^2}{a^2}z^2$, and consequently ($o$)

$$v=\sqrt{b^2-\frac{b^2}{a^2}z^2}=\frac{b}{a}\sqrt{a^2-z^2}$$

which value substituted in (o_1) will give

$$\frac{b^2}{a^2}(a^2-z^2)=x^2+y^2 \; .\; .\; .\; . \; (b_1).$$

The equation of the surface generated by the ellipse revolved about its transverse axis.

Finally, let the hyperbola be referred to the same axes AZ , AV , and first suppose the transverse axis to be taken on AV. The equation will be (63 . (g)) $\frac{v^2}{a^2}-\frac{z^2}{b^2}=1$, from which $v^2=\frac{a^2}{b^2}(b^2+z^2)$ and consequently (o)

$$v=\frac{a}{b}\sqrt{b^2+z^2}$$

and (o_1)

$$\frac{a^2}{b^2}(b^2+z^2)=x^2+y^2 \quad . \; . \; . \quad (c).$$

equation of the surface generated by the hyperbola revolved about its conjugate axis.

Suppose the transverse axis on AZ, the equation will be $\frac{z^2}{a^2}-\frac{y^2}{b^2}=1$, hence $v^2=\frac{b^2}{a^2}(z^2-a^2)$; consequently (o) and (o_1)

$$v=\frac{a}{b}\sqrt{z^2-a^2}$$

$$\frac{b^2}{a^2}(z^2-a^2)=x^2+y^2 \quad . \; . \; . \; . \; . \quad (c_1)$$

equation of the surface generated by the hyperbola revolved about its transverse axis.

CONSTRUCTION OF EQUATIONS.

REMARKS.

105. The finding of the roots or unknown quantities of a determined equation by geometrical construction of right lines or curves, is called *construction of the equations.* Now such a resolution may be obtained by means of the intersections both of straight lines and curves; and if the equation is of the first degree, by the intersection of two straight lines; if the equation is of the second degree, by the intersections of the circle and the straight line; if the equation is of the third and fourth degrees, by the intersections of lines of the second order. So, after having spoken of the properties of these lines, it seems proper to apply them to the resolution of some problems which depend on the construction of equations of a degree superior to the second. Yet, before coming to this application, let us construct the equations of the first and second degree, which will afford a suitable introduction to the same application.

Construction of any determined equation of the first degree.

106. The general formula of any equation of the first degree is

$$x = \mathrm{C} \ldots (d)$$

in which x is given by the determined value of C. Suppose now

$$y = ax + b \ldots . (e)$$

any undetermined equation, that is, an equation in which the value of y depends on those attributed to x, or vice versa. Again, suppose we substitute in (e) the determined value of x given by (d), and let a' , b' be quantities different from a and b, but such as to give the same value of y (already given by (e)) by the following :

$$y = a'x + b' \ldots . . (e_1)$$

substituting here also $x = \mathrm{C}$. Now (e) and (e_1) are the equations of two straight lines ; and supposing these lines referred to the same system of axes, they must be different from each other ; because, since a and a' are different from each other, the inclinations or angles formed by the lines with the axis X are (10) different ; and since b and b' are also different from each other, the points of the axis of the ordinates met by the two lines are equally different from each other. But if we suppose an abscissa equal to C, since in this case the ordinate y given by both equations (e) , (e_1) is the same, the point determined by such co-ordinates must necessarily be that of the common intersection between the two straight lines. But two straight lines inclined to each other can meet in only one point ; hence, if after having derived from (d) and (e) the equation (e_1) in the manner described, and after having constructed the two lines with reference to the same system of axes, we draw from the point of common intersection the ordinate to the axis of the abscissas, the corresponding abscissa is the value of x given by (d). Now it appears that the resolution of the equations of the first degree does not require so long a process ; yet we know from this the principle on which depends the method of construction, which is always the same although applied in different manners.

Construction of any determined equation of the second degree.

107. Let $x^2 + mx = n \dots\dots (o)$

be any equation of the second degree. Supposing x_0 , x_1 to be the roots or values by which the equation is fulfilled according to the general properties of the equations, we will have

$$m = x_0 + x_1 \ , \ n = x_0 . x_1 \dots . (o_1)$$

Again, suppose AX (fig. 71) to represent the axis of the abscissas, and let Ab , Ab' be the linear values of x_0 , x_1; from b and b' draw be , $b'e'$ parallel to the axis Y, next with A as centre, and Ad' as radius, describe a circular arc $d'd$ . . . The equation of this circle referred to the rectangular axes AX , AY is

$$x^2 + y^2 = r^2 \dots (o_2)$$

an equation which will be fulfilled with the values x_0 , x_1 , or roots of the equation (o). Moreover, suppose the straight line BD to pass through d' and d the points of the circle corresponding to the abscissas x_0 , x_1 , and let the equation of the straight line be

$$y = ax + c \dots . (o_3)$$

It is evident that we cannot suppose in the two equations (o_2) , (o_3) the same co-ordinates, without supposing at once the abscissas x_0 , x_1 to be the roots of the former equation (o). Hence, any determined equation of the second degree may be resolved into two undetermined equations, the one of the circle, the other of the straight line fulfilled at once by the roots of the proposed equation. Therefore, the only thing to be done in order to resolve (o), is to determine the dependence of the constant quantities of (o_2) (o_3) upon the given m and n. It is now evident that as far as we suppose in the formulas (o_2) (o_3) the roots of the equation (o), every other equation derived from them must contain the same roots. In this supposition let us square the second of those equations, and let us substitute the value of y^2 deduced from it in the former (o_2), we will obtain

$$x^2 + a^2 x^2 + 2acx + c^2 = r^2$$

or

$$x^2 (1 + a^2) + 2acx = r^2 - c^2$$

and

$$x^2 + \frac{2ac}{1+a^2} x = \frac{r^2 - c^2}{1+a^2} \dots (o_4)$$

which is an equation of the second degree having the same roots as the proposed equation (o); and, consequently, according to the properties of the equations

$$\frac{2ac}{1+a^2} = x_0 + x_1 \ , \ \frac{r^2 - c^2}{1+a^2} = x_0 \, x_1$$

hence (o_1),

$$\left. m = \frac{2ac}{1+a^2} \ , \ n = \frac{r^2 - c^2}{1+a^2} \right\} \dots (o_5)$$

the equations between the known quantities m, n and the unknown a, c, r. Now, from these last formulas we have

$$c = \frac{m(1+a^2)}{2a}$$

$$r^2 = (1+a^2)\, n + c^2$$

$$= (1+a^2)\, n + \frac{m^2 (1+a^2)^2}{4a^2}$$

$$= \frac{1+a^2}{4a^2} [4a^2 n + m^2 (1+a^2)]$$

and substituting these values in (o_2), (o_3)

$$\left. \begin{array}{l} x^2 + y^2 = \dfrac{1+a^2}{4a^2} [4a^2 n + m^2 (1+a^2)] \\ y = ax + \dfrac{m(1+a^2)}{2a} \end{array} \right\} \dots (o_6)$$

the equations of the geometrical loci, from which intersections and corresponding abscissas we have the roots of the proposed equation (o). Let it be remarked that the constant a remains undetermined, and consequently the resolution may be performed with an infinite number of combinations. Now, suppose to be taken a determined value of a, that is, (since a is the tangent of the angle formed by the rectilinear locus and the axis X,) suppose a determined angle formed by the rectilinear locus with X, the last term of the second (o_6), as well as the second member of the first, shall be also determined. In this supposition let AX, AY (fig. 72) be the rectangular axis; since (10) $\frac{m(1+a^2)}{2a}$ is the linear length of the axis Y contained between the origin A and the point met by the rectilinear locus; let us take Al corresponding to that length, and from l let us draw ll', making with X an angle whose tangent is equal to a. Again, since $\frac{1+a^2}{4a^2}[4a^2n+m^2(1+a^2)]$ is the square of the radius of the circle, let us draw the perpendicular Ag from A to the rectilinear locus ll', and from the same A let us take on Ag a linear length corresponding to the radius. Three cases can happen in this construction: the linear length of the radius shall be either greater than Ag, or equal, or less. In the former case the rectilinear locus shall cut the circle in two different points; in the second it will be tangent; in the third no intersection will occur between the circle and the straight line. Therefore, in the formercase two different real roots fulfil the equation (o); in the second two equal real roots; in the third no real root can fulfil the equation.

PROBLEM I.

To find out the side of a cube whose solidity will be equal to twice that of a given cube.

108. This problem, so celebrated among ancient geometricians, consists in finding two mean proportionals between the side of the

given cube and the double of the same side. For let a be the side of the given cube, and let y , y_1 be two mean proportionals between a and $2a$, that is,

$$a : y :: y : y_1 :: y_1 : 2a$$

which proportion decomposed into two

$$a : y :: y : y_1$$

$$y : y_1 :: y_1 : 2a$$

gives first

$$y_1 = \frac{y^2}{a} \; . \; . \; . \; . \; . \; (o_0)$$

and

$$y = \frac{y_1{}^2}{2a} = \frac{y^4}{2a^3}$$

consequently,

$$y^3 = 2a^3 \; . \; . \; . \; . \; . \; (o_1)$$

that is to say, the cube of the first of the two mean proportionals y , y_1 is double the given cube. Consequently, if we are able to find the first mean proportional, the problem will be resolved. But let us consider the equation (o_1) without reference to the proportionals; and let us decompose that equation into two undetermined, in the following way: Suppose the real value of y, by which is fulfilled (o_1), to be substituted in the undetermined equation

$$y^2 = 4ax \; . \; . \; . \; . \; (o_2)$$

And again, let this peculiar value, as taken from (o_2), be substituted in (o_1), we will obtain

$$y \, . \, 4ax = 2a^3$$

hence,

$$xy = \frac{a^2}{2} \; . \; . \; . \; . \; . \; (o_3)$$

another undetermined equation. Now (o_2) and (o_3) cannot admit the same variables x , y, except when the variable y is that which fulfils (o_1); because, first, the equation (o_3) is derived from (o_1) and (o_2) in this supposition. And it is to be observed, that the relation between x and y afforded by (o_2) is such as to give a greater or less value to y, according to the greater or less value of x; therefore, if we suppose y_1 and x_1 to be two corresponding values of (o_2), and different from y, x, and if $xy = \frac{a^2}{2}$, the product $x_1\ y_1$ must necessarily be greater or less than $\frac{a^2}{2}$; consequently, as long as $\frac{a^2}{2}$ remains in the second member of (o_3), we cannot suppose there the same variables of (o_2) without supposing y to be that peculiar value which resolves the equation (o_1). Now from these observations it follows, that the geometrical loci (o_2), (o_3), if referred to the same system of axes, will cross each other in only one point, and the ordinate y, corresponding to that intersection, is the required value which fulfils the equation (o_1) and the required side of the cube. Let us now examine the nature of the geometrical loci (o_2), (o_3). The former is (63) a parabola having the parameter equal to $4a$; the second is an equilateral hyperbola (102) referred to the asymptotes, which (94, C.) are perpendicular to each other. Therefore, let AX, AY be (fig. 73) a system of rectangular axes, and let lAl' be the parabola (o_2), and $m\text{B}m'$ the hyperbola of which AX, AY are the asymptotes; from p, the point of intersection, draw pq perpendicular to AX, pq will be the side of the required cube.

PROBLEM II.

To divide a given angle into three equal parts.

109. The trisection of an angle is another problem whose solution has been much sought for by ancient mathematicians; yet, it is to be understood of the geometrical trisection of any angle, because, with regard to the peculiar case of the right angle the solution is easy. For suppose BAC (fig. 74) to be a right angle, and take Am at pleasure, then construct the equilateral triangle Amn, next draw Aq perpendicular to mn, the lines Ap, Aq divide BAC into three equal parts, because the angle $n\text{A}m = 60°$;

hence, $nAC = 30°$, and $nAo = oAm = 30°$. Moreover, the trisection of any angle, trigonometrically, is likewise easy; for let (fig. 75) BAC be any given angle, and with A as centre, and Am as radius, describe the circle mpn; it is plain that the division of the angle BAC into three equal parts, depends on the equal division of the arc nm. Now suppose np to be the chord corresponding to the third part of the arc mn, and draw Ap in the triangle nAp, the sides An, Ap, as well as the angle contained by them, are known quantities; hence, we may derive the third side np or chord, and, consequently, the trisection of the arc.

But let us come to the exact and geometrical division which will afford us an example of the construction of an equation of the fourth degree; but in order to proceed without interruption, it is requisite to mention that a and b being any two circular arcs of a circle, having the radius equal to unity, we know from trigonometry that

(1) $\sin(a+b) = \sin a \cos b + \sin b \cos a$

(2) $\sin a = 2 \sin \frac{1}{2} a \cos \frac{1}{2} a$

(3) $\cos a = 1 - 2 \sin^2 \frac{1}{2} a$

(4) $\cos^2 a = 1 - \sin^2 a$ (*)

Now let mpn $(= a)$ be the arc corresponding to the given angle, and mn $(= c)$ the chord. Let v be the third part of the arc a, and y the corresponding chord to be determined. Supposing the radius A$m = r$, we have from trigonometry $\frac{1}{2} c = r \sin \frac{1}{2} a$ $\frac{1}{2} y = r \sin \frac{1}{2} v$; hence,

$$\left.\begin{aligned} \sin \tfrac{1}{2} a &= \frac{C}{2r} \\ \sin \tfrac{1}{2} v &= \frac{y}{2r} \end{aligned}\right\} \ldots\ldots (o)$$

again, since v is the third part of a,

(*) Devies' Legend. may be consulted. Trig. xix, xx.

$$a = 3v$$

consequently $\qquad \frac{1}{2}a = \frac{3}{2}v = v + \frac{1}{2}v$

and

$$\sin \tfrac{1}{2} a = \sin (v + \tfrac{1}{2} v) \ldots . . (o_1)$$

but (1)

$$\sin (v + \tfrac{1}{2} v) = \sin v \cos \tfrac{1}{2} v + \cos v \sin \tfrac{1}{2} v$$

and (2), (3) $\sin v = 2 \sin \frac{1}{2} v \cos \frac{1}{2} v$, $\cos v = 1 - 2 \sin^2 \frac{1}{2} v$

hence $\sin (v + \frac{1}{2} v) = 2 \sin \frac{1}{2} v \cos^2 \frac{1}{2} v + (1 - 2 \sin^2 \frac{1}{2} v) \sin \frac{1}{2} v$

again (4) $\qquad \cos^2 \frac{1}{2} v = 1 - \sin^2 \frac{1}{2} v$

hence

$$\sin (v + \tfrac{1}{2} v) = 2 \sin \tfrac{1}{2} v\, (1 - \sin^2 \tfrac{1}{2} v) + (1 - 2 \sin^2 \tfrac{1}{2} v) \sin \tfrac{1}{2} v$$

$$= 2 \sin \tfrac{1}{2} v - 2 \sin^3 \tfrac{1}{2} v + \sin \tfrac{1}{2} v - 2 \sin^3 \tfrac{1}{2} v$$

$$= 3 \sin \tfrac{1}{2} v - 4 \sin^3 \tfrac{1}{2} v$$

therefore, substituting this value in (o_1)

$$\sin \tfrac{1}{2} a = 3 \sin \tfrac{1}{2} v - 4 \sin^3 \tfrac{1}{2} v$$

from which

$$4 \sin^3 \tfrac{1}{2} v + \sin \tfrac{1}{2} a - 3 \sin \tfrac{1}{2} v = o \ldots . (o_2)$$

and substituting the preceding values (o)

$$\frac{4\, y^3}{2^3 r^3} + \frac{C}{2\, r} - \frac{3\, y}{2\, r} = o$$

consequently

$$y^3 + cr^2 - 3\, yr^2 = o \ldots \ldots (o_3)$$

the equation in which the only unknown quantity is y, that is, the chord of the arc v. Therefore, the resolution of the problem depends upon the construction of that equation. But before proceeding to such construction, let us transform the equation into the following

$$y^4 - 3\,y^2\,r^2 + cy\,r^2 = o \;.\;.\;.\;.\; (o_4)$$

Again, supposing $y^2 = rx$, (provided x be properly determined,) we may substitute the value of y^2 in the first term of (o_4), hence

$$x^2 - 3\,y^2 + cy = o$$

an undetermined equation which affords the required value of y when there is substituted the corresponding value of x. Observe, now, that since we suppose $y^2 = rx$, we must also suppose $y^2 - rx = o$ and $\mu y^2 - \mu rx = o$ (μ being any number whatever,) consequently the last formula will remain unvaried by adding to it the difference $\mu y^2 - \mu rx$, and we may say of the equation

$$x^2 - 3\,y^2 + cy + \mu y^2 - \mu rx = o$$

that to some value of x must correspond the required y independent of any value of μ. But this equation represents different geometrical loci according to the different value of μ. Therefore, substituting to μ two different values, we shall obtain the expression of two different geometrical loci, which, if referred to the same system of axes, must admit of the same ordinate when the abscissa is properly chosen ; or, in other words, such geometrical loci must cross each other, and the ordinate corresponding to the point of intersection is the required value of y. Observe, now, that the different geometrical loci represented by the last equation, which may be reduced to the form

$$x^2 + (\mu - 3)\,y^2 + cy - \mu rx = o \;.\;.\;.\; (o_5)$$

are either the parabola, or ellipse, or hyperbola, according to that μ is equal, greater, or less than 3, and it represents the circle when $\mu = 4$; because in the first case it becomes

$$x^2 + cy - 3\,rx = o$$

and then [43 (i_1)] $A = 1$, $B = C = O$, consequently [45 . (i_{10})]

$P_1 = \frac{1}{2} + \frac{1}{2} = 1$, $P_2 = \frac{1}{2} - \frac{1}{2} = o$, and [47 . 1] the curve admits of only one axis, which is the case of the parabola.

In the second supposition, let, for instance, $\mu = 5$: the formula (o_5) becomes

$$x^2 + 2\,y^2 + cy - 5\,rx = o$$

and then $A = 1$, $B = 2$, $C = o$, consequently $P_1 = 2$, $P_2 = 1$ and $C^2 - AB = -2$, that is to say, the curve (47) admits of two axes, and it does not admit (57) of asymptotes. Therefore it is an ellipse.

In the third case supposing for example $\mu = 2$ the same (o_5) becomes

$$x^2 - 1\,.\,y^2 + cy - 2\,rx = o$$

hence $A = 1$, $B = -1$, $C = o$, consequently $P_1 = 1$, $P_2 = -1$, and $C^2 - AB = +1$, that is, the curve admits of two axes as well as the asymptotes, which is the quality exclusively proper to the hyperbola.

Suppose, finally, $\mu = 4$ in which case (o_5) becomes

$$x^2 + y^2 + cy - 4\,rx = o$$

and $A = 1$, $B = 1$, $C = o$, therefore $P_1 = P_2 = 1$, and $P_1 - P_2 = o$ which (47 . 48) is the case of the circle.

Suppose, now, that the construction or geometrical determination of the unknown y is to be performed with the circle and ellipse. To this end let us transform the preceding equation of the ellipse into the following

$$x^2 - 5\,rx + \left(\frac{5r}{2}\right)^2 + 2y^2 + cy + \left(\frac{c}{2\sqrt{2}}\right)^2 = \left(\frac{5\,r}{2}\right)^2 + \left(\frac{c}{2\sqrt{2}}\right)^2$$

or $$\left(x - \frac{5\,r}{2}\right)^2 + \left(y\sqrt{2} + \frac{c}{2\sqrt{2}}\right)^2 = \left(\frac{5\,r}{2}\right)^2 + \left(\frac{c}{2\sqrt{2}}\right)^2$$

from which, since

$$\left(y\sqrt{2}+\frac{c}{2\sqrt{2}}\right)^2=\left[\sqrt{2}\left(y+\frac{c}{2\sqrt{2}\cdot\sqrt{2}}\right)\right]^2$$

$$=\left(\sqrt{2}\right)^2\left(y+\frac{c}{2\,.\,2}\right)^2=2\left(y+\frac{c}{4}\right)^2;$$

$$\frac{\left(x-\frac{5\,r}{2}\right)^2}{\frac{25}{4}r^2+\frac{c^2}{8}}+\frac{2\left(y+\frac{c}{4}\right)^2}{\frac{25}{4}r^2+\frac{c^2}{8}}=1$$

and, finally,

$$\frac{\left(x-\frac{5\,r}{2}\right)^2}{\frac{25}{4}r^2+\frac{c^2}{8}}+\frac{\left(y-\frac{c}{4}\right)^2}{\frac{25}{8}r^2+\frac{c^2}{16}}=1\;.\;.\;.\;.\;.\;(o_6)$$

Let us now transform the equation of the circle into

$$x^2-4\,rx+(2\,r)^2+y^2+cy+\left(\frac{c}{2}\right)^2=(2\,r)^2+\left(\frac{c}{2}\right)^2$$

or $\quad (x-2\,r)^2+\left(y+\frac{c}{2}\right)^2=4\,r^2+\frac{c^2}{4}\;.\;.\;.\;.\;.\;(o_7)$

If now the geometrical loci represented by (o_6) (o_7) are referred to the system AX , AY (fig. 76) of rectangular axes ; let us take A′, a point of which the co-ordinates are $x=\mathrm{A}m=2\,r$, $y=m\mathrm{A}'=-\frac{c}{2}$, and with A′ as centre, and $\sqrt{4\,r^2+\frac{c^2}{4}}$ as radius, describe the circle *lrds*, which is, that corresponding to (o_7) and referred to the axes AX , AY , (49). In the same way take A″ of which the co-ordinates with reference to AX , AY are $x=\mathrm{A}n=\frac{5\,r}{2}$, $y=n\mathrm{A}''=-\frac{c}{4}$: draw A″X″ , A″Y″ parallel to AX , AY , and taking $A''p = \sqrt{\frac{25}{4}r^2 + \frac{c^2}{8}}$, $A''q = \sqrt{\frac{25}{8}r^2 + \frac{c^2}{16}}$ as semi-axes, describe the ellipse *pqts*. This is the ellipse corresponding to the equation (o_6), and referred to the axes AX , AY. Because, suppose the ellipse referred to the axes $A''X''$, $A''Y''$, the corresponding equation is $\frac{x''^2}{\frac{25}{4}r^2 + \frac{c^2}{8}} + \frac{y''^2}{\frac{25}{4}r^2 + \frac{c^2}{16}} = 1$; but (8) $x'' = x - An = x - \frac{5r}{2}$; $y'' = y + A''n = y + \frac{c}{4}$, and substituting, we derive the equation (o_6).

If now from the intersections r , s we draw rf , sg perpendicular to the axis X, either of them will be the required chord v. It is to be observed, that since the equation is of the fourth degree, there must be four intersections, except when the roots of the equation are equal to each other.

BOOK IV.

SURFACES OF THE SECOND ORDER.

REMARKS.

110. According to the observations made at the beginning of the preceding book, (42 . 43), and according to analogy, the equation of the second degree

$$Ax^2 + By^2 + Cz^2 + 2\,Dyz + 2\,Exz + 2\,Fxy + 2\,Gx + 2\,Hy + 2\,Kz = Q \ldots (h_1)$$

containing the rectangular co-ordinates x, y, z, and the constant quantities A , B , C , ... is the most general formula of the surfaces of the second order. Hence the determination of the general as well as peculiar properties of these geometrical loci depends upon the discussion of the same equation (h_1) which may still be reduced to the same form as that given [(44) i_3] of the general equation of the lines.

Simpler form given to the general equation and common properties.

111. Let (fig. 77) AX , AY , AZ be any system of rectangular axes, and let $A'm$ be any straight line in space. Draw from A′, A′X′, A′Y′, A′Z′ parallel to the axes, and produce the same parallel, so as to meet the planes XAY , XAZ , YAZ in P′, Q′, R′. Again, draw from m to the same planes mP , mR , mQ parallel to A′P′, A′Q′, A′R′, and suppose mP to meet X′A′Y′ in p , mR to meet Y′A′Z′ in r , and mQ to meet X′A′Z′ in q , then evidently

$$A'P' = pP\,, \quad A'Q' = qQ\,, \quad A'R' = rR$$

because they are lines contained by parallel planes. Again,

$$mP = mp + A'P', \quad mQ = mq + A'Q', \quad mR = mr + A'R'$$

now
$$mp = A'm \cos A'mp = A'm \cos mA'Z'$$
$$mq = A'm \cos A'mq = A'm \cos mA'Y'$$
$$mr = A'm \cos A'mr = A'm \cos mA'X'$$

hence
$$\left.\begin{aligned} mP &= A'P' + A'm \,.\, \cos mA'Z' \\ mQ &= A'Q' + A'm \,.\, \cos mA'Y' \\ mR &= A'R' + A'm \,.\, \cos mA'X' \end{aligned}\right\} \ldots (e_1)$$

and
$$\left.\begin{aligned} A'P' &= mP - A'm \,.\, \cos mA'Z' \\ A'Q' &= mQ - A'm \,.\, \cos mA'Y' \\ A'R' &= mR - A'm \,.\, \cos mA'X' \end{aligned}\right\} \ldots (e_2)$$

Suppose, now, (fig. 78) any two points n, n' of any surface of the second order referred to a system of rectangular axes. The straight line nn' $(= 2g)$ will of course be a chord of that surface; let this line be divided into two equal parts in o, and let $x_0 = or$, $y_0 = oq$, $z_0 = op$ be the co-ordinates of the point o; and suppose the angles formed by nn' with the axes X, Y, Z to be represented by α, α', α''. The co-ordinates ns, nt, nv; $n's'$, $n't'$, $n'v'$ of the points n, n' are the co-ordinates x, y, z of any two points of the surface. Considering, now, on' or g with reference to the axes, we will have (e_1)

$$z = z_0 + g \cos \alpha''$$
$$y = y_0 + g \cos \alpha'$$
$$x = x_0 + g \cos \alpha$$

Considering on we will have (e_2)

$$z = z_0 - g \cos \alpha''$$
$$y = y_0 - g \cos \alpha'$$
$$x = x_0 - g \cos \alpha$$

Substituting, now, in the different terms of (h_1) the values given by these two systems of equations, we will derive

$$Ax^2 = Ax_\circ^2 + 2Agx_\circ \cos \alpha + Ag^2 \cos^2 \alpha$$

And

$$Ax^2 = Ax_\circ^2 - 2Agx_\circ \cos \alpha + Ag^2 \cos^2 \alpha$$

$$By^2 = By_\circ^2 + 2Bgy_\circ \cos \alpha' + Bg^2 \cos^2 \alpha'$$

And

$$By^2 = By_\circ^2 - 2Bgy_\circ \cos \alpha' + Bg^2 \cos^2 \alpha'$$

$$Cz^2 = Cz_\circ^2 + 2Cgz_\circ \cos \alpha'' + Cg^2 \cos^2 \alpha''$$

And

$$Cz^2 = Cz_\circ^2 - 2Cgz_\circ \cos \alpha'' + Cg^2 \cos^2 \alpha''$$

$$2Dyz = 2Dy_\circ z_\circ + 2Dgy_\circ \cos \alpha'' + 2Dgz_\circ \cos \alpha' + 2Dg^2 \cos \alpha' \cos \alpha''$$

And

$$2Dyz = 2Dy_\circ z_\circ - 2Dgy_\circ \cos \alpha'' - 2Dgz_\circ \cos \alpha' + 2Dg^2 \cos \alpha' \cos \alpha''$$

$$2Exz = 2Ex_\circ z_\circ + 2Egx_\circ \cos \alpha'' + 2Egz_\circ \cos \alpha + 2Eg^2 \cos \alpha \cos \alpha''$$

And

$$2Exz = 2Ex_\circ z_\circ - 2Egx_\circ \cos \alpha'' - 2Egz_\circ \cos \alpha + 2Eg^2 \cos \alpha \cos \alpha''$$

$$2Fxy = 2Fx_\circ y_\circ + 2Fgx_\circ \cos \alpha' + 2Fgy_\circ \cos \alpha + 2Fg^2 \cos \alpha \cos \alpha'$$

And

$$2Fxy = 2Fx_\circ y_\circ - 2Fgx_\circ \cos \alpha' - 2Fgy_\circ \cos \alpha + 2Fg^2 \cos \alpha \cos \alpha$$

$$2Gx = 2Gx_\circ + 2Gg \cos \alpha$$

And

$$2Gx = 2Gx_\circ - 2Gg \cos \alpha$$

$$2Hy = 2Hy_\circ + 2Hg \cos \alpha'$$

And

$$2Hy = 2Hy_\circ - 2Hg \cos \alpha'$$

$$2Kz = 2Kz_\circ + 2Kg \cos \alpha''$$

And

$$2Kz = 2Kz_\circ - 2Kg \cos \alpha''$$

Making, compendiously

$$\left.\begin{array}{r}A\cos^2\alpha+B\cos^2\alpha'+C\cos^2\alpha''+2D\cos\alpha'\cos\alpha''+\\ 2E\cos\alpha\cos\alpha''+2F\cos\alpha\cos\alpha'=R\\ (Ax_o+Fy_o+Ez_o+G)\cos\alpha+(Fx_o+By_o+Dz_o+H)\cos\alpha'\\ +(Ex_o+Dy_o+Cz_o+K)\cos\alpha''=R'\\ Ax_o^2+By_o^2+Cz_o^2+2Dy_oz_o+2Ex_oz_o+2Fx_oy_o+\\ 2Gx_o+2Hy_o+2Kz_o=R''\end{array}\right\}(h_2)$$

We will obtain from (h_1), and the preceding substitutions,

$$\left.\begin{array}{r}Rg^2+2R'g+R''=Q\\ Rg^2-2R'g+R''=Q\end{array}\right\}(h_3)$$

Or, representing both equations by the only one, $Rg^2\pm 2R'g+R''=Q$, which has [44 (i_3)] the same form as that corresponding to the lines of the second order. And from the discussion of the formulas (h_3) we are enabled to derive the properties of the surfaces in the same way in which, from the discussion of (i_3), we deduced the properties of the lines.

Diametral plane.

112. The second (h_3), subtracted from the first, gives

$$4R'g=o,\quad\text{or}\quad R'=o;$$

and, consequently, (h_2)

$$(Ax_o+Fy_o+Ez_o+G)\cos\alpha+(Fx_o+By_o+Dz_o+H)\cos\alpha'$$
$$+(Ex_o+Dy_o+Cz_o+K)\cos\alpha''=o$$

from which

$$x_o(A\cos\alpha+F\cos\alpha'+E\cos\alpha'')+y_o(F\cos\alpha+B\cos\alpha'+D\cos\alpha'')$$
$$+z_o(E\cos\alpha+D\cos\alpha'+C\cos\alpha'')+G\cos\alpha+H\cos\alpha'+K\cos\alpha''=o$$

and, consequently,

$$z_\circ [\text{E} \cos \alpha + \text{D} \cos \alpha' + \text{C} \cos \alpha''] =$$

$$- x_\circ [\text{A} \cos \alpha + \text{F} \cos \alpha' + \text{E} \cos \alpha'']$$
$$- y_\circ [\text{F} \cos \alpha + \text{B} \cos \alpha' + \text{D} \cos \alpha'']$$
$$- \quad [\text{G} \cos \alpha + \text{H} \cos \alpha' + \text{K} \cos \alpha'']$$

and making, for brevity,

$$-\frac{\text{A} \cos \alpha + \text{F} \cos \alpha' + \text{E} \cos \alpha''}{\text{E} \cos \alpha + \text{D} \cos \alpha' + \text{C} \cos \alpha''} = m$$

$$-\frac{\text{F} \cos \alpha + \text{B} \cos \alpha' + \text{D} \cos \alpha''}{\text{E} \cos \alpha + \text{D} \cos \alpha' + \text{C} \cos \alpha''} = n$$

$$-\frac{\text{G} \cos \alpha + \text{H} \cos \alpha' + \text{K} \cos \alpha''}{\text{E} \cos \alpha + \text{D} \cos \alpha' + \text{C} \cos \alpha''} = q$$

we will have

$$z_\circ = mx_\circ + ny_\circ + q \; . \; . \; . \; . \; (h_4)$$

Let it be now remarked that m, n, q depend only upon the constant quantities A, B, C, . . . and the angles α, α', α'' formed by the chord nn' with the axes X, Y, Z. Therefore the same equation (h_4) would have been obtained, considering any chord parallel to nn', and the only difference would be in the co-ordinates $x_\circ$, $y_\circ$, $z_\circ$ of the middle point of the chord. Therefore the formula (h_4) is the equation of the series of the middle points of a system of parallel chords. But (29) the geometrical locus corresponding to (h_4) is a plane; hence any system of parallel chords in any surface of the second is bisected by a plane, which is termed *diametral plane;* and in the case in which the system of chords is perpendicularly bisected, the plane then is called *principal plane.*

PROPOSITION.

In every surface of the second order there is always at least one system of chords bisected by a principal plane.

113. The equations of the chord nn' passing through o and referred to the axes X, Y, Z are [(35) (40)]

$$x - x_\circ = \frac{\cos\alpha}{\cos\alpha'}(y - y_\circ)$$

$$x - x_\circ = \frac{\cos\alpha}{\cos\alpha''}(z - z_\circ)$$

hence, the equations of the straight line passing through the origin of the co-ordinates, and parallel to nn', will be

$$x = \frac{\cos\alpha}{\cos\alpha'}y \,,\; x = \frac{\cos\alpha}{\cos\alpha''}z$$

Supposing now the system perpendicularly bisected in this case, we will have

$$\left.\frac{\cos\alpha}{\cos\alpha'} = \frac{m}{n} \,,\; \frac{\cos\alpha}{\cos\alpha''} = -m \right\} \ldots (h_5)$$

for the equation (h_4) may be transformed into

$$x_\circ = \frac{1}{m}z_\circ - \frac{n}{m}y_\circ - \frac{q}{m}$$

and comparing this equation and those of the chord passing through the origin of the co-ordinates with the equations of the plane (36), and of the perpendicular drawn to it, we deduce $\frac{\cos\alpha}{\cos\alpha'} = -1 : -\frac{n}{m} \,,\; \frac{\cos\alpha}{\cos\alpha''} = -1 : \frac{1}{m}$. Hence, if for every surface of the second order there is such a system of parallel chords, whose angles α, α', α'' with the axes, fulfil the conditions (h_5), every surface admits of at least one principal plane. But the angles α, α', α'', which any chord makes with the axes, must be such as to give (25, C. V.)

$$\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$$

hence, the real values of α, α', α'' by which are to be fulfilled (h_5), must be derived from this last equation. To know now if this is really the case with regard to any surface of the second order, let $\frac{\cos\alpha}{\cos\alpha'}$ and $\frac{\cos\alpha}{\cos\alpha''}$ be represented by $\frac{1}{v}$ and $\frac{1}{v'}$, so as to have

$$\cos\alpha' = v\cos\alpha\,,\ \cos\alpha'' = v'\cos\alpha$$

and since

$$\frac{m}{n} = \frac{A\cos\alpha + F\cos\alpha' + E\cos\alpha''}{F\cos\alpha + B\cos\alpha' + D\cos\alpha''}$$

$$-m = \frac{A\cos\alpha + F\cos\alpha' + E\cos\alpha''}{E\cos\alpha + D\cos\alpha' + C\cos\alpha''}$$

so by substituting

$$\left.\frac{m}{n} = \frac{A + Fv + Ev'}{F + Bv + Dv'}\,,\ -m = \frac{A + Fv + Ev'}{E + Dv + Cv'}\right\} \ldots (l)$$

and the equations (h_5) will become

$$\frac{1}{v} = \frac{A + Fv + Ev'}{F + Bv + Dv'}\,,\ \frac{1}{v'} = \frac{A + Fv + Ev'}{E + Dv + Cv'}$$

the former of which gives

$$F + Bv + Dv' = (A + Fv + Ev')\,v$$

hence,

$$v' = \frac{Av + Fv^2 - Bv - F}{D - Ev} \ldots . (l_1)$$

the second gives

$$E + Dv + Cv' = (A + Fv + Ev')\,v'$$

hence,

$$Ev'^2 + [Fv + A - C]\,v' = Dv + E$$

or

$$Ev'^2 + \frac{(Fv + A - C)\,(D - Ev)}{D - Ev}\,v' = Dv + E$$

and substituting the value of v' derived from the former

$$\frac{E\,[Av + Fv^2 - Bv - F]^2}{(D - Ev)^2} +$$

$$\frac{[Fv + A - C]\,[D - Ev]\,[Av + Fv^2 - Bv - F]}{(D - Ev)^2} = Dv + E$$

hence, (*)

---

*The same formula may be transformed into

$$E\,[Fv^2 + (Av - Bv - F))]^2 + [FDv + AD - CD - EFv^2 - AEv + CEv]$$
$$[Av + Fv^2 - Bv - F] - (Dv + E)\,[D - Ev]^2 = o$$

and considering each term separately,

$$E[Fv^2 + (Av - Bv - F))]^2 = EF^2v^4 + 2EF\,(Av - Bv - F)v^2 + E\,(Av - Bv - F)^2$$

$$[FDv + AD - CD - EFv^2 - AEv + CEv]\,[Av + Fv^2 - Bv - F] =$$
$$[-EFv^2 + (FD - AE + CE)v + AD - CD]\,[Fv^2 + Av - Bv - F]$$
$$= -EF^2v^4 + F\,(FD - AE + CE)\,v^3 + F\,(AD - CD)\,v^2 + (Av -$$
$$Bv - F)\,[-EFv^2 + (FD - AE - CE)\,v + AD - CD]$$

the last term,

$$(-Dv + E)\,[D - Ev]^2 = -[DE^2v^3 - (2D^2E - E^3)\,v^2 + (D^3 - 2E^2D)\,v - D^2E]$$

making the sum

$$[F\,(FD - AE + CE) - DE^2]\,v^3 + (Av - Bv - F)\,[EFv^2 + E\,(Av - Bv - F) + (FD - AE + CE)\,v + AD - CD] + [F\,(AD - CD) + (2D^2E - E^3)]\,v^2 + (D^3 - 2E^2D)\,v + D^2E = o$$

the second term of which may be further transformed into

$$[(A - B)\,v - F]\,[EFv^2 + (FD + CE - EB)\,v + AD - CD - EF]$$

or

$$(A - B)\,EFv^3 + (A - B)\,(FD + CE - EB)\,v^2 + (A - B)\,(AD - CD -$$
$$EF)\,v - EF^2v^2 - F\,(FD + CE - EB)\,v - F\,(AD - CD - EF)$$

hence the same sum shall become

$$(F^2D - BEF + CEF - DE^2)\,v^3 + [(A - B)\,(FD + CE - EB) + F\,(AD - CD - EF) + E\,(2D^2 - E^2)]\,v^2 + [EF\,(2B - A - C) + AD\,(A - C - B) + D\,(D^2 - 2E^2 - F^2 + CBD)]\,v - F\,(AD - CD - EF) + D^2E = o$$

and consequently

$$v^3 + \&c. \ldots . = o$$

$$v^3 + \frac{[(A-B)(FD+CE-EB)+F(AD-CD-EF)+E(2D^2-E^2)]}{F^2D-BEF+CEF-DE^2}v^2$$

$$+\frac{[EF(2B-A-C)+AD(A-C-B)+D(D^2-2E^2-F^2)+CBD]}{F^2D-BEF+CEF-DE^2}v$$

$$-\frac{F(AD-EF-CD)+D^2E}{F^2D-BEF+CEF-DE^2}=o$$

An equation of the third degree, which, according to the general properties of equations, may be resolved by a real value of v, which substituted in (l_1) will give a corresponding real value of v'; hence, the two equations (l) may be resolved by real values of v and v', or what is the same, there are always such real angles α, α', α'', by which the formulas (h_4) may be fulfilled. It remains now for us to investigate, if in every case the same angles can fulfil the other condition, $\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$; that is to say, if the equations

$$\frac{\cos\alpha}{\cos\alpha'} = \frac{1}{v}, \frac{\cos\alpha}{\cos\alpha''} = \frac{1}{v'}$$

$$\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$$

are at once fulfilled by the same angles α, α', α''. To this end observe, that from the first and second of these equations we have $\cos^2\alpha' = v^2\cos^2\alpha$, $\cos^2\alpha'' = v'^2\cos^2\alpha$; and substituting these values in the third

$$\cos^2\alpha\,(1+v^2+v'^2) = 1$$

hence,

$$\cos\alpha = \frac{1}{\sqrt{1+v^2+v'^2}}$$

Substituting, now, this last value in the two former equations, we obtain

$$\cos\alpha' = \frac{v}{\sqrt{1+v^2+v'^2}}, \quad \cos\alpha'' = \frac{v'}{\sqrt{1+v^2+v'^2}}$$

which are the required values; because v and v' being real values, the angles α, α', α'' must be also real; and dividing $\cos\alpha$ by $\cos\alpha'$, and $\cos \alpha$ by $\cos \alpha''$, we have in the former case $\frac{1}{v}$, in the latter $\frac{1}{v'}$, and the sum of the squares $\cos^2 \alpha$, $\cos^2 \alpha'$, $\cos^2 \alpha''$, equal to the unity. Hence, the conditions required may be always fulfilled, and in every surface of the second order there is a system of chords perpendicularly bisected by a principal plane.

Tangent plane and normal line.

114. Let l (fig. 79) be any point of any surface of the second order, through which conceive a plane passing tangent the curve surface. Describe on that plane the straight line mn passing through the same point l, which of course must be a straight line tangent the surface in l. Let α, α', α'' be the angles formed by mn with the axes X, Y, Z; and observe that the diametral plane bisecting the system of chords parallel to mn must pass through l. For, suppose any chord parallel to mn to be moved in such a manner as to remain constantly parallel to itself, and so moved as to become coincident with mn, it is plain that as far as the chord is below mn, its middle point shall be on the diametral plane, and that the middle point, as well as the extremities of the chord, shall unite in a common point when the chord becomes tangent. Hence l, the point of contact of the tangent mn, is a point of the diametral plane bisecting the system of chords, whose angles with X, Y, Z are α, α', α'', but (h_4) is the equation of this plane; hence the co-ordinates x, y, z of l must fulfil the same equation; that is, we will have

$$z = mx + ny + q \ldots . (o)$$

Suppose the co-ordinates of mn to be represented by X, Y, Z; since mn must pass through l, the equations of mn will be (40)

$$\frac{X - x}{\cos \alpha} = \frac{Y - y}{\cos \alpha'}, \quad \frac{X - x}{\cos \alpha} = \frac{Z - z}{\cos \alpha''}$$

hence, $$\frac{(X-x)^2}{\cos^2\alpha} = \frac{(Y-y)^2}{\cos^2\alpha'} = \frac{(Z-z)^2}{\cos^2\alpha''}$$

and supposing these ratios to be represented by r, $(X-x)^2 = r\cos^2\alpha$, $(Y-y)^2 = r\cos^2\alpha'$, $(Z-z)^2 = r\cos^2\alpha''$; and consequently,

$$\frac{(X-x)^2 + (Y-y)^2 + (Z-z)^2}{\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha''} = r$$

each of the preceding ratios is equal to the last in which it is to be remarked that the denominator (25 . c . 5) is equal to unity. Hence

$$\frac{(X-x)^2}{\cos^2\alpha} = (X-x)^2 + (Y-y)^2 + (Z-z)^2$$

$$\frac{(Y-y)^2}{\cos^2\alpha'} = (X-x)^2 + (Y-y)^2 + (Z-z)^2$$

$$\frac{(Z-z)^2}{\cos^2\alpha''} = (X-x)^2 + (Y-y)^2 + (Z-z)^2$$

and consequently

$$\cos\alpha = \frac{X-x}{\sqrt{(X-x)^2 + (Y-y^2) + (Z-z)^2}}$$

$$\cos\alpha' = \frac{Y-y}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}$$

$$\cos\alpha'' = \frac{Z-z}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}$$

hence, by substituting (112)

$$m = -\frac{\mathrm{A}(X-x) + \mathrm{F}(Y-y) + \mathrm{E}(Z-z)}{\mathrm{E}(X-x) + \mathrm{D}(Y-y) + \mathrm{C}(Z-z)}$$

$$n = -\frac{\mathrm{F}(X-x) + \mathrm{B}(Y-y) + \mathrm{D}(Z-z)}{\mathrm{E}(X-x) + \mathrm{D}(Y-y) + \mathrm{C}(Z-z)}$$

$$q = -\frac{\mathrm{G}(X-x)+\mathrm{H}(Y-y)+\mathrm{K}(Z-z)}{\mathrm{E}(X-x)+\mathrm{D}(Y-y)+\mathrm{C}(Z-z)}$$

and substituting these values in (o)

$$\begin{aligned}\mathrm{E}z(X-x)+\mathrm{D}z(Y-y)+\mathrm{C}z(Z-z) = \\ &-[\mathrm{A}x(X-x)+\mathrm{F}x(Y-y)+\mathrm{E}x(Z-z)] \\ &-[\mathrm{F}y(X-x)+\mathrm{B}y(Y-y)+\mathrm{D}y(Z-z)] \\ &-[\mathrm{G}\ (X-x)+\mathrm{H}\ (Y-y)+\mathrm{K}\ (Z-z)]\end{aligned}$$

hence

$$\begin{aligned}(Z-z)[\mathrm{C}z+\mathrm{E}x+\mathrm{D}y+\mathrm{K}] = -(X-x)[\mathrm{A}x+\mathrm{F}y+\mathrm{E}z+\mathrm{G}] \\ -(Y-y)[\mathrm{F}x+\mathrm{B}y+\mathrm{D}z+\mathrm{H}]\end{aligned}$$

and making compendiously

$$-\frac{\mathrm{A}x+\mathrm{F}y+\mathrm{E}z+\mathrm{G}}{\mathrm{C}z+\mathrm{E}x+\mathrm{D}y+\mathrm{K}} = m',\ -\frac{\mathrm{F}x+\mathrm{B}y+\mathrm{D}z+\mathrm{H}}{\mathrm{C}z+\mathrm{E}x+\mathrm{D}y+\mathrm{K}} = n'$$

we will have

$$(Z-z) = m'(X-x)+n'(Y-y) \ \ldots\ldots\ (h_6).$$

The equation of the tangent plane. For in the same manner as we considered the line *mn* of that plane, we could consider any other line of the same plane passing through l, and the difference would have been only in the angles formed by the same lines with the axes. But the equation (h_6) is independent of such angles; hence the co-ordinates X, Y, Z may be those of any straight line on the tangent plane passing through l, that is, the equation (h_6) is the equation of the tangent plane in the point (x, y, z).

115. The perpendicular line drawn to the tangent plane, and passing through the point of contact, is called the *normal* cor-

responding to that plane. Now, since (h_6) may be transformed into

$$(X-x)=\frac{1}{m'}(Z-z)-\frac{n'}{m'}(Y-y)$$

by comparing the present equation with that already considered, (41), and supposing X', Y', Z' to be the co-ordinates of the normal, we will obtain

$$\left.\begin{array}{l} X'-x=\quad\frac{m'}{n'}(Y'-y) \\ X'-x=-m'(Z'-z) \end{array}\right\}\ldots(h_7)$$

for the equations of the normal.

Division of the surfaces of the second order in surfaces having a centre and surfaces without a centre.

Modifications of the most general formula.

116. We proved (113) that in every surface of the second order there is at least a principal plane. Suppose, now, the system of axes AX, AY, AZ to be taken (fig. 80) in such a manner as to have the axes X, Z on that principal plane. In this case the general equation (h_1) shall be converted into

$$A'x^2+B'y^2+C'z^2+2E'xz+2G'x+2K'z=Q'\ldots.(o)$$

in which there are no longer any terms involving y to the first power. For, supposing that there are such terms, and considering the late equation with regard to y, we could represent it by

$$y^2+2Sy-T=o$$

while, if there are no terms with y to the first power, the same equation shall take the form

$$y^2-T=o.$$

Of which equations the first resolved according to the known ule, affords

$$y = -s \pm \sqrt{s^2 + T}$$

the second $\quad y = \pm \sqrt{T}$

Now s and T depend on the variables x , z , hence to every different value of these variables correspond two values for y, which are different from each other, if we suppose in the equation (o) some terms with y to the first power; and equal if there are not such terms. But the co-ordinates y are the system of chords bisected by the principal plane ZAX; hence, supposing the axes AX, AZ on this plane, the equation (o) becomes the general equation of the surfaces of the second order.

Suppose, now, a plane Y′A′Z′ of which A′Z′ , A′Y′ are the intersections with XAZ , XAY. Let A′X′ be drawn perpendicular to A′Z′, and on the same plane ZAX. Supposing, moreover, the intersection A′Y′ parallel to the axis AY, so that A′Y′ be perpendicular to AX , the lines A′X′ , A′Y′ , A′Z′ will constitute a system of rectangular axes, and the relation between the co-ordinates of any point with regard to X , Y , Z , and with regard to X′ , Y′ , Z′ will be given by the known formulas (27 , C . II). Still it is to be remarked, that since the axis Y′ is parallel to the corresponding axis Y, we will have

$$\cos(xy') = \cos(yx') = \cos(yz') = \cos(zy') = \cos 90^\circ = o$$

$$\cos(yy') = 1$$

again, supposing A′L parallel to AZ , since XA′Z′ = Z′A′X′ + X′A′X = XA′L + LA′Z′ ;

$$\cos(xz') = \cos(90^\circ + (xx')) = -\sin(xx')$$

$$= \cos(90^\circ + (zz')) = -\sin(zz')$$

and since $\quad$ LA′X = LA′X — XA′X′ = X′A′Z′ — LA′Z′

$$\cos(zx') = \cos(90^\circ - (xx')) = \sin(xx')$$

$$= \cos(90^\circ - (zz')) = \sin(zz')$$

12

and since $XA'X' = LA'Z'$

$$\cos(zz') = \cos(xx')$$

the first of the co-ordinates x_o, y_o, z_o of the new origin A′ with reference to A, is equal to AA′, the other two equal to zero: hence to have the co-ordinates x, y, z of any point, with regard to the former system, given by x', y', z', with regard to the latter; it is sufficient to substitute the preceding value in the formulas (27) of transformation. In this manner we will obtain

$$x = x_o + x' \cos(xx') - z' \sin(xx')$$

$$y = y'$$

$$z = x' \sin(xx') + z' \cos(xx')$$

from which

$$x^2 = x_o{}^2 + 2x_o x' \cos(xx') - 2x_o z' \sin(xx') + x_1{}^2 \cos^2(xx') -$$
$$2x'z' \cos(xx') \sin(xx') + z'^2 \sin^2(xx')$$

$$xz = x_o x' \sin(xx') + x_o z' \cos(xx') + x'^2 \sin(xx') \cos(xx') +$$
$$x'z' [\cos^2(xx') - \sin^2(xx')] + z'^2 \sin(xx') \cos(xx')$$

$$y^2 = y'^2$$

$$z^2 = x'^2 \sin^2(xx') + 2x'z' \sin(xx') \cos(xx') + z'^2 \cos^2(xx').$$

Substituting, now, these values in the preceding formula (o), we will derive the equation of the surface with regard to the system X′, Y′, Z′ having the following form:

$$A''x'^2 + B'y'^2 + C''z'^2 + 2E''x'z' + 2G''x' + 2K''z' = Q'' \ldots . (o_1)$$

in which

$$A'' = A' \cos^2(xx') + 2E' \sin(xx') \cos(xx') + C' \sin^2(xx'),\ B' = B'$$

$$C'' = A' \sin^2(xx') + 2E' \sin(xx') \cos(xx') + C' \cos^2(xx')$$

$$E'' = -A' \cos(xx') \sin(xx') + E' [\cos^2(xx') - \sin^2(xx')] + C' \sin(xx') \cos(xx')$$

$$G'' = A'x_o \cos(xx') + E'x_o \sin(xx') + G' \cos(xx') + K' \sin(xx')$$

$$K'' = -A'x_o \sin(xx') + E'x_o \cos(xx') - G' \sin(xx') + K' \cos(xx')$$

$$Q'' = -Ax_o{}^2 - 2\,G'x_o + Q'.$$

Suppose, now $G'' = E'' = o$, that is,

$$\left.\begin{array}{l}(C' - A')\,[\sin(xx') \cos(xx')] + E'\,[\cos^2(xx') - \sin^2(xx')] = o \\ (A'x^{\circ} + G')\,[\cos(xx')] + (E'x_o + K')\,[\sin(xx')] = o\end{array}\right\}(o_2)$$

which in every case is possible, provided in every case both equations may be fulfilled by real values of (xx') and x_o. For in this case taking $A'X'$ inclined to AX by an angle equal to (xx') and AA' equal to x_o corresponding to the equations (o_2), we will have $G'' = E'' = o$, and consequently (o_1) converted into

$$A''x'^2 + B'y'^2 + C''z'^2 + 2\,K''z' = Q'' \ldots (h_8)$$

which, as the preceding, is a general equation of surfaces of the second order. It still remains to prove that the equations (o_2) may be always fulfilled by real values of (xx') and x_o. To this end let us divide the former by $\cos^2(xx')$, and the latter by $\cos(xx')$, which consequently will become, the first,

$$(C' - A')\,tg\,(xx') + E'\,(1 - tg^2(xx')) = o$$

or

$$tg^2(xx') + \frac{A' - C'}{E}\,tg\,(xx') - 1 = o$$

the second,

$$A'x_o + G' + (E'x_o + K')\,tg\,(xx') = o$$

Now the first is an equation of the second degree, which resolved according to the known rule, gives

$$tg\,(xx') = -\frac{A' - C'}{2E} \pm \sqrt{1 + \left(\frac{A' - C'}{2E}\right)^2}$$

from which we have two real values for (xx'), which, excepting the case of $A' = C'$, are different from each other, and which substituted in the second equation will give the corresponding real value for x_0. Hence, it is always possible to reduce the general equation of the surfaces of the second order to the form (h_8), which, considered with regard to x', may be also represented by the simplest formula

$$x'^2 = O\,, \text{ or } x' = \pm\sqrt{O}$$

in which O is a function of the variables y', z'. Hence, to every value of y', z' correspond two equal values for x'; that is to say, all the chords parallel to the axis $A'X'$ are perpendicularly bisected by the plane $Z'A'Y'$, which is consequently a principal one, but XAZ is also a principal plane and perpendicular to $Z'A'Y'$; herefore, in every surface of the second order there are always at least two principal planes, and these are perpendicular to each other.

Reduction of the late general formula to two equivalent formulas.

117. It is evident that as far as the equation (h_8) represents a surface of the second order, we cannot suppose A'', or B'' equal to zero, for in this case the corresponding geometrical locus would be a line. For the same reason, we cannot suppose C'' and K'' at once equal to zero; hence, the only suppositions which may be made with regard to the coefficients are, that only one of the two C'', K'' is equal to zero, or neither is equal to zero; and in the former case the equation will be deprived either of the third or of the fourth term; but, as we shall presently see, supposing all the coefficients different from zero, the equation (h_8) may be deprived of the fourth term; hence, two cases only may be supposed, the equation (h_8) deprived of either the third or of the fourth term. Let us now observe how the equation may be deprived of the fourth term, still remaining the equation of the same geometrical locus. Suppose (fig. 81) $A'X'$, $A'Y'$, $A'Z'$ to be the system to which the surface represented by (h_8) is referred. Pro-

duce $Z'A'$ so far as to have $A'A'' = \frac{K''}{C''}$; draw $A''X''$ parallel to $A'X'$, and $A''Y''$ parallel to $A'Y'$; it is plain that the co-ordinates y' and x' of the surface referred to the former system will remain unvaried, if the same surface be referred to the system $A''X''$, $A''Y''$, $A''Z'$, and that the co-ordinates z' will become $z'' - \frac{K''}{C''}$. Therefore, supposing the surface referred to the new system, it will be sufficient to substitute in (h_8) x'' and y'', instead of x' and y', and $z'' - \frac{K''}{C''}$ instead of z', to have the corresponding equation. But by making such a substitution we deduce

$$A''x''^2 + B'y''^2 + C''z''^2 - 2K''z'' + \frac{K''^2}{C''} + 2K''z'' - \frac{2K''^2}{C''} = Q''$$

from which

$$A''x''^2 + B'y''^2 + C''z''^2 = \frac{K''^2}{C} + Q' \ldots\ldots (o)$$

In the other case, in which the equation (h_8) is deprived of the third term, or converted into $A''x'^2 + B'y'^2 + 2K''z' = Q''$; let as before $A'X'$, $A'Y'$, $A'Z'$ be the axes to which the corresponding geometrical locus is referred, and take $A'A''' = \frac{Q''}{2K''}$; from A''' draw $A'''X''$ parallel to $A'X'$, and $A'''Y'''$ parallel to $A'Y'$, the co-ordinates x''', y''' of the surface referred to the new system shall be equal to x', y', and the co-ordinates z''' shall be equal to $z' - \frac{Q''}{2K'''}$; hence, $z' = z''' + \frac{Q''}{2K'''}$. Therefore, to have the equation of the surface referred to the axes $A'''X'''$, $A'''Y''$, $A'''Z'''$, it is sufficient to substitute in the late equation of the same surface referred to the axes X', Y', Z' ; x''', y''', instead of x', y', and $z''' + \frac{Q''}{2K'''}$, instead of z', which consequently will become

$$A'''x'''^2 + B'y'''^2 + 2K'''z''' + Q'' = Q''$$

from which

$$A'''x'''^2 + By'''^2 + 2K'''z'' = o \ldots\ldots\ldots (o_1)$$

Now, since all the surfaces of the second order are represented by (h_8), and since every surface represented by (h_8) may be represented also by the two (o) and (o_1), hence, all the surfaces of the second order shall be represented by the general equations

$$Mx^2 + Ny^2 + Pz^2 = V \,.\,.\, (h_9)$$

$$Mx^2 + Ny^2 + 2Sz = o \,.\,.\, (h_{10})$$

Surfaces having centres.

118. The centre of a surface is that point by which every chord passing through it is bisected. Now the surfaces corresponding to the equation (h_9) have this point in the origin of the axes to which the surface is referred; because, by comparing (h_9) with (h_1), we find $A = M$, $B = N$, $C = P$, $Q = V$, $D = E = F = G = H = K = o$; hence (112),

$$m = -\frac{M\cos\alpha}{P\cos\alpha''}\,,\; n = -\frac{N\cos\alpha'}{P\cos\alpha''}\,,\; q = o$$

and (h_4) will become

$$z_\circ = -\frac{M\cos\alpha}{P\cos\alpha''}x_\circ - \frac{N\cos\alpha'}{P\cos\alpha''}y_\circ$$

equation of a plane passing (29, C. III) through the origin of the co-ordinates. But (h_4) is the equation of any diametral plane; hence, with regard to the surfaces corresponding to the formula (h_9) all the diametral planes shall pass through the origin of the axes to which the surface is referred, and, consequently, there is the centre of the surface. For let us conceive any chord mn (fig. 82), for instance, passing through that origin, the diametral plane by which this chord with its parallel system is bisected, passes through the origin where of course the chord mn is bisected.

Corollary. Since all the diametral planes pass through the origin of the co-ordinates, if the surface admits of several principal planes, all shall pass through the same origin ; and it is to be observed, that besides the planes XAZ, ZAY which (117) are the principal planes, XAY is also a principal plane. For, considering the equation (h_9) with regard to z, it may be represented by $z^2 = \mathrm{T}$, or $z = \pm\sqrt{\mathrm{T}}$, in which T depends upon x and y. Therefore, to every value of x and y there are two corresponding equal values for z; that is to say, all the chords parallel to AZ are bisected by the plane XAY ; but AZ is perpendicular to XAY, therefore, XAY is a principal plane.

Surfaces without centre.

119. All the surfaces corresponding to the equation (h_{10}) are without centre. For, by comparing (h_1) with (h_{10}), we have $\mathrm{A} = \mathrm{M}$, $\mathrm{B} = \mathrm{N}$, $\mathrm{K} = \mathrm{S}$, $\mathrm{C} = \mathrm{D} = \mathrm{E} = \mathrm{F} = \mathrm{G} = \mathrm{H} = \mathrm{Q} = 0$; hence (112)

$$m = -\frac{\mathrm{M}\cos\alpha}{0} = -\infty$$

$$n = -\frac{\mathrm{N}\cos\alpha'}{0} = -\infty$$

$$q = -\frac{\mathrm{S}\cos\alpha''}{0} = -\infty$$

and consequently (h_4) the general equation of the diametral plane becomes

$$z_\circ = -\frac{\mathrm{M}\cos\alpha}{0}x_\circ - \frac{\mathrm{N}\cos\alpha'}{0}y_\circ - \frac{\mathrm{S}\cos\alpha''}{0} \quad \ldots\ldots (o)$$

Now (29) the coefficient, of $x_\circ$ and $y_\circ$ are the tangents of the angles which the intersections of the plane (o) with the planes XAZ, YAZ make with the axes X and Y. Therefore, since these two tangents are infinite, the intersections must be perpendicular to the axes X and Y, which may happen in two different ways. Either if the diametral plane passes through AZ (fig. 83), as, for instance, the plane mAZ, or through two lines sr, for instance, and qp parallel to the same axis; because in both cases the intersections of the diametral plane with XAZ and YAZ are perpendicular to the axes X, Y. Still there is a difference to be remarked in the two cases: that is, the intersection Am of the diametral plane with XAZ must pass through the origin of the co-ordinates in the former case, and sq never meets that origin in the second. It is also to be observed that in both cases the diametral planes are perpendicular to XAY. Now the planes XAZ, YAZ are principal planes; therefore in the supposition of the diametral plane $rsqp$ not passing through AZ, there will not be any point common to the three planes; hence there will be no point through which pass the chords bisected by the three planes. Again, in the supposition of the diametral planes passing through AZ, there cannot be any determined point through which the chords corresponding to those different planes are bisected; because every point of the common intersection can be such a point. Hence the surfaces corresponding to the equation (h_{10}) are said to be without centre.

Scholium. Let us remark that the equation (h_4) may be transformed into $y_0 = \frac{1}{n} z_0 - \frac{m}{n} x_0 - \frac{q}{n}$, and substituting in the equations (112)

$$\frac{1}{n} = -\frac{\text{E} \cos \alpha + \text{D} \cos \alpha' + \text{C} \cos \alpha''}{\text{F} \cos \alpha + \text{B} \cos \alpha' + \text{D} \cos \alpha''},$$

$$\frac{m}{n} = \frac{\text{A} \cos \alpha + \text{F} \cos \alpha' + \text{E} \cos \alpha''}{\text{F} \cos \alpha + \text{B} \cos \alpha' + \text{D} \cos \alpha''},$$

$$\frac{q}{n} = \frac{\text{G} \cos \alpha + \text{H} \cos \alpha' + \text{K} \cos \alpha''}{\text{F} \cos \alpha + \text{B} \cos \alpha' + \text{D} \cos \alpha''},$$

the values A, B, . . . above determined; since, then, we have

$$\frac{1}{n} = 0, \quad \frac{m}{n} = \frac{\text{M} \cos \alpha}{\text{N} \cos \alpha'}, \quad \frac{q}{n} = \frac{\text{S} \cos \alpha''}{\text{N} \cos \alpha'}$$

the same (h_4) will become

$$y_\circ = -\frac{\text{M} \cos \alpha}{\text{N} \cos \alpha'} x_\circ - \frac{\text{S} \cos \alpha''}{\text{N} \cos \alpha'}$$

Now α, α', α'' are the angles formed by any chord with the axes. Hence such values of α', α'' may be always found giving the last term of the equation different from zero. But an equation between the co-ordinates $x_\circ$, $y_\circ$ of a diametral plane is the equation of the intersection of this plane with XAY; that is, the equation of a straight line referred to the axes X, Y, which will not pass through the origin of the co-ordinates as far as $-\frac{\text{S} \cos \alpha''}{\text{N} \cos \alpha'}$ is different from zero. Hence in every surface represented by (h_{10}) there are such diametral planes not passing through the common intersection AZ of the two principal XAZ, YAZ.

Different species of the surfaces having centres.

120. In the equation $\text{M}x^2 + \text{N}y^2 + \text{P}z^2 = \text{V}$ we must suppose all the coefficients M, N, P different from zero, otherwise the equation can no longer represent any surface. Therefore all the different species of the surfaces represented by (h_9) depend upon the signs of the same coefficients. And, first, we may suppose all the coefficients positive, in which case V of course is positive. Secondly, supposing V still positive, we may suppose either two coefficients positive and one negative, or two negative and one positive. Thirdly, supposing $\text{V} = o$, we can again suppose two coefficients negative and one positive, and vice versa. These are all the possible cases. Because, in the supposition of all the coefficients negative, we must admit of V negative; but an equation in which both members are negative does not differ from that in which both members are positive. In the suppo-

sition of V negative, we must suppose two coefficients having the sign different from that of the third, and since the equation remains the same by changing the signs in both members, so the last case does not differ from the second.

FIRST SPECIES.

121. Let all the coefficients be positive, and let us divide both members of the equation (h_9) by V, we will obtain

$$\frac{M}{V}x^2 + \frac{N}{V}y^2 + \frac{P}{V}z^2 = 1$$

and representing the ratios $\frac{V}{M}$, $\frac{V}{N}$, $\frac{V}{P}$ by a^2, b^2, c^2, we will have

$$\frac{M}{V} = \frac{1}{a^2}, \quad \frac{N}{V} = \frac{1}{b^2}, \quad \frac{P}{V} = \frac{1}{c^2}$$

and consequently the preceding equation will be converted into

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \, . \, . \, . \, . \, . \, (h_{11})$$

To have, now, the intersections of the corresponding surface with the plane XAZ, it is sufficient to suppose in (h_{11}) the co-ordinate $y = o$, because for every point of the plane XAY the co-ordinates y are equal to zero. Hence the equation of the intersection will be

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

But this equation (63 (g)) is that of an ellipse ; therefore the intersection between the plane XAZ and the surface is an ellipse, of which the semi-axes are a and c. In the same manner if we put in (h_{11}) successively $x = o$, $z = o$, we shall obtain the equations

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \, , \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

of the intersections between the surface and the planes ZAY, XAY, which are two ellipses, the former having the semi-axes b, c, the latter a and b. These intersections are termed *principal.*

Suppose, again, a plane parallel to the plane, for instance, ZAY. The intersection of this plane with the surface is a curve of which all the co-ordinates x are equal to the distance of the two parallel planes. Hence, supposing that distance to be equal to d. The equation (h_{11}) will become that of the parallel intersection by substituting there d to x; which, in the present supposition, is to be considered as a constant quantity. But by such a substitution the formula (h_{11}) becomes

$$\left.\begin{aligned} &\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{d^2}{a^2} = \frac{a^2 - d^2}{a^2} \\ \text{or} \quad &\frac{y^2}{\frac{b^2}{a^2}(a^2 - d^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - d^2)} = 1 \end{aligned}\right\} \ldots (o)$$

which is either the equation of an ellipse, or of a point, or an imaginary one. Because as far as we suppose $d < a$, that is, the distance of the cutting plane from YAZ, less than the semi-axis a, the denominators of the variables y, z are both positive and represent the square of the semi-axes of an ellipse. If we suppose $d = a$ the former (o) becomes $\frac{y^2}{b^2} + \frac{z^2}{c^2} = o$, which cannot be fulfilled but by $y = z = o$. But, if two co-ordinates are equal to zero, and the third has only one determined value, the locus referred to the axes is a point; hence in the second hypothesis the equation (o) is that of a point; that is, the plane parallel to YAZ is a tangent plane of the surface. Suppose, finally, $d > a$: in this case we must also suppose that the sum of two negative terms is equal to $+1$, which being absurd, we conclude that any plane passing beyond the extremity of the semi-axis a and parallel to ZAY, can neither cut nor touch the surface.

The same may be proved in the same way of the sections of the surface produced by planes parallel to ZAX and XAY. This surface is termed *ellipsoid.*

Scholium I. Suppose $a = b = c$ the formula (h_{11}) will become

$$x^2 + y^2 + z^2 = a^2$$

which (30 , C . I) is the equation of the sphere having the radius a.

Scholium II. Suppose only $a = c$ or only $b = c$, in the former case (h_{11}) becomes

$$\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{a^2}{b^2}(b^2 - y^2) = x^2 + z^2$$

which (104) is the equation of the surface generated by an ellipse turned about Y.

In the latter case (h_{11}) becomes

$$\frac{y^2 + z^2}{b^2} + \frac{x^2}{a^2} = 1$$

or

$$\frac{b^2}{a^2}(a^2 - x^2) = y^2 + z^2$$

which is the equation of the surface generated by an ellipse turned about X.

SECOND SPECIES.

122. The second species correspond to the case in which two coefficients, for instance, N and P, are negative, and the third, M, positive, and from the case in which two coefficients, for example, M and P, are positive, and the third negative. Now, since V

is still positive, supposing in the former case the ratios $\frac{V}{M}, \frac{V}{N}, \frac{V}{P}$ represented by a^2, $-b^2$, $-c^2$, and consequently in the latter by $a^2, -b^2, +c^2$, we will also have, in the first case,

$$\frac{M}{V}=\frac{1}{a^2}, \frac{N}{V}=-\frac{1}{b^2}, \frac{P}{V}=-\frac{1}{c^2}$$

in the second

$$\frac{M}{V}=\frac{1}{a^2}, \frac{N}{V}=-\frac{1}{b^2}, \frac{P}{V}=\frac{1}{c^2}$$

and the formula (h_9) shall become

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \ldots \ldots (h_{12})$$

or
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ldots \ldots (h_{13})$$

Following now with regard to (h_{12}) the same process observed in the preceding number, we shall obtain the principal sections of the corresponding surface, supposing successively $x=o$, $y=o$, $z=o$, hence the equations

$$\frac{y^2}{a^2}+\frac{z^2}{c^2}=-1, \frac{x^2}{a^2}-\frac{z^2}{c^2}=1, \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

of which the first is an imaginary one, the others are (63) equations of hyperbolas. That is, there is no section between the plane ZAY and the surface, and the sections between the planes ZAX, XAY and the same surface are hyperbolas, having the common transverse axis $2a$. To have the sections of the surface with planes parallel to XAY, XAZ, it is sufficient (121) to substitute instead of z and y a linear value d equal to the distance of the parallel plane from the principal : in this way we derive the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{d^2}{c^2}+1, \frac{x^2}{a^2}-\frac{z^2}{c^2}=\frac{d^2}{b^2}+1$$

$$\text{or} \qquad \frac{x^2}{\frac{a^2}{c^2}(d^2+c^2)} - \frac{y^2}{\frac{b^2}{c^2}(d^2+c^2)} = 1$$

$$\frac{y^2}{\frac{a^2}{b^2}(d^2+b^2)} - \frac{z^2}{\frac{c^2}{b^2}(d^2+b^2)} = 1$$

equations of hyperbolas, the first of which has its transverse semi-axis equal to $\frac{a}{c}\sqrt{d^2+c^2}$ and the conjugate equal to $\frac{b}{c}\sqrt{d^2+c^2}$. The second has the transverse semi-axis equal to $\frac{a}{b}\sqrt{d^2+b^2}$ and the conjugate equal to $\frac{c}{b}\sqrt{d^2+b^2}$. Therefore, all the sections formed by such parallel planes are hyperbolas. And it is to be remarked, that the transverse axis of these hyperbolas are all greater than $2a$ or $2b$. For since $\sqrt{d^2+c^2} > \sqrt{c^2} = c$ and $\sqrt{d^2+b^2} > \sqrt{b^2} = b$; so $\frac{a}{c}\sqrt{d^2+c^2} > \frac{a}{c}\,c = a$ and $\frac{a}{b}\sqrt{d^2+b^2} > \frac{a}{b}\,b = a$.
Let us come to the sections formed by the planes parallel to YAZ. For this purpose it is sufficient to substitute d instead of x in the formula (h_{12}), which becomes

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{d^2}{a^2} - 1$$

$$\text{or} \qquad \frac{y^2}{\frac{b^2}{a^2}(d^2-a^2)} + \frac{z^2}{\frac{c^2}{a^2}(d^2-a^2)} = 1$$

which is an imaginary equation when $d < a$: represents a point when $d = a$: an ellipse when d is greater than a; therefore, taking on the axis X from the origin of the co-ordinates in the positive direction of the axis as well as in the negative, two portions equal to a, and supposing two planes parallel to YAZ to pass through the extremities of the same portions, the surface will not enter within the space contained by these two planes, but shall begin from the points common to the planes, and to the axis X , and beyond the planes, will extend indefinitely itself.

123. Supposing, now, successively $x = o$, $y = o$, $z = o$ in the equation (h_{18}), we obtain the intersections of the surface represented by that formula and the principal planes, which are

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1\,, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,, \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$$

that is, an ellipse and two hyperbolas, the first on the plane XAZ , and the others on the planes XAY , YAZ.

Supposing, moreover, in the same (h_{18}) successively d instead of x , y , z , we derive the equations of the parallel sections,

$$\left.\begin{aligned}
&\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 + \frac{d^2}{b^2} \text{ or } \frac{x^2}{\frac{a^2}{b^2}(b^2+d^2)} + \frac{z^2}{\frac{c^2}{b^2}(b^2+d^2)} = 1 && (o)\\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} \text{ or } \frac{x^2}{\frac{a^2}{c^2}(c^2-d^2)} - \frac{y^2}{\frac{b^2}{c^2}(c^2-d^2)} = 1 && (o_1)\\
&\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1 - \frac{d^2}{a^2} \text{ or } \frac{z^2}{\frac{c^2}{a^2}(a^2-d^2)} - \frac{y^2}{\frac{b^2}{a^2}(a^2-d^2)} = 1 && (o_2)
\end{aligned}\right\}$$

hence all the sections parallel to the plane XAZ are elliptical, and the axes increase more and more with the distance d. With regard to the sections parallel to the planes XAY , YAZ , three cases are to be distinguished, which being the same for (o_1) as well as for (o_2), we will only consider the former. We may have $d < c$ or $d = c$ or $d > c$ in the first case; the second (o_1) represents a hyperbola, of which $\frac{a^2}{c^2}(c^2 - d^2)$ is the conjugate diameter.

In the second, the former (o_1) becomes $\frac{x^2}{a_2} - \frac{y^2}{b^2} = o$ or $x^2 = \frac{a^2}{b^2} y_2$

or $x = \pm \frac{a}{b} y$, a double equation of a straight line in the third,

since $\dfrac{x^2}{\frac{a^2}{c^2}(c^2 - d^2)} = - \dfrac{x^2}{\frac{a^2}{c^2}(d^2 - c^2)}$ and $- \dfrac{y^2}{\frac{b^2}{a^2}(c^2 - d^2)} = \dfrac{y^2}{\frac{b^2}{a^2}(d^2 - c^2)}$ by substituting these values in the second (o_1) being $d > c$, we have again a hyperbolical section.

Scholium. Suppose in (h_{12}) $b = c$, and in (h_{13}) $a = c$, the former equation will become

$$\frac{x^2}{a^2} - 1 = \frac{y^2 + z^2}{b^2}, \text{ or } \frac{b^2}{a^2}(x^2 - a^2) = y^2 + z^2$$

the equation (104) of the surface generated by the hyberbola turned about its transverse axis. The second (h_{13}) becomes

$$\frac{x^2 + z^2}{a^2} = 1 + \frac{y^2}{b^2}, \text{ or } \frac{a^2}{b^2}(b^2 + y^2) = x^2 + z^2$$

the equation of the surface generated by the hyberbola turned about the transverse axis. The surfaces corresponding to the equations (h_{12}), (h_{13}) are termed *hyperboloids.*

THIRD SPECIES.

124. Let us come to the last species of surfaces having a centre which corresponds (120, 3°) to the case, in which supposing as before two coefficients of the first member of (h_9) with the sign different from the third. We suppose, moreover, the second member $V = o$; and it is plain that it will be the same to make, for instance, M positive, and N and P negative, or the first negative and the remaining positive; because on account of the second member of the equation equal to zero, we may change at pleasure the signs of the first member. In the case of M positive, and N and P negative, we will have, as in the preceding numbers,

$$M = \frac{1}{a^2}, \; N = -\frac{1}{b^2}, \; P = -\frac{1}{c^2}$$

and the equation (h_9) will become

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = o \; \ldots \; (h_{14})$$

from which we derive, as in the preceding cases,

$$-\frac{y^2}{b^2} - \frac{z^2}{c^2} = o, \quad \text{or} \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = o$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = o, \quad \text{or} \quad y = \pm \frac{b}{a}\, x$$

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = o, \quad \text{or} \quad z = \pm \frac{c}{a}\, x$$

the equations of the principal intersections. But the first of these equations can only be fulfilled with $y = z = o$; therefore, the intersection of the surface (h_{14}) with the plane YAZ is a single point on the origin of the co-ordinates; the remaining equations are both equations of two straight lines. Hence, the intersections of (h_{14}) with the planes YAX, XAZ are two straight lines passing through the origin of the axes, and equally inclined on both sides to the same axes.

To have the sections parallel to the principal planes, let us substitute d in (h_{14}) successively, instead of x, y, z, we will obtain

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{d^2}{a^2}, \quad \text{or} \quad \frac{y^2}{\frac{d^2}{a^2} b^2} + \frac{z^2}{\frac{d^2}{a^2} c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{d^2}{c^2}, \quad \text{or} \quad \frac{x^2}{\frac{d^2}{c^2} a^2} - \frac{y^2}{\frac{d^2}{c^2} b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = \frac{d^2}{b^2}, \quad \text{or} \quad \frac{x^2}{\frac{d^2}{b^2} a^2} - \frac{z^2}{\frac{d^2}{b^2} c^2} = 1$$

therefore, the sections parallel to YAZ are elliptical, and those parallel to the planes XAY, XAZ hyperbolical.

Since the surface passes through the origin of the co-ordinates, and extends itself indefinitely on both sides of the plane YAZ, we may conceive that surface as having the form of a double cone with the vertex at the origin of the axes and elliptical bases. In this hypothesis every plane passing through the axis AX will make with the surface rectilinear sections passing through the origin of the co-ordinates; hence, every other plane passing through the same origin, and touching the surface of the cone in any point, shall touch the same cone indefinitely. Inversely, if every plane, touching the surface, passes through the origin of the co-ordinates, the surface shall of course be conical; because, suppose (fig. 84) the surface not to be conical, and let us represent it by AL, it is evident that that plane only which touches the surface in A can pass through the origin of the co-ordinates. For draw from A the chord Am to any point of the surface; now every plane passing through Am must necessarily cut the surface, and the plane touching the surface in m cannot pass through the origin of the co-ordinates. Therefore, if with regard to the surface represented by (h_{14}) every tangent plane passes through the origin of the axes, the form of the surface will be conical.

The general equation $(114, h_6)$ of the plane touching the surfaces of the second order in any point varies according to the different values of the co-efficients m', n'. Now, by comparing (h_{14}) with (h_1), we have $A = \frac{1}{a^2}$, $B = -\frac{1}{b^2}$, $C = -\frac{1}{c^2}$ and $D = E = F = G = H = K = o$; hence, (114)

$$m' = \frac{\frac{1}{a^2}x}{\frac{1}{c^2}z} = \frac{c^2 x}{a^2 z},\ n' = -\frac{\frac{1}{b^2}y}{\frac{1}{c^2}z} = -\frac{c^2 y}{b^2 z}$$

consequently, the equation (h_6) of the tangent plane becomes

$$Z - z = \frac{c^2 x}{a^2 z}(X - x) - \frac{c^2 y}{b^2 z}(Y - y)$$

or $$\frac{z}{c^2}(\mathrm{Z}-z)=\frac{x}{a^2}(\mathrm{X}-x)-\frac{y}{b^2}(\mathrm{Y}-y)$$

from which

$$\frac{\mathrm{Z}.z}{c^2}-\frac{z^2}{c^2}=\frac{\mathrm{X}.x}{a^2}-\frac{x^2}{a^2}-\frac{\mathrm{Y}.y}{b^2}+\frac{y^2}{b^2}$$

and

$$\frac{\mathrm{Z}.z}{c^2}+\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=\frac{\mathrm{X}.x}{a^2}-\frac{\mathrm{Y}.y}{b^2}$$

but (h_{14}) $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=o$; hence,

$$\mathrm{Z}=\frac{c^2\,x}{a^2\,z}\,\mathrm{X}-\frac{c^2\,y}{b^2\,z}\,\mathrm{Y}$$

equation of a plane passing (29, C. III) through the origin of the axes. Hence, the tangent plane of any point of the surface (h_{14}) shall pass constantly through the origin of the axes, and the form of that surface is conical.

Relation between the surfaces of the second and third species.

125. Suppose in the equations (h_{12}), (h_{14}) the same values of a, b, c, and let the corresponding surfaces be referred to the same system of axes; let the double cone corresponding to (h_{14}) be represented (fig. 85) by $m\mathrm{A}m'n'n$, and the surface corresponding to (h_{12}) by $f\mathrm{R}d$, $f'\mathrm{R}'d'$, the cone will be an asymptotical surface to the other. To demonstrate this it is sufficient to prove that the elliptical sections of the two surfaces, produced by the planes parallel to YAZ, are continually approaching to each other. Now the general formula by which such sections are represented with regard to (h_{12}), is (122)

$$\frac{y^2}{\frac{b^2}{a^2}(d^2-a^2)}+\frac{z^2}{\frac{c^2}{a^2}(d^2-a^2)}=1$$

and that which represents the same sections with regard to (h_{14}), is (124)

$$\frac{y^2}{\frac{b^2}{a^2}\,d^2} + \frac{z^2}{\frac{c^2}{a^2}\,d^2} = 1$$

that is to say, the square semi-axes of the former sections are $\frac{b^2}{a^2}(d^2 - a^2)$, $\frac{c^2}{a^2}(d^2 - a^2)$, and those corresponding to the latter $\frac{b^2}{a^2}d^2$, $\frac{c^2}{a^2}d^2$. Let us divide each semi-axis of the first sections by the corresponding semi-axis of the second, we will obtain in both cases

$$\frac{d^2 - a^2}{d^2} = 1 - \frac{a^2}{d^2}$$

but since a is a constant quantity, and d continually increasing, the ratio $\frac{a^2}{d^2}$ approaches continually to zero. And the ratio between the square of the semi-axes is likewise is approaching to unity; but this cannot happen unless the axes of the ellipses, and, consequently, the ellipses themselves, nearer and nearer approach to each other. Observe, that since

$$\frac{b^2}{a^2}(d^2 - a^2) < \frac{b^2}{a^2}d^2 \text{ and } \frac{c^2}{a^2}(d^2 - a^2) < \frac{c^2}{a^2}d^2$$

the semi-axes of the sections of (h_{12}) are less than the corresponding semi-axes of (h_{14}), and, consequently, the whole surface (h_{12}) must be contained within the cone (h_{14}).

The same thing, although with some variation with regard to the relative position of the two surfaces, may be proved of the surfaces corresponding to (h_{13}) (h_{14}). First, observe that supposing in (h_{13}) instead of $\frac{1}{a^2}$ positive and $\frac{1}{b^2}$ negative, $\frac{1}{a^2}$ negative and $\frac{1}{b^2}$ positive, the elliptical sections of the corresponding surface will be represented (123) by

$$\frac{y^2}{\frac{b^2}{a^2}(d^2+a^2)} + \frac{z^2}{\frac{c^2}{a^2}(d^2+a^2)} = 1$$

therefore the square semi-axes of the ellipses are, in this case,

$$\frac{b^2}{a^2}(d^2+a^2)\,,\ \frac{c^2}{a^2}(d^2+a^2) \text{ and } \frac{b^2}{a^2}d^2\,,\ \ \frac{c^2}{a^2}d^2$$

which, divided as before, by each other, give

$$\frac{d^2+a^2}{d^2} = 1 + \frac{a^2}{d^2}$$

From which equation we are enabled to derive the same conclusion as in the preceding case; that is to say, the cone shall approach continually to the surface (h_{13}). But in the present case the semi-axes of the sections of (h_{14}) are less than those of (h_{13}) for $\frac{b^2}{a^2}(d^2+a^2) > \frac{b^2}{a^2}d^2$, $\frac{c^2}{a^2}(d^2+a^2) > \frac{c^2}{a^2}d^2$; hence the double cone $mAnm'$ . . . is contained within the surface $pqgg'$ . . .

Different species of the surfaces without a centre.

126. Surfaces of the second order, without a centre, are (119) generally represented by the equation $Mx^2 + Ny^2 + 2Sz = o$, in which we must, of course, suppose the coefficients M, N, 2S different from zero, for the reason already given (120); and consequently the different species of the surfaces depend upon the different signs of the coefficients. Now two only different cases can happen. S may be either positive or negative; and, first, we may suppose the remaining M and N, both with the same, or, secondly, with different sign. For suppose, for instance, all the coefficients positive: it is plain that without supposing, at the same time, z negative, we will never obtain the first member of the equation equal to the second; and if we suppose M and N positive and S negative, we never will obtain the first member equal to the second without supposing z positive. Therefore in

both cases the equation will represent the same surface, and the only difference shall be in the different position of the surface with regard to the plane XAY. The same is to be said if we suppose both coefficients M and N negative. Hence when the coefficients M, N have the same sign, we may suppose S indifferently positive or negative. But it is moreover to be remarked that the surface represented by (h_{10}), supposing M and N positive is the same as that represented by the same equation in which M, N are negative. For, since the second member of the equation is zero, without making any alteration we may change the signs of the first member, and the difference between the two cases is only apparent. Therefore the first case to be considered is that of the equal signs of M and N. Again, suppose S positive, and one of the remaining coefficients positive, the other negative, we shall obtain the same surface as we would obtain with S negative, provided the co-ordinate z be taken with contrary sign. Hence, when M, for example, is positive and N negative, we may suppose S either positive or negative; and since, in the case of M negative and N positive, we may change at pleasure the signs of the first member, the other case to be considered is that of the unequal signs of M and N, supposing S indifferently positive or negative.

FIRST SPECIES

127. Let M and N be represented, as in the preceding numbers by $\frac{1}{a^2}$, $\frac{1}{b^2}$, and S by $\frac{1}{c}$, the equation (h_{10}) will become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2z}{c}=o\ \ldots\ldots\ (o)$$

in which, making $\frac{1}{a^2}$ as well as $\frac{1}{b^2}$ positive, suppose $\frac{1}{c}$ negative, and let us examine the surface represented by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2z}{c}=o\ \ldots\ldots\ (h_{15})$$

The principal sections shall be obtained as usual by substituting successively x, y, z equal to zero; in this manner we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=o,\quad x^2=\frac{2a^2}{c}z,\quad y^2=\frac{2b^2}{c}z$$

The first of which equations can only be fulfilled with $x=y=o$, and consequently represents a single point at the origin of the axes; the second and third (63 (g)) are equations of parabolas of which $2\frac{a^2}{c}$, $2\frac{b^2}{c}$ are the parameters. That is, the section of the surface (h_{15}) made by the plane XAY is a point; the sections of the same surface with XAZ, YAZ are two parabolas.

Let us come to the parallel sections; and first let us substitute d to z, to have the sections parallel to the plane XAY. The equation (h_{15}) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2d}{c}$$

or

$$\frac{x^2}{2\frac{da^2}{c}}+\frac{y^2}{2\frac{db^2}{c}}=1$$

that is, the sections parallel to XAY are elliptical. And since substituting successively d to y and x we obtain

$$\frac{x^2}{a^2}+\frac{d^2}{b^2}=\frac{2z}{c},\quad \frac{d^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$$

or

$$x^2=\frac{2a^2}{c}\left(z-\frac{cd^2}{2b^2}\right),\quad y^2=\frac{2b^2}{c}\left(z-\frac{cd^2}{2a^2}\right)$$

that is, the sections parallel to the planes XAZ, YAZ are parabolical sections; from which it follows that the surface represented by (h_{15}) touches the plane XAY at the origin of the axes, and then extends itself indefinitely on the positive side, because (h_{15}) may be fulfilled only with z positive.

Scholium. Suppose in (h_{15}) $a = b = c$, we have

$$\frac{x^2 + y^2}{a^2} - \frac{2z}{a} = o, \quad \text{or} \quad x^2 + y^2 = 2az$$

equation of the surface generated by the parabola turned (104) about its axes.

SECOND SPECIES.

128. To have the equation of the surfaces of the second species it is sufficient (126) to suppose in the formula (o) of the preceding number $\frac{1}{b^2}$ negative and $\frac{1}{a^2}$ positive. That is, the surface of the second species is represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2z}{c} = o \,.\,.\, (h_{16})$$

from which, as in the preceding cases, we derive the equations

$$\left.\frac{x^2}{a^2} - \frac{y^2}{b^2} = o, \quad \text{or } y = \pm \frac{b}{a} x \quad \right\} (o)$$

$$\left.\frac{x^2}{a^2} + \frac{2z}{c} = o, \quad \text{or } x^2 = -\frac{2a^2}{c} z \quad \right\} (o_1)$$

$$\left.-\frac{y^2}{b^2} + \frac{2z}{c} = o, \quad \text{or } y^2 = \frac{2b^2}{c} z \quad \right\} (o_2)$$

that is, the principal section of the surface made by the plane XAY, consists in two straight lines passing through the origin of the axis; the principal sections made by the planes XAZ, YAZ are two parabolas having $-\frac{2a^2}{c}$, $\frac{2b^2}{c}$ for parameters, and in the first of which the abscissas must be taken on the negative Z.

Now, substituting successively d for x, y, z in (h_{16}) to have the parallel sections, we obtain, first,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2d}{c} = o, \quad \text{or} \quad \frac{y^2}{\frac{2d}{c} b^2} - \frac{x^2}{\frac{2d}{c} a^2} = 1$$

that is, the sections parallel to the plane XAY are hyberbolical, and the transverse axis is converted into the conjugate, when d from positive becomes negative, and vice versa; but in every case the equation of the corresponding asymptotes will be represented by

$$y = \pm \frac{b}{a} x$$

because, suppose d positive and equal to Am (fig. 86), in this case the square of the transverse semi-axis is $\frac{2d}{c} b^2$ and that of the conjugate, $\frac{2d}{c} a^2$; and considering Y as axis of the abscissas, the equation of the asymptotes nm, mp, corresponding to the hyperbola on the plane nmp, is (91)

$$x = \pm \frac{\sqrt{\frac{2d}{c} a^2}}{\sqrt{\frac{2d}{c} b^2}} y = \pm \frac{a}{b} y$$

in which $\pm \frac{a}{b}$ represents the trigonometrical tangents of the angles formed by the asymptotes with the axis of the abscissas; but from this equation we derive the equivalent

$$y = \pm \frac{b}{a} x$$

Again, suppose d negative, and equal to Am', the square of the transverse axis will be in this case $\frac{2d}{c} a^2$, and that of the conjugate, $\frac{2d}{c} b^2$, and having X for axis of the abscissas, the equation of the asymptotes $n'm'$, $p'm'$ corresponding to the hyperbola on the plane $n'm'p'$ is

$$y = \pm \frac{\sqrt{\frac{2d}{c} b^2}}{\sqrt{\frac{2d}{c} a^2}} x$$

that is

$$y = \pm \frac{b}{a} x$$

but this equation is the same as that of the principal intersection made by XAY; hence, all the asymptotes $mn \ldots m'n' \ldots$ $mp \ldots m'p' \ldots$ are parallel to the same principal sections AL, AR, and all passing through ZZ'; therefore, all the asymptotes parallel to AL, are on the plane determined by ZZ' and AL, and all the asymptotes parallel to AR, are on the plane determined by ZZ' and AR, which planes are consequently the asymptotical planes of the surface. The equations corresponding to the remaining parallel sections are

$$\frac{x^2}{a^2} = \frac{d^2}{b^2} - \frac{2z}{c}$$

$$-\frac{y^2}{b^2} = -\frac{d^2}{a^2} - \frac{2z}{c}$$

which reduced to the form

$$x^2 = \frac{2a^2}{c}\left(\frac{cd^2}{2b^2} - z\right)$$

$$y^2 = \frac{2b^2}{c}\left(\frac{cd^2}{2a^2} + z\right)$$

show that the sections parallel to the planes XAZ, YAZ are of parabolic figure. The surfaces corresponding to (h_{15}), (h_{16}) are termed *paraboloids*.

Common properties of the surfaces of the second order.

PROPOSITION I.

The section of any surface of the second order made by a plane, must be either rectilinear or a line of the second order.

129. The surfaces, as considered in the preceding numbers, are referred to the principal planes; and all the surfaces of the se-

cond order could be represented by the formulas (h_9), (h_{10}). But suppose a plane cutting the surface in any direction whatever, and let this plane be taken as that of a new system of axes; for instance, the plane X′A′Y′. To the surface referred to the new system shall correspond an equation between the co-ordinates x', y', z'; and since (27, *Sch.*) passing from one to another system of axes, the degree of the new equation does not exceed that of the former, therefore, in the equation between the co-ordinates x', y', z', there will be no terms with the co-ordinates except those of the first and second degree. Now, to have the equation of the section made by the plane X′A′Y′, it is sufficient to substitute zero for z'; hence, the equation of that line shall be an equation of which the variables x', y' do not surpass the second degree; but such an equation can only represent straight lines, or lines of the second order. Therefore, every section of the surfaces of the second order made by a plane, is either rectilinear or a line of the second order.

PROPOSITION II.

Any straight line can meet a surface of the second order in no more than two points.

130. From the equations of any straight line in space [35 (*o*)] we may obtain the value of x, as well as of y given by z. Now it is evident that supposing the points common to the straight line, and any surface represented (110) by (h_1), the co-ordinates of the line, as well as those of the surface, must be the same. Hence, we may substitute in this case in (h_1) the values of x and y, derived from the equations of the straight line, and so we shall obtain an equation with the variables z alone, which we may consequently represent by

$$\alpha z^2 + \beta z + \gamma = \rho$$

Moreover, it is evident that the co-ordinates z, common to any straight line and to any surface, must fulfil this equation; but an equation of the second degree is fulfilled by two real values only; hence, only two different co-ordinates z can be substituted in the preceding equation. In the same manner we can demonstrate, that no more than two different values of y and x can be substituted in the analogous equations; hence, only two points may be common both to any surface of the second order and any straight line.

Equations of conical and cylindrical surfaces.

131. Suppose any curve to be described in space by the extremity of a straight line which constantly passes through a fixed point with the other variable extremity, the surface generated by the motion of that line is termed a conical surface; but if the straight line, while describing any curve with one extremity, remains constantly parallel to itself, the surface generated by this line is cylindrical.

General equation of any conical surface.

132. The equations of any curve in space are (33) generally represented by

$$x = f(z)\ ,\ x = f(y)$$

or

$$z = \varphi(x)\ ,\ y = \psi(x)\ \ldots\ldots\ (e)$$

and the equations of any straight line passing through a given point are (35, *Sch.*)

$$x - x_\circ = a\,(y - y_\circ)\ ,\ x - x_\circ = a'\,(z - z_\circ)$$

or

$$\left.y - y_\circ = \frac{1}{a}\,(x - x_\circ)\ ,\ z - z_\circ = \frac{1}{a'}\,(x - x_\circ)\right\}\ \ldots\ (e_1)$$

Suppose mn (fig. 87) to represent a peculiar position of the generating line, and let n be a point of the curve in space: moreover, suppose a and a' to be taken in such a manner as to have the equations (e_1) corresponding to the position mn of the generating line. It is evident that the co-ordinates of the point n will fulfil at once the equations (e) and (e_1), and of course in this hypothesis the values y, z given by (e) may be substituted in (e_1). Hence,

$$\psi(x) - y_0 = \frac{1}{a}(x - x_0)$$

$$\varphi(x) - z_0 = \frac{1}{a'}(x - x_0).$$

From which, by eliminating x, may be deduced an equation between the quantities a, a' and the constant co-ordinates x_0, y_0, z_0 of the point m. Let this equation be represented by

$$F(a, a') = o \ . \ . \ . \ (e_2)$$

Since from (e_1) we have

$$a = \frac{x - x_0}{y - y_0},\ a' = \frac{x - x_0}{z - z_0}$$

in which x, y, z are the co-ordinates of any point of mn; substituting these values of a and a' in (e_2), we shall obtain the equation

$$F\left[\frac{x - x_0}{y - y_0}; \frac{x - x_0}{z - z_0}\right] = o \ . \ . \ . \ . \ (e_3)$$

between the constant co-ordinates x_0, y_0, z_0 and the co-ordinates of any point of mn. Now, the formula (e_3) is independent of the coefficients a, a', which are different according to the different position of mn. Therefore, whatever be the position of that line, the variable x, y, z shall represent the co-ordinates of any of its points, or, what is the same, the co-ordinates x, y, z contained in (e_3) are the co-ordinates of any point of the conical surface. Therefore (e_3) is the required equation.

Corollary. Supposing the curve described in a plane of co-ordinates, for instance, YAZ. Instead of the two equations (e) of the curve, we will have only one represented, for instance, by

$$y = \chi(z) \ . \ . \ . \ . \ (f_0).$$

It is evident that all the co-ordinates x of the points common to the curve and the generating line are equal to zero. Hence with regard to such points, the equations (e_1) become

$$y = y_\circ - \frac{1}{a} x_\circ, \; z = z_\circ - \frac{1}{a'} x_\circ$$

but such values are contained in $(f_\circ)$, hence they may be substituted there, in which case we will obtain

$$y_\circ - \frac{1}{a} x_\circ = \chi \left(z_\circ - \frac{1}{a'} x_\circ \right)$$

and substituting again instead of a and a', the corresponding values derived from (e_1), we will have

$$\frac{y^\circ (x - x_\circ) - x_\circ (y - y_\circ)}{x - x_\circ} = \chi \left[\frac{z_\circ (x - x_\circ) - x_\circ (z - z_\circ)}{x - x_\circ} \right]$$

or

$$\frac{y_\circ x - x_\circ y}{x - x_\circ} = \chi \left(\frac{z_\circ x - x_\circ z}{x - x_\circ} \right) \ldots (f_1).$$

EXAMPLE.

133. Let the equation $(f_\circ)$ represent an ellipse, since (63 (g)) $\frac{y^2}{A^2} + \frac{z^2}{B^2} = 1$, and, consequently,

$$y = \pm \frac{A}{B} \sqrt{B^2 - z^2}$$

The function χ signifies in this case that the square of the variable z is to be subtracted from the square of the constant B, and the square root of this difference is to be multiplied by the ratio $\frac{A}{B}$. But to have the equation of the conical surface, it is necessary to submit $\frac{z_\circ x - x_\circ z}{x - x_\circ}$ of the formula (f_2) to the same operations of z; hence

$$\frac{y_\circ x - x_\circ y}{x - x_\circ} = \pm \frac{\text{A}}{\text{B}} \sqrt{\text{B}^2 - \left(\frac{z_\circ x - x_\circ z}{x - x_\circ}\right)^2}$$

from which

$$(y_\circ x - x_\circ y)^2 = \frac{\text{A}^2}{\text{B}^2} [\text{B}^2 (x - x_\circ) - (z_\circ x - x_\circ z)^2]$$

the required equation of the surface. It is here to be remarked, that according to the equation of the ellipse, we must suppose the centre of that curve at the origin of the co-ordinates. Therefore, supposing, moreover, the cone to be right, the vertex or the point through which the generating line constantly passes shall be on the axis X, and of course the co-ordinates $y_\circ$, $z_\circ$ of that vertex must be equal to zero. In this supposition the preceding formula becomes

$$(x_\circ y)^2 = \frac{\text{A}^2}{\text{B}^2} [\text{B}^2 (x - x_\circ)^2 - (x_\circ z)^2]$$

or

$$\frac{y^2}{\text{A}^2} = \frac{(x - x_\circ)^2}{x_\circ^{\,2}} - \frac{z^2}{\text{B}^2}$$

and transposing to the vertex the origin of the co-ordinates, or, what is the same, substituting $x' + x_\circ$ for x, if every thing else remains as before, the last formula will become

$$\frac{y^2}{\text{A}^2} = \frac{x'^2}{x_\circ^{\,2}} - \frac{z^2}{\text{B}^2}$$

or, since we may use indiscriminately x or x' and $x_\circ$ is a constant quantity like A and B, substituting x to x' and C to $x_\circ$, we have

$$\frac{y^2}{\text{A}^2} = \frac{x^2}{\text{C}^2} - \frac{z^2}{\text{B}^2}$$

from which

$$\frac{x^2}{\text{C}^2} - \frac{y^2}{\text{A}^2} - \frac{z^2}{\text{B}^2} = o$$

equation corresponding (120) to the third species of surfaces having centres.

General equation of any cylindrical surface.

134. Suppose (132) the equations of the curve in space to be

$$z = \phi(x) \text{ , } y = \psi(x) \text{ . . . . } (e)$$

and the equations of the generating line (35 (o))

$$x = ay + b \text{ , } x = a'z + b'$$

or $$\left. y = \frac{1}{a}x - \frac{b}{a} \text{ , } z = \frac{1}{a'}x - \frac{b'}{a'} \right\} \text{ . . . } (e_1).$$

In the present supposition of the describing line constantly parallel to itself, the co-efficients $\frac{1}{a}$, $\frac{1}{a'}$ will be constant quantities, while $\frac{b}{a}$, $\frac{b'}{a'}$ will vary for every different position of the line. Now, observe, here, as in the similar case (132) that for the points common to the generating line and the curve, the co-ordinates of (e) and (e_1) must be the same. Hence in this hypothesis

$$\psi(x) = \frac{1}{a}x - \frac{b}{a} \text{ , } \phi(x) = \frac{1}{a'}x - \frac{b'}{a'}$$

from which by eliminating x, we are enabled to derive an equation between the constant quantities a , a' and the variables b , b', which we represent by

$$\chi(b \text{ , } b') = o \text{ . . . } (e_2).$$

But from (e_1) $b = x - ay$, $b' = x - a'z$, in which x , y , z are the co-ordinates of the generating line, which will be differently situated according to the different values of b and b'. Now, in every case, by substituting the last values of these two variables in (e_2), we will obtain

$$\chi(x - ay \text{ , } x - a'z) = o \text{ . . . } (e_3).$$

Which giving the same relation between the co-ordinates of the generating line in every one of its positions, is the general equation of the cylindrical surface.

Corollary. Supposing the curve to be described on the plane XAZ, and represented by

$$y = \phi(z) \quad . \quad . \quad . \quad . \quad (f_0).$$

Since in this case all the co-ordinates x of the points common to the curve and the generating line are equal to zero, the equations (e_1) with regard to such points will become

$$y = -\frac{b}{a}, \; z = -\frac{b'}{a'}$$

and since in this same case the values of y, z given by these equations fulfil the equation (f_0); so they may be substituted there, in which case

$$-\frac{b}{a} = \phi\left(-\frac{b'}{a'}\right)$$

and substituting for b and b' the corresponding values

$$y - \frac{x}{a} = \phi\left(z - \frac{x}{a'}\right) \; . \; . \; . \; (f_1)$$

EXAMPLE.

135. Suppose the equation (f_0) to be that of a parabola, that is,

$y = \pm\sqrt{2pz}$ hence (133) $y - \dfrac{x}{a} = \sqrt{2p\left(z - \dfrac{x}{a'}\right)}$

and $$\left(y - \frac{x}{a}\right)^2 = 2p\left(z - \frac{x}{a'}\right)$$

the equation of a surface produced by parallel motion of a straight line describing a parabola with one of its extremities.

2nd [illegible] Mathematics

Mr. John Pendergrast, Teacher

1854 & 55.

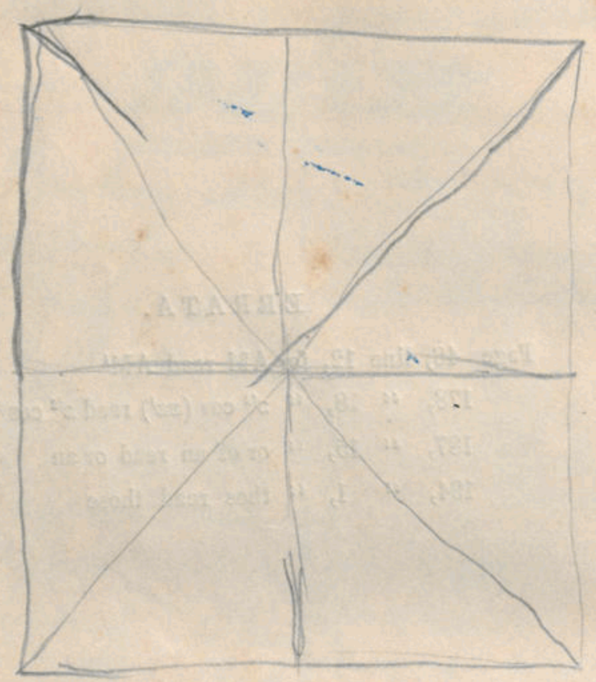
Henry Bowling - Md
John Bowling - "
William Blandford "
William J. Clarke - Geo
Henry A. Cecil - Ky
William B. Choice - S.Ca
Edgar D'aquin - La
Eugene Digges - Md
Jas. D. Dougherty - Pa
William J. Hill - Md
Theodore Jenkins - "
Charles B. Kenny - Pa
Henry Loughborough D.C
Francis A. Lancaster - Pa
George C. Merrick - Md
Jas. Miltenberger La
Lucius B. Northrop - S.Ca
John E. Plater - D.C.
James R. Randall - Md
B. R. Riordan - D.C.
Chas Shields - Ireland
Carroll Smith - Pa
Edward Scott - S.Ca

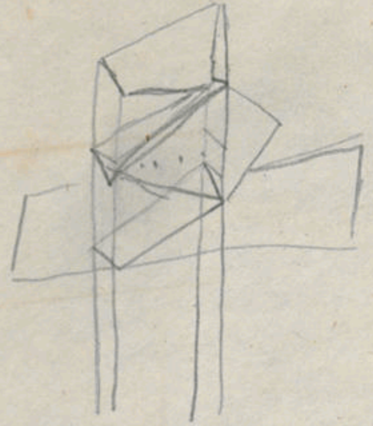
Left during the year
1854
Henry Loughborough - D.C
James Miltenberger - La
1855
Edward Scott - S.Ca
Carroll Smith - Pa
William J. Clarke - Geo
George C. Merrick - Md
Eugene Digges - "
John E. Plater - D.C
Chas Shields - Ireland
Henry A. Cecil - Ky

Theodore

ERRATA.

Page 46,	line 12,	for	AM	read AM′
178,	" 18,	"	$z'^2 \cos (xx')$	read $z'^2 \cos^2 (xx')$
187,	" 15,	"	or of an	read or an
194,	" 1,	"	thos	read those





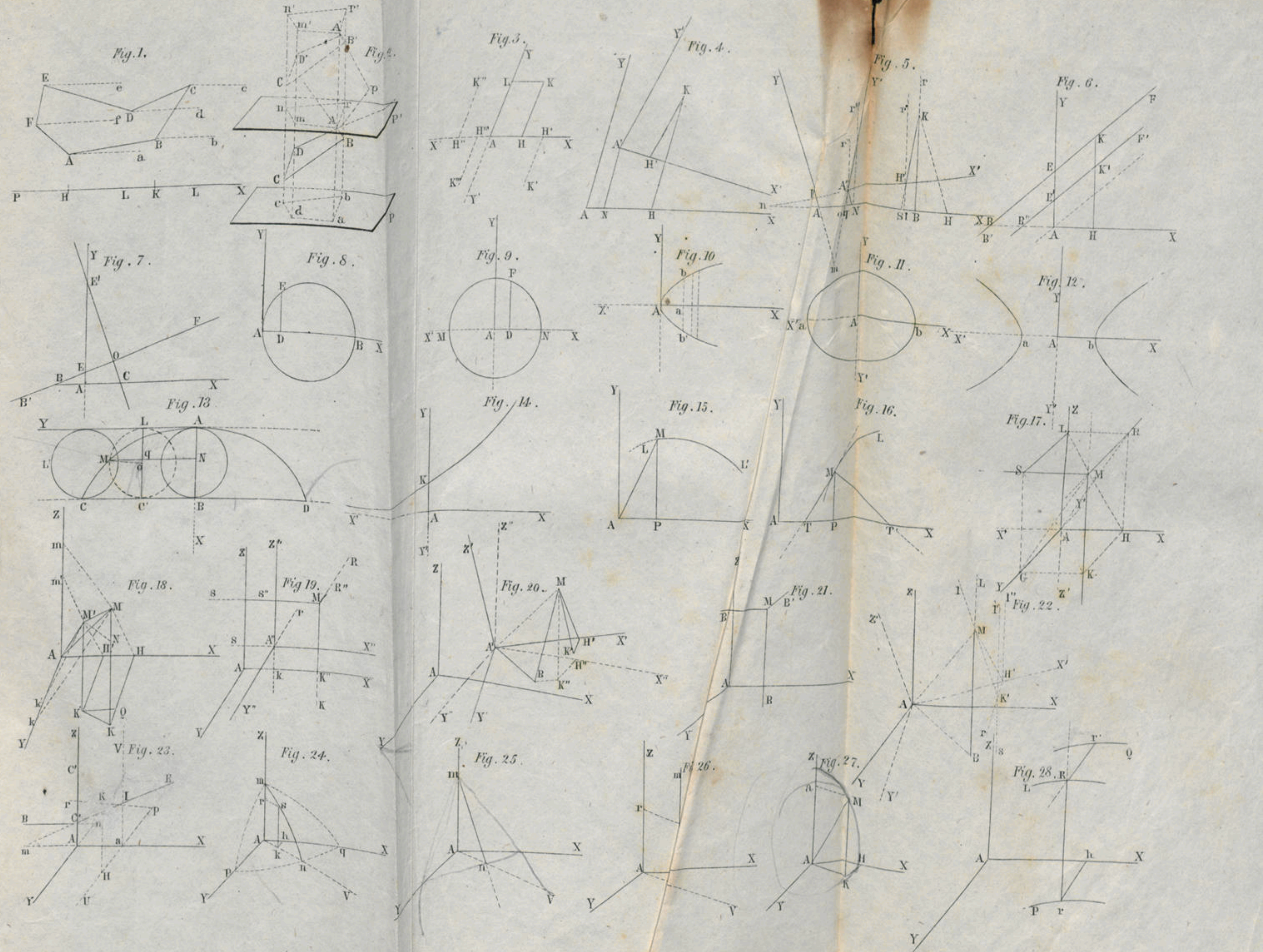


Fig. 1.
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Fig. 11.
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Fig. 18.
Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26.
Fig. 27.
Fig. 28.

veluti ovis ad occisionem ducitur, he shall not open his mouthe & when he shall not answer

$mz + my + q - x$
$mz + my + q - x$

$m^2z^2 + m^2zy + mzq - mzx$
$m^2zy + m^2y^2 + myq - myx$
$mzq + myq + q^2 - qx$
$+ mzx + myx + qx - x^2$

$m^2z^2 + 2m^2zy + 2mzq + m^2y^2 + 2myq + q^2 - x^2$

Fig. 31.

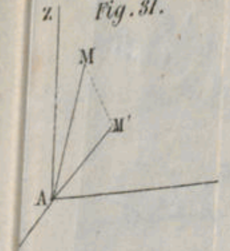


Fig. 37.

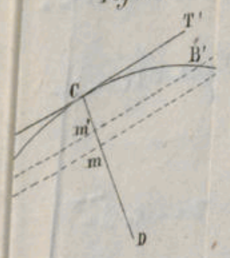


Fig. 45.

Fig. 52.

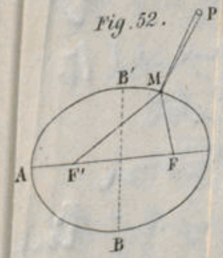
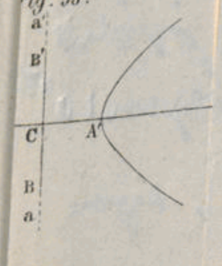


Fig. 58.



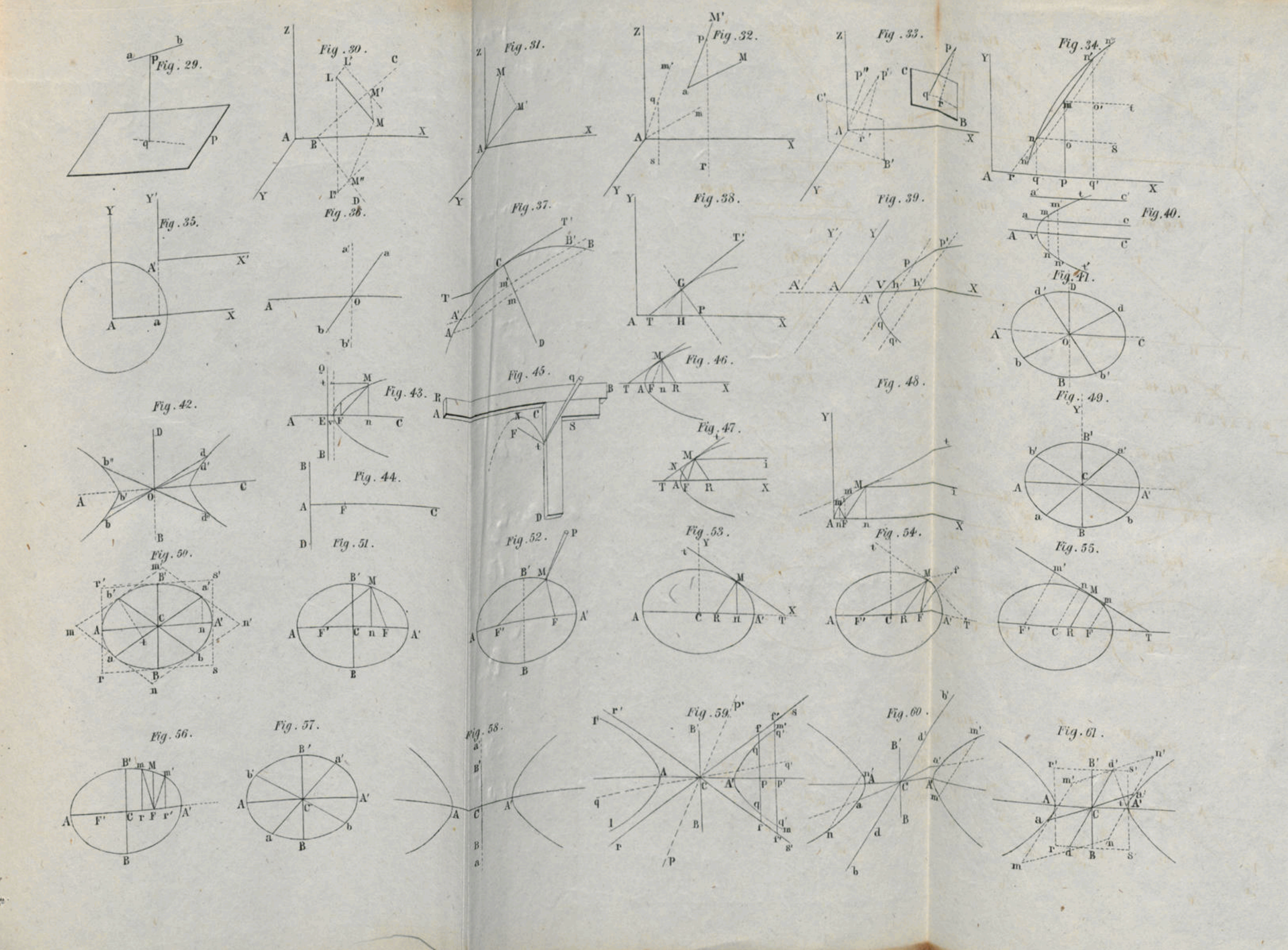


Fig. 29.
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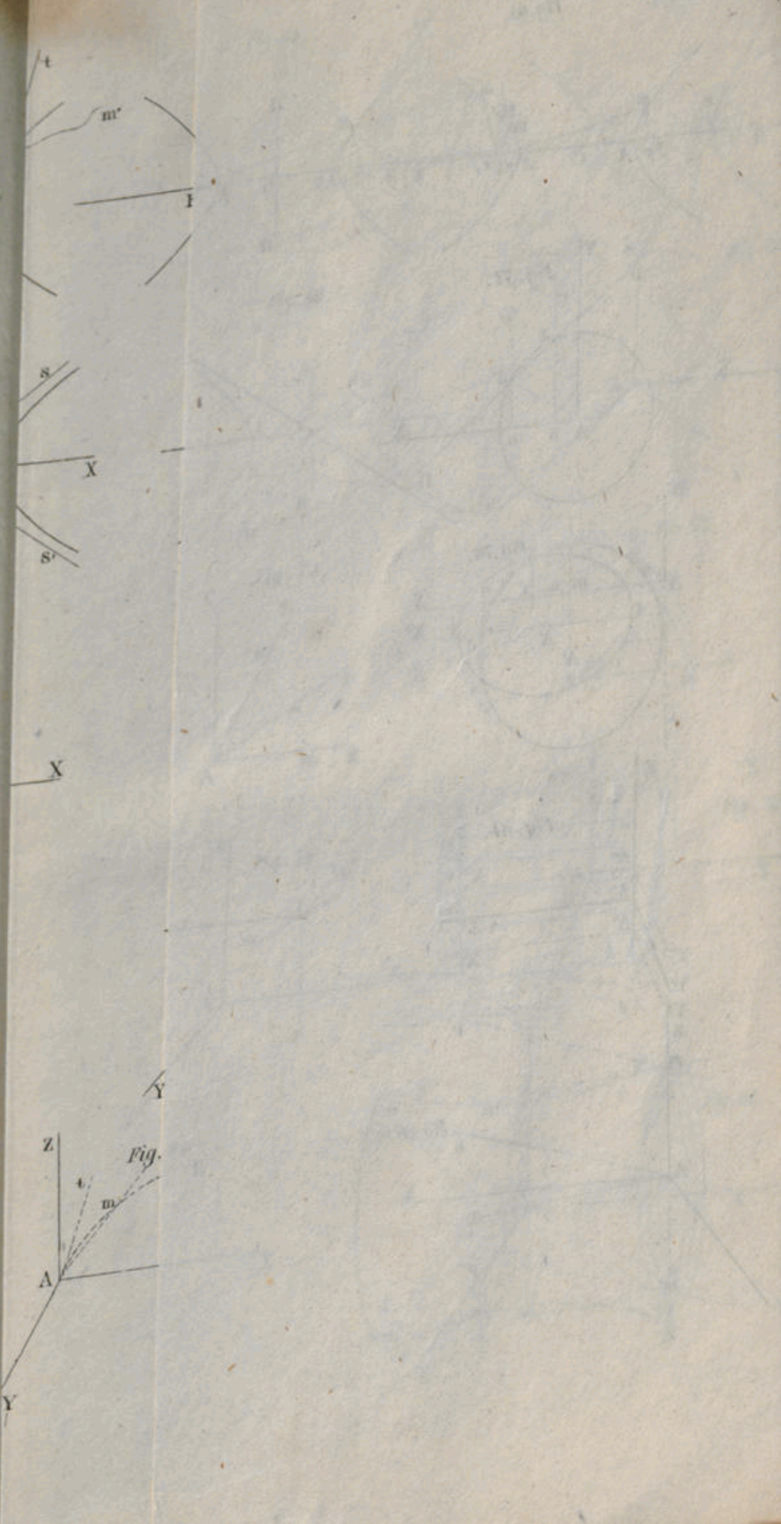
Fig. 55.
Fig. 56.
Fig. 57.
Fig. 58.
Fig. 59.
Fig. 60.
Fig. 61.

Mensuration of solids
1st book
1st to find surface of right prism
2nd find surface of right pyramid
3rd find solidity of a prism.
4th find " " " sphere

Analytical Geometry
prop 1st — Definitions & remarks of relative position of points from 4th to 6th pages
2 formulas for passing from one system of axes to another from 5th to 6th
3. what is an equation
4 Equation of a straight line
5 Equations of periphery of circle &c & analysis or discussion of Equations begin 10th and 11th
6th polar coordinates begin page 17 end of book

2nd Book
Two different manner of ascertaining &c - page 17
2 ~~formula~~
Equation of the plane begin p 28 end p 29



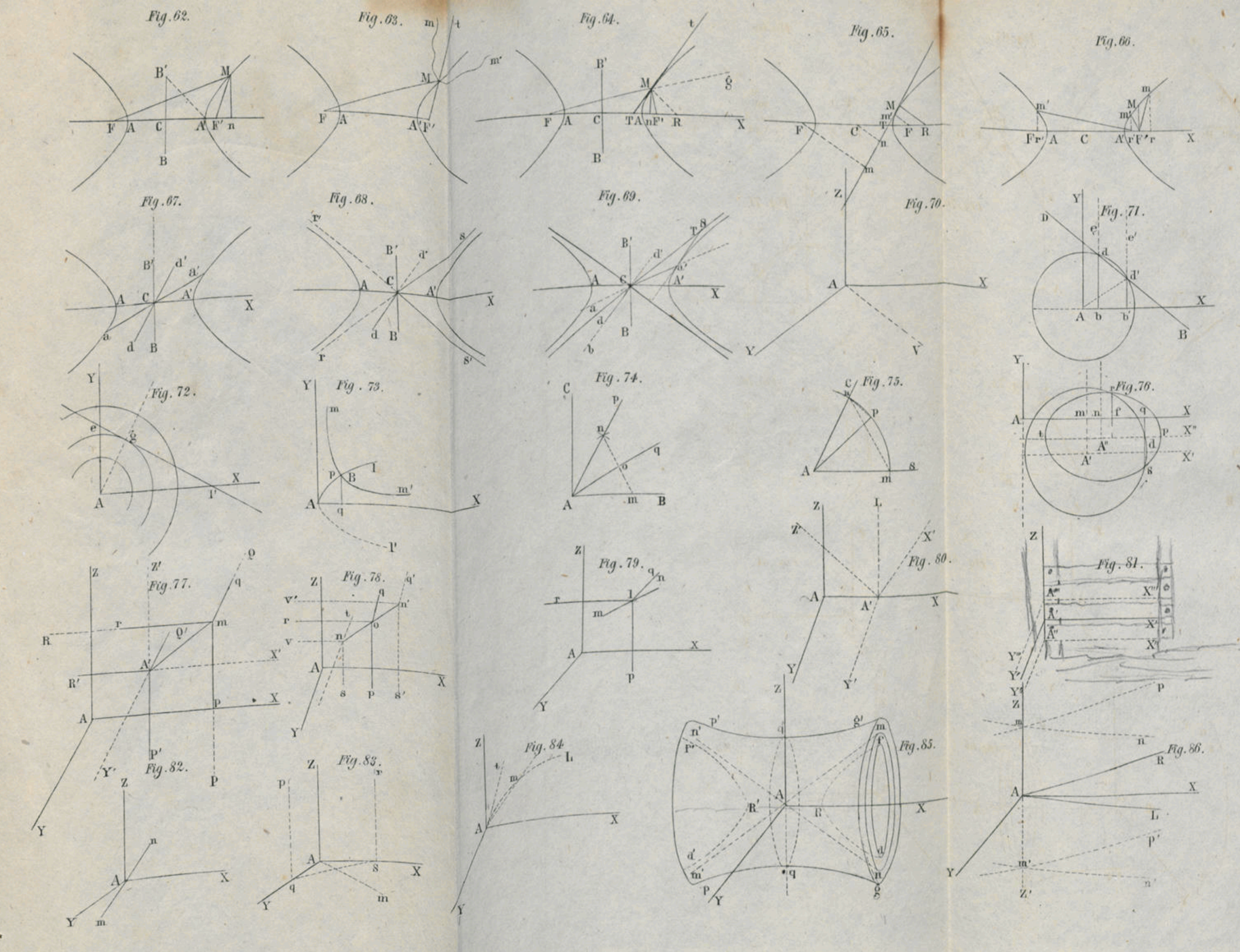


Fig. 62.
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Fig. 78.
Fig. 79.
Fig. 80.
Fig. 81.
Fig. 82.
Fig. 83.
Fig. 84.
Fig. 85.
Fig. 86.

no where does he name them all together
one name nor any others than those ...
A from B they were the 1st 36. But he call
did he call others Barbarians [illegible] Nor any more
[illegible] not having as yet been distinguished
by one name in opposition to the rest as appears
[illegible] therefore the Hellenes as
seperately [illegible]
as many as understood each other according to [illegible]
& afterwards called all by the
same name in a body did nothing

2nd MATHEMATICS

1854 & 55 John Pendergrast S. J. Teacher.

William B. Blandford - Md	4
John Bowling - - Md	12
Henry Bowling - - Md	5
William J. Clarke - Geo	+
William B. Choice - S. Ca	2
Henry A. Cecil - Ky	+
Edgar D'aquin - La	6
Jas D. Dougherty - - Pa	1
Eugene Digges + - Md	+
William J. Hill - - Md	11
Theodore Jenkins - Md	10
Charles B. Kenney - Pa	8
Henry Loughborough - D.C.	+
Frances A. Lancaster - Pa	9
James Miltenberger La	+
George C. Merrick - - Md	+
Lucius B. Northrop - S. Ca	7
John C. Plater - D.C.	+
James R. Randall - Md	3.
Rotchford B. Riordan DC	13
Charles Shields - Ireland	+
Carrol Smith - Pa	+
[illegible] - S. Ca	+