MANUAL

OF

GEOMETRICAL AND INFINITESIMAL

ANALYSIS.

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PREFACE.

THIS manual, prepared with the view of its serving as an introduction to the study of Physical Science, was only intended for a class of students intrusted to the care of the compiler. The suggestion of friends that the work might prove advantageous to others induces him to offer it to the public.

Works of analysis — some of them voluminous — are not wanting; nor does our little book pretend to give a complete development of its subject. For this reason we call it a manual, which excludes all discussions the results of which are seldom or never called into use in the applications. It is hoped, however, that it will sufficiently serve the purpose intended.

A detailed Index will contribute to render the manual more useful. It will also give a better idea of the nature of this little work.

We leave it to the reader to judge whether, without detriment to lucidity, our efforts to combine comprehensiveness with brevity and exactness have been successful.

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WOODSTOCK COLLEGE, MD., *January* 18, 1871.

☞ *N. B. When Theorems of Algebra, Trigonometry, &c., are mentioned, reference is made to books previously published by the author of this Manual.*

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PRINCIPLES OF

ANALYTICAL GEOMETRY.

I. *Rectilinear and polar co-ordinates of a point on a plane surface.*

LET XX′, YY′ (Fig. 1) be two indefinite straight lines cutting each other in A and forming any angle. Take any point M on the plane of the two lines, or, as they are called, *axes*, and from M draw MH, MK parallel to the axes. The position of the point M is evidently determined relatively to the axes by these parallels, which are called *rectilinear co-ordinates* of the point M. MH, or its equal AK, represented by x, is called the *abscissa*, and MK, represented by y, is called the *ordinate*. XX′ is the *axis of abscissas*, YY′ the *axis of ordinates*, A the *origin of co-ordinates*. The axes are called *orthogonal* when they cut each other at right angles; otherwise, *oblique*. In every case, taking for positive the abscissas from A toward X and the ordinates from A toward Y, the abscissas from A toward X′ and the ordinates from A toward Y′ must be considered negative. It is thus plain that, varying x and y from $-\infty$ to $+\infty$, we may, by means of them, determine the position of all and each of the points of the plane.

Polar co-ordinates offer the same advantage. Let A be a point on the plane, and let XAX′ be a fixed straight line or axis on the same plane and passing through A. Let again M be any point of the plane, and AM, represented by ρ, be its distance from A. The angle MAX, which we call ω, together with ρ, determines the position of the point M on the plane; and these are called *polar co-ordinates* of the point M.

The point or centre A is called *pole*, the fixed line XAX′ *polar axis*, the distance AM *radius vector*. To determine the position of all and each of the points of the plane, it is necessary to vary the radius vector from 0 to ∞ and the angle ω from 0 to 360°.

II. *Orthogonal and oblique rectilinear co-ordinates.*

Let (Fig. 2) AX, AY be orthogonal axes, and A′X′, A′Y′ oblique axes. Any point M referred to the first system shall have $x = \text{AK}$ and $y = \text{KM}$ for its co-ordinates, and, referred to the second system, $x' = \text{A'K'}$, $y' = \text{K'M}$. Call x_0, y_0 the co-ordinates AB, BA′ of the origin A′ referred to the orthogonal axes. The angles which A′X′, A′Y′ form with AX are, for brevity sake, represented by $(x'x)$, $(y'x)$. Draw now from A′ and K′, A′C, K′D parallel to AX, and K′C parallel to AY, we shall have

$$x = \text{AB} + \text{A'C} + \text{K'D},\ y = \text{BA'} + \text{CK'} + \text{DM};$$

i. e., (1) $\begin{cases} x = x_0 + x' \cos (x'x) + y' \cos (y'x) \\ y = y_0 + x' \sin (x'x) + y' \sin (y'x). \end{cases}$

By means of these formulas, which give us the orthogonal co-ordinates by the oblique, we may pass from one to another system of axes.

Orthogonal and polar co-ordinates.

Let (Fig. 2) A be the pole, and AX the polar axis, and M any point on the plane XAY, having $\text{AM} = \rho$ for radius vector, and $\text{MAK} = \omega$ for corresponding angular co-ordinate. From the triangle MKA, right-angled in K, we obtain

$$(2) \quad \begin{cases} x = \rho \cos \omega \\ y = \rho \sin \omega. \end{cases}$$

And by means of these formulas we may pass from the orthogonal to the polar co-ordinates.

III. *Equation of the straight line.*

As each point on the plane of the rectilinear axes has its co-ordinates, so the different points of a line described on that plane have their own co-ordinates. Now it happens that the value of the ordinate, given by the corresponding abscissa, is found to be given under the same form for each point of the line; for instance, we may find $y = ux$, or, more generally, $y = f(x)$ for each and all points of the line referred to the axes; $y = f(x)$ is, in this case, called the equation of the line or of the *geometrical locus* described on the plane of the axes, whatever the form of the line may be. To give an example: Let (Fig. 3) RR be a straight line on the plane XOY of the orthogonal axes, and let $x =$ OK, $y =$ KM be the co-ordinates of one of its points M. We shall have $\frac{MK}{AK} =$ tgMAX, and calling a this tangent,

$$\frac{y}{x + AO} = a \quad \text{or} \quad y = ax + a \cdot AO.$$

Now we have from the right-angled triangle ABO, BO = $a \cdot$ AO. BO is the ordinate of RR corresponding to the origin of the axes, and which we represent by b. Hence

$$(3)\ y = ax + b.$$

But M is any point of RR′; hence this equation of the first degree represents the corresponding co-ordinates of any point of the straight line, and is, consequently, the equation of the same line. We infer from this,

1st. If the straight line passes through the origin of the axes, the equation of the line is

$$y = ax.$$

2d. If the same line, whether it passes through the origin or not, passes through a point, C, for instance, having $x_,$ and $y_,$ for its co-ordinates, together with the above equation, we shall have $y_, = ax_, + b$ or $y_, = ax_,$; and in both cases

$$y - y_, = a(x - x_,)$$

the equation of a straight line passing through a given point $(x_,, y_,)$.

3d. If R′R′ represents another straight line parallel to RR, calling b' the ordinate OB′, corresponding to the origin, since B′A′X = BAX, the equation of the parallel will be

$$y = ax + b'.$$

4th. If NN′ is drawn perpendicular to RR, calling b'' the ordinate, OI corresponding to the origin, its equation will be

$$y = tg\mathrm{N'LX} \cdot x + b''.$$

But on account of the right-angled triangle ACL, and because $tg\mathrm{N'LX} = -\, tg\mathrm{N'LO}$,

$$tg\mathrm{N'LX} = -\, cot\mathrm{MAO} = -\frac{1}{tg\mathrm{MAO}} = -\frac{1}{a};\ \text{hence}$$

$$y = -\frac{1}{a}x + b''$$

is the equation of a straight line perpendicular to one represented by $y = ax + b$. It follows, therefore, that in order that two straight lines represented by the equations

$$y = mx + c, \quad y = m'x + c'$$

be perpendicular to one another, it is necessary and sufficient that $m' = -\frac{1}{m}$, or that the equation

$$mm' + 1 = 0$$

be verified.

IV. *Equation of the circle.*

Let r be the radius of the circle AMB (Fig. 4), C the centre, which is at once the origin of the orthogonal axes CX, CY. Let M be any point of the periphery, and CK, KM the co-ordinates x, y of that point. Drawing the radius CM, the right-angled triangle CMK gives us

$$y^2 = r^2 - x^2,$$

which being verified for any point of the periphery, is the equation of the periphery, referred to the orthogonal axes having their origin in the centre.

But let the origin be the extremity A of the diameter AB, the axes AX, AY being still orthogonal, and AK, KM the co-ordinates x and y of the point M. Now MK is mean geometrical proportional between AK, KB, and therefore $y^2 = x\,(2r - x)$, or

$$y^2 = 2rx - x^2;$$

which being verified for any point of the periphery, is consequently the equation of the periphery, referred to the orthogonal axes having their origin at the extremity of a diameter.

But let the origin of the orthogonal axes be anywhere, for instance in O, and let OH, HM be the co-ordinates x, y of any point M of the periphery referred to the orthogonal axes OX, OY, and parallel to CX, CY. Call a and b the co-ordinates OD, DC of the centre of the circle. We have, from the right-angled triangle CKM, $\overline{CK}^2 + \overline{MK}^2 = \overline{CM}^2$; but $CK = OH - OD = x - a$, $MK = MH - DC = y - b$ and $CM = r$; hence

$$(x - a)^2 + (y - b)^2 = r^2,$$

an equation of the second degree, like the preceding, and which, being verified for every point of the periphery relatively to the axes OX, OY, is therefore the equation of the periphery, referred to these axes.

V. *Equation of the parabola.*

Let (Fig. 5) the straight lines DD′, CX be perpendicular to each other, take a point F anywhere on CX, and let A be the middle point of CF. Let now the curved line MHAM′ pass through A, and as the point A is equally distant from F and DD′(the distance of a point from a straight line is known to be the perpendicular from the point to the line), so let all the points of MHAM′ be equally distant from DD′ and from F. This curve line is the *parabola*, the point F is called the *focus*, DD′ is called *directrix*, CX the *axis* of the curve, and the point A of intersection between the axis and the curve the *vertex*.

Let now A be taken for the origin of the orthogonal axes

AX, YAY′, to which we refer the curve, and let AK, KM be the co-ordinates of any of its points M. Join F with M, and let the variable FM, called *radius vector*, be represented by ρ, and let the constant CF be represented by p. We shall have first,

$$AF = AC = \tfrac{1}{2}p;$$

and since MF is equal to MN perpendicular to DD′, and MN = AK + AC,

$$\rho = x + \tfrac{1}{2}p,$$

from which $$\rho^2 = x^2 + px + \tfrac{1}{4}p^2,$$

but from the right-angled triangle MKF $\overline{MF}^2 = \overline{MK}^2 + (KA - AF)^2$; hence also

$$\rho^2 = y^2 + x^2 - px + \tfrac{1}{4}p^2.$$

From this and the preceding value of ρ^2 we obtain easily

$$y^2 = 2px,$$

which, being verified for each and all the points of the curve, is the equation of the parabola referred to the orthogonal axes having their origin in the vertex, the axis of the curve being axis of abscissas. The constant $2p$ is called *parameter* or measure of the curve. In fact, it is plain from the above equation that, supposing the same values for x, the branches of the curve will open more or less according to the magnitude of the parameter. And indeed, from the *analysis* of the above equation we may infer the properties of the curve.

VI. *Analysis of equations and geometrical loci.*

To find out the curve or the *geometrical locus* to which a given equation belongs, and to find out from the same equation the properties of the corresponding geometrical locus, is called the analysis of the equation, and the name of *analytical geometry* is accordingly given to the branch of science which has for object this analysis. This process is evidently the inverse of the preceding, as shown in the foregoing examples.

The equation $y^2 = 2px$ of the parabola, already obtained, may also be written as follows:

$$y = \pm \sqrt{2px},$$

in which p is positive. To find out the geometrical locus of this equation draw the axes XX′, YY′ at right angles and intersecting each other in A. Take then from A different abscissas x, and substituting their values in the last equation, the resulting values for y will be the ordinates corresponding to the abscissas and marking the geometrical locus with their extremities. It is plain, 1st, that with $x = o$, y also $= o$. 2d. No real ordinates correspond to negative abscissas. 3d. Two real ordinates correspond to each positive abscissa, equal in length, but opposite in sign, and these ordinates increase with x from o to ∞; i. e., the geometrical locus corresponding to the equation $y = \pm \sqrt{2p \cdot x}$ is a curve which cuts the axis of abscissas at the origin and touches the axis of ordinates at the same point; it has a double branch, one on each side of the positive axis of abscissas, and equal to one another, departing more and more from this axis as they do from the axis of ordinates by their increase.

VII. *Properties of the parabola.*

It follows besides, that AX, called also axis of the curve, bisects all the chords parallel to the tangent of the vertex A. Taking from A on XX′, AF = AC = $\frac{1}{2}p$, and drawing from C, DD′ the directrix parallel to YY′, it follows that each point of the curve is equally distant from the focus and the directrix. In fact, $\overline{MF}^2 = \overline{MK}^2 + (AK - AF)^2 = y^2 + (x - \frac{1}{2}p)^2 = 2px + x^2 + \frac{1}{4}p^2 - px = x^2 + px + \frac{1}{4}p^2 = (x + \frac{1}{2}p)^2$; hence $MF = x + \frac{1}{2}p$. But MN perpendicular to $DD' = KA + AC = x + \frac{1}{2}p$; hence MF = MN.

Resuming again the equation $y^2 = 2px$, observe that it is resolvable into the proportion

$$x : y :: y : 2p;$$

i. e., *the parameter in the parabola is a third proportional to any abscissa and the corresponding ordinate.*

Hence, calling q the ordinate drawn from the focus, since then $\frac{1}{2}p : q : q : 2p$, and consequently $q = p$, it follows that *the parameter of the parabola is equal to the double ordinate passing through the focus.*

VIII. *Tangent and other properties of the parabola.*

Join (Fig. 6) the focus F with N, the foot of the normal drawn to the directrix from any point M of the curve. Join also F with M, and draw from M, ME perpendicular to NF, produced to T, to meet the axis AX of the curve, as also on the opposite side toward P. This line TP is the tangent of the point M of the parabola. To show this, it is enough to prove, first, that none of the points of TP are equally distant from the focus and the directrix except M; secondly, that all the points of TP on either side of M are outside of the branch.

Because MF = MN and ME is perpendicular to NF, the triangles MEN, MEF are equal; hence NE = EF, and drawing from any point P of TP, PN, PF, these two oblique lines also are equal to one another. But drawing from P, PQ perpendicular to the directrix, since PQ < PN, it follows also that PF > PQ. The point P is then not equally distant from the focus and the directrix, and thereby not on the curve. Now such a point may be within the branches of the parabola or outside of them. In either case drawing from it a perpendicular to the axis AX of the curve, this perpendicular has all its points equidistant from the directrix, but only one of them is at once equidistant from the directrix and the focus, and this one is the point of intersection with the curve. All the points between the curve and its axis are nearer to, and all those outside of the branch are farther off from the focus than the directrix. But PF > PQ; hence all the points of TP on either side of M are outside of the branch, and TP touches the curve in M.

Produce NM to X′, the angle PMX′ = MTX; but PMX′ = TMN = TMF, hence the angles at M and T of the triangle MTF are equal to each other, and consequently FM = FT. But FM = MN = CK; therefore TF = CK, and consequently TC = FK. Now AC = AF (= $\frac{1}{2}p$), hence AT = AK; but AK is the abscissa x of the point M, hence TK = $2x$. The segment TK from the point T of the axis met by the tangent, to the point of the same axis corresponding to the ordinate of M, the point of contact, is called the *subtangent* of that same point. Therefore *the subtangent of any point of the parabola is equal to twice the abscissa of the same point.*

It follows from this that we may draw a tangent to any point M of the parabola by drawing first a perpendicular to the axis from that point, and taking on the axis a point twice the distance from the foot of the perpendicular than the vertex of the curve is. The straight line joining this point with M touches the curve on that point.

We have remarked in the preceding process that the angles PMX′, TMF are equal to each other; i. e.,

For any point of the parabola the parallel to the axis and the radius vector form equal angles with the tangent.

Draw now from M, MR perpendicular to the tangent. The segment MR of this perpendicular, between the point of contact and the axis, is called the normal of that point. Now from the equality of the last-mentioned angles it follows that FMR and RMX′ are also equal; i. e.,

The angle formed by the radius vector of any point of the parabola and a parallel to the axis drawn from that point, is bisected by the normal.

IX. *Equation of the parabola referred to different axes.*

Taking (Fig. 5) the origin of the axes, in the focus F, and considering FX as the negative axis of ordinates, and FX, perpendicular to the first as positive axis of abscissas, representing besides by $x_{,}$, $y_{,}$ the abscissas and ordinates of the new system corresponding to any point H of the curve, whose co-ordinates x, y relatively to the first system of axes are AB, BH; since AB = AF — FB and BH is ordinate with regard to the first, and abscissa with regard to the second system, we shall have

$$x = \tfrac{1}{2}p - y_{,} \text{ and } y = x_{,}.$$

These values substituted in the equation of the parabola $y^2 = 2\,px$ referred to the first system of axes, will give us

$$x_{,}^2 = p^2 - 2\,py_{,}.$$

Let us now take a *diameter* for axis of abscissas and the corresponding tangent for axis of ordinates.

A parallel A′X′ (Fig. 7) to the axis AX, drawn from any point A′ of the parabola, is called diameter. Take A′X′ as axis of abscissas and TY′ tangent in A′ as axis of ordinates, and let the co-ordinates of the curve referred to the new system be represented by x', y'. Call x_0, y_0 the co-ordinates AB, BA′ of the origin A′ of the new axes referred to the first system of axes and α the angle Y′TX, which the tangent Y′T forms with the axis of the parabola; we shall have (II.) $(x'x) = 0$, $(y'x) = \alpha$, and

$$x = x_0 + x' + y' \cos \alpha, y = y_0 + y' \sin \alpha.$$

Now, from the right-angled triangle A′BT, A′B or $y_0 =$ A′T $\sin \alpha$, and BT = A′T $\cos \alpha$, hence $\frac{y_0}{\text{BT}} = tg\ \alpha$, and $y_0 = \text{BT}\ tg\ \alpha$, but (VIII.) BT = $2x_0$; hence

$$y_0 = 2x_0\ tg\ \alpha.$$

Besides (V) $y_0^2 = 2px_0$, therefore, dividing this formula by the preceding,

$$y_0 = \frac{p}{tg\ \alpha} = \frac{p \cos \alpha}{\sin \alpha};$$

and since, from $y^2_0 = 2px_0$, $x_0 = \frac{y^2_0}{2p}$; we have also

$$x_0 = \frac{p \cos^2 \alpha}{2 \sin^2 \alpha};$$

hence, $x = \frac{p \cos^2 \alpha}{2 \sin^2 \alpha} + x' + y' \cos \alpha$, $y = \frac{p \cos \alpha}{\sin \alpha} + y' \sin \alpha$.
Substituting these values in the equation $y^2 = 2px$ of the parabola referred to the original axes, we obtain

$$y'^2 = 2 \frac{p}{\sin^2 \alpha} x'.$$

From which, calling p' the constant factor $\frac{p}{\sin^2 \alpha}$,

$$y'^2 = 2p'x'.$$

The coefficient $2p'$ is called parameter relatively to the diameter A′X′. It follows from this equation that all the chords parallel to the tangent TA′T are bisected by the diameter.

X. *Polar equation of the parabola.*

Take (Fig. 5) the pole in the focus, and, for polar axis, the axis of the curve from F toward the vertex A. Let M be any point of the curve. The polar co-ordinates of this point are $\rho =$ FM and MFA $= \omega$. Now MF equal to MN, perpendicular to the directrix, is also equal to KA + AC $= x + \frac{1}{2}p$, hence $\rho = x + \frac{1}{2}p$.

Now $x =$ AF + FK $= \frac{1}{2}p + \rho \cos$ MFK $= \frac{1}{2}p - \rho \cos \omega$. Substituting this value of x in the preceding equation, we obtain

$$\rho = p - \rho \cos \omega \text{ or } \rho = \frac{p}{1 + \cos \omega},$$

which is the equation of the parabola referred to the focus by means of polar co-ordinates.

XI. *Equation of the ellipse referred to its own axes.*

Let (Fig. 8) the straight line AA′ be equally divided in C, and let two points F, F′ be taken on it on each side of C, and at an equal distance from it. Let also the curve line ABA′B′ pass through A and A′, and let the sum of the distances of each of its points from F and F′ be equal to AA′. This curve is the *ellipse.* C is its *centre.* AA′, represented by $2a$, is called *transverse axis,* and BB′, perpendicular to AA′, and terminated by the curve, is called *conjugate axis,* and represented by $2b$. The points F, F′ are called *foci,* and the ratio $\frac{CF}{CA}$ (< 1), or its equivalent ratios $\frac{CF}{a}, \frac{CF'}{a}$ is called the *eccentricity,* which is represented by e. Thus

$$CF = CF' = ea.$$

Taking now C for the origin of the orthogonal axes, and CAX, CBY for positive axes of abscissas and of ordinates, let M be any point of the curve, and CK, MK be the corresponding abscissa and ordinate x and y of M. Join M with F and F′, MF, MF′, called *radius vectors,* are represented by ρ and ρ'. From the triangles MFK, MF′K, right-angled in K, we have

$$(r) \begin{cases} \rho^2 = y^2 + (x - ea)^2 \\ \rho'^2 = y^2 + (x + ea)^2. \end{cases}$$

Subtracting the first of these equations from the second we obtain the difference $\rho'^2 - \rho^2 = 4\,eax$; i. e.,

$$(\rho' + \rho)(\rho' - \rho) = 4\,eax.$$

Now $\rho' + \rho = 2\,a$; hence $\rho' - \rho = 2\,ex$; hence also $(\rho' + \rho) + (\rho' - \rho) = 2a + 2\,ex$; but $(\rho' + \rho) + (\rho' - \rho) = 2\rho'$; therefore

$$\rho' = a + ex \text{ and } \rho'^2 = a^2 + 2eax + e^2x^2.$$

Substituting this last value in the second (r), we obtain

$$a^2 + e^2x^2 = y^2 + x^2 + e^2a^2;$$

and consequently,

$$(1)\ y^2 = (1 - e^2)(a^2 - x^2),$$

which is the equation of the ellipse referred to the axes of the curve. Draw now from the foci the radius vectors FB, F′B, we shall have $FB = F'B = a$. Now $\overline{BC}^2 = \overline{BF}^2 - \overline{CF}^2$ and $BC = b$, $CF = ea$; therefore $b^2 = a^2 - e^2a^2 = a^2(1 - e^2)$; hence

$$(2)\quad (1-e^2)=\frac{b^2}{a^2}.$$

This value substituted in the preceding formula (1) gives for the same equation of the ellipse

$$(3)\quad y^2=\frac{b^2}{a^2}(a^2-x^2).$$

XII. *Analysis and corresponding geometrical locus of the equation.*

Taking the square root of the last equation, we obtain

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2};$$

the geometrical locus of which referred to rectangular axes cannot give but the ellipse having its centre in the origin of the axes, $2a$ for the transverse and $2b$ for the conjugate axis, which coincide with the axes of reference. In fact, making $x=o$ we obtain $y=\pm b$, and making $x=\pm a$, $y=o$; giving to x different values from o to $\pm a$, to each of these values correspond two values for y, one positive and one negative, equal in length and diminishing when x increases. But no real value of y corresponds to any value of x greater than a. The curve is therefore re-entering and symmetrical. Moreover, if $a>b$, $\sqrt{a^2-b^2}$ has a real linear value and less than a, thus $\frac{\sqrt{a^2-b^2}}{a}<1$, calling e this ratio, and taking on the transverse axis $\mathrm{CF}=\mathrm{CF'}=\sqrt{a^2-b^2}$, we shall have

$$\frac{\mathrm{CF}}{a}=\frac{\mathrm{CF'}}{a}=e.$$

Drawing now MF, MF′ from any point M of the curve, whose co-ordinates x and y are CK, KM, we shall have

$$\overline{\mathrm{MF}}^2=y^2+(\mathrm{CK}-\mathrm{CF})^2=\frac{b^2}{a^2}(a^2-x^2)+\left(x-\sqrt{a^2-b^2}\right)^2.$$

$$\overline{\mathrm{MF'}}^2=y^2+(\mathrm{CK}+\mathrm{CF'})^2=\frac{b^2}{a^2}(a^2-x^2)+\left(x+\sqrt{a^2-b^2}\right)^2.$$

Now $\frac{b^2}{a^2}(a^2 - x^2) = b^2 - \frac{b^2}{a^2}x^2$ and $(x \mp \sqrt{a^2 - b^2})^2 = x^2 \mp$

$$2x\sqrt{a^2 - b^2} + a^2 - b^2.$$

Substituting these values in the last members of the preceding equations, we deduce

$$x^2\left(\frac{a^2 - b^2}{a^2}\right) \mp 2x\sqrt{a^2 - b^2} + a^2 = \left(a \mp \frac{x\sqrt{a^2 - b^2}}{a}\right)^2.$$

Hence $\qquad \mathrm{MF} = a - x\frac{\sqrt{a^2 - b^2}}{a} = a - ex$

$$\mathrm{MF}' = a + x\frac{\sqrt{a^2 - b^2}}{a} = a + ex;$$

and therefore $\mathrm{MF} + \mathrm{MF}' = 2a$; i. e.,

The sum of the radius-vectors of any point of the curve is equal to the transverse axis, which is the characteristic property of the ellipse.

It is plain from the equation of the ellipse that all the chords parallel to the conjugate axis, as MKM′, are bisected by the transverse axis, and all the chords parallel to the transverse axis like MHM′ are bisected by the conjugate axis, and all of them form right angles with the bisecting axes. This inference may be rendered more evident, relatively to the chords parallel to the transverse axis, by transforming the equation, into the following:

$$x = \pm\frac{a}{b}\sqrt{b^2 - y^2}.$$

XIII. *Parameter of the ellipse.*

For the ellipse, as for the parabola, the parameter is the double ordinate passing through the focus. Either of the *foci* will evidently give the same parameter.

The equation of the ellipse which we have taken to analyze does not differ from the formula (1) XI. To obtain the parameter it is enough to make $x = \mathrm{CF} = ea$ in the equation; and taking the equation (1) for this purpose, we obtain (the parameter being represented by $2p$)

$$p^2 = (1 - e^2)(a^2 - e^2a^2) = a^2(1 - e^2)^2;$$

and from the formula (2) XI.,

$$p = a(1 - e^2) = \frac{b^2}{a};$$

from which $2a : 2b = 2b : 2p.$

Thus, *The parameter, in the ellipse, is the third proportional after the transverse and the conjugate axis.*

XIV. *Tangent and normal.*

Produce (Fig. 9) the radius vector F′M to N so that MN = MF. Join F with N, and draw from M, MET perpendicular to FN. This perpendicular is the tangent of the point M, for any other point of it is out of the curve. Let P be one of these points, join it with F, F′ and with N. The equal right-angled triangles FME, NME give us EF = EN, hence also PF = PN. Now from the triangle F′NP, F′P + PN > F′N; hence also F′P + FP > F′N; but F′N = F′M + MF = $2a$, therefore F′P + FP > $2a$; the point P therefore is outside of the ellipse and PT touches the ellipse in M.

Draw from M, MR perpendicular to the tangent. This perpendicular is called the *normal* of the point M. Since MR is parallel to NF, the angles RMF, MFN are equal, as also the angles F′MR, MNF; but MFN = MNF, therefore RMF = F′MR; i. e., *The normal of any point of the ellipse bisects the angle formed by the radius vectors of that point.*

It follows then that F′MP = FMT; i. e.,

The radius-vectors of any point of the ellipse form equal angles with the tangent.

XV. *Diameters of the ellipse — Conjugate diameters.*

A straight line passing through the centre of the ellipse and terminated on both sides by the periphery, is called *diameter*.

Let now (Fig. 10) DCD′ be one of these diameters passing through the middle point O of the chord MM′, which we shall represent by $2c$. Draw from M, O, M′ the perpendiculars MK, OH, M′K′ to the transverse axis AA′, and, from O and M′, the perpendiculars ON, M′N′ to MK, OH. Call $x_,$, $y_,$ the co-ordinates CH, HO of the middle point O of the chord referred to the orthogonal axes A′AX, B′BY, and let β be the angle which MM′ forms with the positive axis of abscissas. From the equation (3) XI. of the ellipse referred to these same axes, we have

$$(1) \quad y^2 = b^2 - \frac{b^2}{a^2}x^2;$$

therefore $\overline{KM}^2 = b^2 - \frac{b^2}{a^2}\overline{CK}^2$, $\overline{K'M'}^2 = b^2 - \frac{b^2}{a^2}\overline{CK'}^2$; but

$KM = HO + NM = y_, + c \sin\beta$, $CK = CH - ON = x_, + c\cos\beta$.
$K'M' = HO - ON' = y_, - c\sin\beta$, $CK' = CH + N'M' = x_, - c\cos\beta$.

Making a substitution of these values in the preceding equations, we obtain

$$(y_, + c \sin \beta)^2 = b^2 - \frac{b^2}{a^2}(x_, + c \cos \beta)^2$$

$$(y_, - c \sin \beta)^2 = b^2 - \frac{b^2}{a^2}(x_, - c \cos \beta)^2;$$

and taking the second of these formulas from the first,

$$4\, y_, c \cdot \sin \beta = -4\frac{b^2}{a^2}x_, c \cdot \cos \beta;$$

hence also $\qquad (2) \quad y_, = -\frac{b^2 \cot \beta}{a^2}x_,.$

Now the angle β does not change for any chord parallel to MM′, hence the last equation would be obtained in equal manner for the co-ordinates of the middle point of any chord parallel to MM′; hence the same equation represents the geometrical locus passing through the middle points of a system of parallel chords. But the geometrical locus represented by the equation is (III. 1st) a straight line passing through the origin of the axes, hence all the chords parallel to MM′ are bisected by the diameter DD′; and to bisect any system of parallel chords in the ellipse, it suffices to draw a diameter from the middle point of any one of them.

It follows, from what precedes, that the diameter EE′ parallel to MM′ is also bisected by DD′, and, as we shall see in the next article, EE′ bisects in its turn all the chords parallel to DD′. These diameters, each one of which bisects the chords parallel to the other, are called *conjugate diameters*.

Calling now α, the angle which DCD′ forms with the positive axis CX, the equation of D′CD referred to the axes A′X, B′Y is (III.) $y = tg\alpha x$; but the equation of the same line, as we have seen above, is also $y = -\frac{b^2 \cot \beta}{a^2} x$, hence $tg\alpha = -\frac{b^2 \cot \beta}{a^2}$, and consequently

$$(3) \quad tg\alpha \, tg\beta = -\frac{b^2}{a^2}.$$

Therefore, the condition to be verified, in order that two diameters be conjugate diameters, is that the product of the tangents of their respective angles with the positive axis of abscissas be equal to the negative quotient of the square of the conjugate semiaxis divided by the square of the transverse semiaxis.

To determine the length of the conjugate semidiameters $CD = a'$, $CE = b'$, observe that the co-ordinates x and y of the extremity D of CD are respectively equal to $a' \cos \alpha$, $a' \sin \alpha$. Now D is one of the points of the ellipse represented by the preceding equation (1); therefore, the co-ordinates of this point substituted in (1) fulfil that equation, i. e., $a'^2 \sin^2 \alpha = b^2 - \frac{b^2}{a^2} a'^2 \cos^2 \alpha$, or

$$a'^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) = b^2 a^2.$$

In like manner the co-ordinates x and y, of the extremity

E of CE, being respectively represented by $b' \cos \beta$ and $b' \sin \beta$, substituted in the same equation (1), give us

$$b'^2 (a^2 \sin^2 \beta + b^2 \cos^2 \beta) = b^2 a^2.$$

Hence, from this and from the preceding formula, we infer

$$(4) \begin{cases} a'^2 = \dfrac{a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \\ b'^2 = \dfrac{a^2 b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}. \end{cases}$$

XVI. *Equation of the ellipse referred to conjugate diameters.*

Let us now refer the curve to the conjugate diameters, taking D′CX′ for axis of abscissas and E′CY′ for axis of ordinates. Representing by x', y' the co-ordinates of the curve referred to this system of axes, we shall have from the formulas (1) II.,

$$x = x' \cos \alpha + y' \cos \beta, \ y = x' \sin \alpha + y' \sin \beta.$$

These values substituted in the equation (1) of the preceding paragraph, first reduced to the following form,

$$(1) \ldots \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

give $\left(\dfrac{\cos^2 \alpha}{a^2} + \dfrac{\sin^2 \alpha}{b^2}\right) x'^2 + \left(\dfrac{\cos^2 \beta}{a^2} + \dfrac{\sin^2 \beta}{b^2}\right) y'^2$

$$+ 2 \left(\frac{\cos \alpha \cos \beta}{a^2} + \frac{\sin \alpha \sin \beta}{b^2}\right) x'y' = 1.$$

Now from the formula (3) of the preceding paragraph XV. we infer $\sin \alpha \sin \beta = -\dfrac{b^2}{a^2} \cos \alpha \cos \beta$, and this value substituted in the last equation destroys its third term. With regard to the first and second term, which are equivalent to $\dfrac{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}{a^2 b^2}$, $\dfrac{a^2 \sin^2 \beta + b^2 \cos^2 \beta}{a^2 b^2}$, the last formulas of the preceding paragraph change them into $\dfrac{1}{a'^2}, \dfrac{1}{b'^2}$; hence the above equation assumes the form

$$(2) \quad \frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

altogether similar to the preceding (1) which represents the ellipse referred to the axes; and as from the nature of that equation we infer that all the chords parallel to one of the axes is necessarily bisected by the other axis, so also it follows from the last equation that all the chords parallel to one of the conjugate diameters are bisected by the other diameter.

XVII. *Polar equation of the ellipse.*

We have found (XI.) that $\rho + \rho' = 2a$ and $\rho' = a + ex$. Taking the focus F (Fig. 9) for pole and AA′ for polar axis, and calling ω the angle MFA. Since $x = CK = CF + FK$ and (XI.) $CF = ea$, $FK = \rho \cos \omega$, we have also $x = ea + \rho \cos \omega$, but from the above equations $\rho = 2a - \rho' = 2a - a - ex = a - ex$, therefore

$$\rho = a - e^2a - e\,\rho \cos \omega;$$

hence

$$(1) \quad \rho = \frac{a(1-e^2)}{1 + e \cos \omega};$$

or (XIII.) on account of $a(1-e^2) = p = \frac{b^2}{a}$,

$$(2) \quad \rho = \frac{p}{1 + e \cos \omega} = \frac{b^2}{a(1 + e \cos \omega)}.$$

Each of these equations, having no other variables but ω and ρ, represents the ellipse referred to the polar co-ordinates.

XVIII. *Theorems concerning the axes and conjugate diameters.*

From the first of the formulas, (4) XV., we obtain

$$a'^2\, a^2 \sin^2 \alpha + a'^2\, b^2 \cos^2 \alpha = a^2\, b^2 = a^2\, b^2 (\sin^2 \alpha + \cos^2 \alpha),$$

which, divided by $\cos^2 \alpha$, and resolved with regard to $tg^2\, \alpha$, gives

$$tg^2\, \alpha = \frac{b^2\,(a^2 - a'^2)}{a^2\,(a'^2 - b^2)}.$$

In like manner we obtain from the second formula

$$tg^2\,\beta = \frac{b^2\,(a^2 - b'^2)}{a^2\,(b'^2 - b^2)},$$

and from these two

$$tg^2\,\alpha\;tg^2\,\beta = \frac{b^4}{a^4}\cdot\frac{a^4 - a^2a'^2 - a^2b'^2 + a'^2b'^2}{a'^2b'^2 - b'^2b^2 - a'^2b^2 + b^4};$$

but from the equation (3) XV. $tg^2\,\alpha\;tg^2\,\beta = \frac{b^4}{a^4}$, hence

$\frac{a^4 - a^2a'^2 - a^2b'^2 + a'^2b'^2}{a'^2b'^2 - b'^2b^2 - a'^2b^2 + b^4} = 1$, and consequently

$$a^4 - a^2\,(a'^2 + b'^2) = b^4 - b^2\,(b'^2 + a'^2);$$

from which $a^4 - b^4 = (a^2 - b^2)\,(a'^2 + b'^2)$. Now $a^4 - b^4 = (a^2 + b^2)\,(a^2 - b^2)$, therefore

$$a'^2 + b'^2 = a^2 + b^2,$$

and

$$4a'^2 + 4b'^2 = 4a^2 + 4b^2;$$

i. e., in the ellipse, *The sum of the squares of the axes is equal to the sum of the squares of the conjugate diameters.*

From the same formulas, (4) XV., we have

$$a^2\sin^2\alpha + b^2\cos^2\alpha = \frac{a^2b^2}{a'^2},\; a^2\sin^2\beta + b^2\cos^2\beta = \frac{a^2b^2}{b'^2};$$

hence

$$a^4\sin^2\alpha\sin^2\beta + a^2b^2\sin^2\alpha\cos^2\beta + a^2b^2\cos^2\alpha\sin^2\beta + b^4\cos^2\alpha\cos^2\beta = \frac{a^4b^4}{a'^2b'^2}.$$

Now from the equation, (3) XV., we have

$$tg\alpha\;tg\beta + \frac{b^2}{a^2} = 0,$$

which squared gives

$$tg^2\,\alpha\;tg^2\,\beta + 2tg\alpha\;tg\beta\,\frac{b^2}{a^2} + \frac{b^4}{a^4} = 0,$$

and also

$a^4 \sin^2 \alpha \sin^2 \beta + 2a^2b^2 \sin \alpha \sin \beta \cos \alpha \cos \beta + b^4 \cos^2 \alpha \cos^2 \beta = 0$;
hence

$$a^4 \sin^2 \alpha \sin^2 \beta + b^4 \cos^2 \alpha \cos^2 \beta = -2a^2b^2 \sin \alpha \sin \beta \cos \alpha \cos \beta.$$

This value substituted in the preceding equation gives us

$$a^2b^2 \sin^2 \alpha \cos^2 \beta + a^2b^2 \cos^2 \alpha \sin^2 \beta - 2a^2b^2 \sin \alpha \sin\beta \cos \alpha \cos \beta = \frac{a^4b^4}{a'^2b'^2}.$$

Suppressing the common factor a^2b^2, and taking the square root, we obtain the formula,

$$\sin \beta \cos \alpha - \cos \beta \sin \alpha = \frac{a \cdot b}{a' \cdot b'}.$$

Now (Trig., p. 253, h'') $\sin \beta \cos \alpha - \cos \beta \sin \alpha = \sin (\beta - \alpha)$; hence

$$a \cdot b = a' \cdot b' \sin (\beta - \alpha).$$

Observe that $a \cdot b$ represents the area of the rectangle constructed on the semi-axes, and $a' \cdot b' \sin (\beta - \alpha)$ the rectangular area of the parallelogram constructed on the conjugate semi-diameters; therefore, *The parallelogram constructed with the conjugate diameters is equivalent to the rectangle of the axes.*

XIX. *Equation of the hyperbola referred to its axes.*

Take (Fig. 11) the point C on the indefinite straight line XX′ and on each side of C two segments CA = CA′, CF (>CA) = CF′. Let also a curve MAM′ pass through A, and a corresponding one through A′, and let the difference of the distances of each point of the curve, on either side, from F and F′ be always the same and equal to AA′. This double curve is the hyperbola, which has C for *centre* and AA′, represented by $2a$, for *transverse axis.* The points F, F′ are called the *foci*, and the points A, A′ of the curve met by the axis are called the *vertices.* Any straight line passing through C and terminated on both sides by the curve is called a *diameter.* $\frac{CF}{CA} = \frac{CF}{a} = \frac{CF'}{a}$ is called the *eccentricity,* which we represent by e, and since $CF > CA$,

$$\frac{CF}{a} = \frac{CF'}{a} = e > 1.$$

Draw now from the centre C, YY′ perpendicular to AA′, and taking XX′ for axis of the abscissas and YY′ for axis of ordinates, let x, y be the co-ordinates CK, MK of any point M of the curve. Since from the above equality we have

$$CF = CF' = ae,$$

calling ρ, ρ' the distances MF, MF′ of the foci from the point M, which distances are called *radius-vectors,* the right-angled triangles MKF, MKF′ give

$$(r) \begin{cases} \rho^2 = y^2 + (x - ae)^2 \\ \rho'^2 = y^2 + (x + ae)^2. \end{cases}$$

Taking the first of these formulas from the second, we have

$$(\rho' + \rho)(\rho' - \rho) = 4aex;$$

but $\rho' - \rho = 2a$; hence $\rho' + \rho = 2ex$; hence also adding to or subtracting from each other these last two equations,

$$(r') \quad \rho' = a + ex, \rho = ex - a.$$

Substituting in the second (r) the value of ρ' last obtained, that formula will become $a^2 + e^2x^2 = y^2 + x^2 + a^2e^2$, from which

$$(1) \quad y^2 = (e^2 - 1)(x^2 - a^2),$$

the equation of the hyperbola referred to the rectangular axes XX′, YY′. To eliminate the eccentricity from this equation, call c the distance CF; we shall then have $c = ae$, and consequently $e = \frac{c}{a}$, and $e^2 - 1 = \frac{c^2 - a^2}{a^2}$. Let now b represent the mean geometrical proportional between $c + a$, $c - a$, we shall have $b^2 = c^2 - a^2$, and therefore

$$(2) \quad e^2 - 1 = \frac{b^2}{a^2}.$$

This value changes the equation (1) into

$$(3) \quad y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

To keep the analogy with the ellipse, taking on each side of YY′ from the centre CB = CB′ = b, the segment BB′ of YY′ is called the *conjugate axis* of the hyperbola.

XX. *Analysis and corresponding geometrical locus of the equation.*

From the last equation (3) we obtain the following:

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

from the analysis of which we infer, 1st, that no real value of y corresponds to the values of x, either positive or negative, from 0 to a. 2d. That the ordinate $y = 0$ corresponds to the abscissas $x = a$, $x = -a$. 3d. Two real ordinates, one positive and one negative, but of equal numerical value, correspond to the values, either positive or negative, of x when its numerical value is greater than a, and the numerical value of y increases indefinitely with that of x. Referring, therefore, the geometrical locus represented by the above equation to the orthogonal axes XX′, YY′, we find it to cut the axis of abscissas at the distances a, $-a$ from the origin, and recede thence from the axis of ordinates divided into four symmetrical branches, two above and two below the axis of abscissas.

Taking on the axis XX′ two points F, F′ equidistant from the centre, the distance being CF = CF′ = $\sqrt{a^2 + b^2}$ ($>a$), so that $\frac{CF}{a} = \frac{CF'}{a} > 1$, and representing as usual by e the ratio $\frac{CF}{a}$, CF = CF′ = ae; from the right-angled triangles MFK, MF′K with a process similar to that followed for the ellipse, we shall find for the positive values of both MF, MF′,

$$MF = ex - a, \quad MF' = ex + a.$$

Hence MF′ — MF = $2a$, which is the characteristic property of the hyperbola. It may be well to remark that the formula from which we obtain the value of MF gives indifferently $a - ex$ and $ex - a$, but as in the case of the ellipse e being < 1 and x never greater than a, the positive value of MF can be given only by $a - ex$, so in the case of the hyperbola for which $e > 1$ and x never less than a the positive value of MF is only obtained by $ex - a$.

XXI. *Parameter of the hyperbola.*

The parameter of the hyperbola, like that of the preceding curves, is the double ordinate which passes through the focus; $2p$ representing the parameter, p is the ordinate corresponding to the abscissa CF = ae. Substituting, therefore, in the equation (1) XIX., ae for x, the same equation gives us

$$p^2 = a^2 (e^2 - 1)^2;$$

hence from the equation (2) of the same paragraph

$$p = a \cdot \frac{b^2}{a^2} = \frac{b^2}{a},$$

as for the ellipse. Therefore, since from this last equation,

$$2a : 2b :: 2b : 2p.$$

For the hyperbola, as for the ellipse, *the parameter is the third proportional to the transverse and the conjugate axis.*

XXII. *Tangent and normal.*

Let M (Fig. 12) be any point of the hyperbola, and MF, MF′ its distances from the foci or radius-vectors. Take on MF′, MN = MF, and connecting F with N, draw MT perpendicular to NF. MT is the tangent of the point M of the curve. Taking, in fact, on either side of M a point P on TMT′, and connecting it with F, F′ and N, we shall have PF = PN and PF′ $<$ PN + NF′; hence PF′ — PF $<$ NF′. Now, NF′ = MF′ — MF = $2a$; hence PF′ — PF $< 2a$, P therefore is not one of the points of the curve. The same result would be obtained for any point of MT taken on the side of T′, hence none of the points of TT′, except M, is on the curve. TT′ is, moreover, altogether on the convex side of the curve, a condition to be fulfilled in order that TT′ be a tangent. Taking, in fact, any point between the transverse axis and the curve, on the concave side, and drawing from it a perpendicular to the axis produced on the other side, until it reaches the curve, connecting then the point of the curve met by this perpendicular and the point from which the perpendicular is drawn with the foci, we shall find the difference of the distances of the last point greater than that of the radius-vectors of the point of the curve; hence the same difference is $> 2a$. Now the difference of the distances of any point of TT′ on either side of M from the foci is $< 2a$; hence all these points and consequently the whole straight line TT′ is on the convex side of the curve.

From the equal triangles NEM, FEM we infer the equality of the angles formed by the tangent with the radius-vectors. To draw, therefore, from any point of the hyperbola a tangent, *divide by a straight line the angle formed by the radius-vectors of that point into two equal parts.* It is easy to see that producing FM to N′ and drawing from M, MR perpendicular to the tangent, (MR is called the *normal* of M,) the normal also divides into two equal parts the angle formed by the radius-vectors; i. e.,

The angles formed by the radius-vectors of any point of the hyperbola are bisected, one by the tangent and the other by the normal.

XXIII. *Asymptotes of the hyperbola.*

The *asymptote* is a straight line approaching indefinitely to a curvilinear branch or branches without ever reaching them. The hyperbola admits of two of these asymptotes, which are the indefinitely produced diagonals UX′, VY′ (Fig. 13) of the rectangle DED′E′, constructed upon the axes $2a$, $2b$ of the curve.

Let, in fact, CK be the abscissa x corresponding to KM, the ordinate y of the curve, and to KL the ordinate $y_{,}$ of VY, the prolonged diagonal E′D of the rectangle. From the equations of the hyperbola referred to its own axes, and of the straight line referred to the same axes, we shall have at once,

$$y = \frac{b}{a}\sqrt{x^2 - a^2},\ y_{,} = \frac{b}{a}\,x\,;$$

and consequently,

$$y_{,} - y = \frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right) = \frac{ab}{x + \sqrt{x^2 - a^2}}.$$

It is plain from the first two equations, that whatever be x the corresponding $y_{,}$ is greater than y, and from the last equation, that the greater is x the smaller is the difference $y_{,} - y$. Hence CY′, even indefinitely prolonged, is altogether outside of the branch AM of the curve, but approaching to it more and more, the more the branch of the curve and the diagonal recede from the centre. The same demonstration is applicable to the other branches. Hence the hyperbola admits of two asymptotes VY′, UX′, each approaching in opposite directions to two of the branches of the curve.

XXIV. *Equation of the hyperbola referred to the asymptotes.*

Representing by α and $-\alpha$ the equal angles which the asymptotes CY′, CX′ form with the axis CX, and representing by x', y' the co-ordinates of any point M of the curve referred to them, as x, y represent the co-ordinates of the same point referred to the axes of the curve; we shall first obtain from the general formulas (1) II.,

$$x = x' \cos\alpha + y' \cos\alpha,\ y = y' \sin\alpha - x' \sin\alpha,$$

these values substituted in the equation (3) XIX., which can easily be reduced to the following:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

will give

$$\left(\frac{\cos^2\alpha}{a^2} - \frac{\sin^2\alpha}{b^2}\right)x'^2 + \left(\frac{\cos^2\alpha}{a^2} - \frac{\sin^2\alpha}{b^2}\right)y'^2 +$$
$$2\left(\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2}\right)x'y' = 1.$$

Now $\frac{\sin^2\alpha}{\cos^2\alpha} = tg^2\,\alpha = \frac{b^2}{a^2}$; hence

$$\cdot\frac{\sin^2\alpha}{b^2} = \frac{\cos^2\alpha}{a^2};$$

and, consequently, $\sin^2\alpha = \frac{b^2}{a^2}\cos^2\alpha$; hence

$$\sin^2\alpha + \cos^2\alpha = \cos^2\alpha\left(1 + \frac{b^2}{a^2}\right) = \frac{\cos^2\alpha}{a^2}(a^2 + b^2).$$

Now $\sin^2\alpha + \cos^2\alpha = 1$; hence $\frac{\cos^2\alpha}{a^2} = \frac{1}{a^2 + b^2}$, and therefore

$$\frac{\sin^2\alpha}{b^2} + \frac{\cos^2\alpha}{a^2} = \frac{2}{a^2 + b^2}.$$

Thus the preceding formula becomes

$$\frac{4x'y'}{a^2 + b^2} = 1,$$

or,

$$x'y' = \frac{a^2 + b^2}{4},$$

which is the equation of the hyperbola referred to the asymptotes. Now $a^2 + b^2$ or c^2 is (XIX.) the square of the distance of each focus from the centre; therefore $x'y' = \left(\frac{c}{2}\right)^2$; i. e., in the equation of the hyperbola referred to the asymptotes the product of the co-ordinates is constant and equal to the square of one-half the distance of each focus from the centre. If the axes $2a$, $2b$ be equal, in which case the hyperbola is called equilateral, the angle formed by the asymptotes is a right angle, and the equation of the curve referred to them becomes

$$x'y' = \frac{a^2}{2}.$$

XXV. *Polar equation of the hyperbola.*

From the characteristic property of the hyperbola expressed by the equation $\rho' - \rho = 2a$, and from the first equation (r') XIX., we obtain

$$\rho = \rho' - 2a = ex - a.$$

Calling now ω the angle AFM (Fig. 11) formed by the radius-vector of any point M of the curve with the axis XX′, which we take for polar axis, we shall have

$$x = \text{CF} + \text{FK} = ae - \rho \cos \omega ;$$

hence

$$\rho = e\,(ae - \rho \cos \omega) - a = a\,(e^2 - 1) - e\,\rho \cos \omega ;$$

therefore,

$$\rho = \frac{a\,(e^2 - 1)}{1 + e \cos \omega}.$$

Now (XXI.) $a\,(e^2 - 1)$ is the semiparameter p of the hyperbola; hence,

$$\rho = \frac{p}{1 + e \cos \omega},$$

which is the polar equation.

XXVI. *General polar equation.*

Comparing this equation with the polar equations of the parabola, X., and with that of the ellipse, (2) XVII., we find that the same formula,

$$\rho = \frac{p}{1 + e \cos \omega},$$

represents the ellipse, the parabola and the hyperbola, according as the eccentricity e is < 1, $= 1$ or > 1; p representing the semiparameter of each curve: the pole is taken in the focus or in one of the foci, and the angle ω commences on the side of the nearest vertex to the pole.

XXVII. *Equation of the cycloid.*

We call cycloid the curve BAB′ (Fig. 14) produced on the plane BEAB′ by a point B of the circular periphery BNE, while the circle, touching constantly the straight line BB′, revolves until the point B comes again into contact with the straight line in B′. The rolling circle BNE is called the *generating circle*, whose diameter we shall represent by $2c$. The straight line BB′, which is $= 2c\,\pi$, is called the *base* of the curve; and AA′ perpendicular to BB′ and passing through the middle point A′ of the base, is called the axis of the cycloid, and the extremity A of this axis, *vertex* of the curve. It is plain that the axis is equal to the diameter $2c$ of the generating circle.

Placing the origin of orthogonal axes, to which the cycloid is to be referred, in the vertex A let the axis AX of abscissas coincide with the axis of the curve, and let AY parallel to BB′ be the axis of ordinates. Let also x, y be the co-ordinates AK, KM of the point M of the cycloid corresponding to the position DMD′ of the generating circle. The diameter DD′, whose extremity D′ is the point of contact with the base, is necessarily perpendicular to the same base, and consequently parallel to AX.

Call now α the arc of the circle having unity for radius, and measuring the same angle measured by MD the supplement of MD′. From the genesis of the cycloid we have

$$BA' = DMD',\ BD' = MD';$$

hence

$$HK = BA' - BD' = DM = c\,\alpha;$$

hence, also,

$$x = DH = c\,v \cdot \sin\alpha = c\,(1 - \cos\alpha),$$

$$y = KH + HM = c\,\alpha + c\sin\alpha = c\,(\alpha + \sin\alpha).$$

Now $\overline{MH}^2 = HD' \cdot HD = (2c - x)\,x = 2cx - x^2$ or $MH = \sqrt{2cx - x^2}$. But $MH = c\sin\alpha$, therefore $\sin\alpha = \dfrac{\sqrt{2cx - x^2}}{c}$;

hence from the second of the last equations

$$y = c \; arc \; (\sin = (\frac{\sqrt{2cx - x^2}}{c}) + \sqrt{2cx - x^2},$$

which is the equation of the cycloid referred to the above-mentioned axes.

XXVIII. *Rectilinear and polar co-ordinates of points in space.*

Conceive three planes XAZ, ZAY, YAX passing through the same point A (Fig. 15); AX, AY, AZ being their mutual intersections. Let now M be a point in space, i. e., placed somewhere out of each of the three planes. The position of M relatively to the three planes or to A is in this case determined by means of three co-ordinates, as follows: Draw from M, MH parallel to AZ, called also axis Z, until it reaches the plane XAY in H, MK′ parallel to AY, or axis Y, until it reaches the plane ZAX in K′ and MD′ parallel to AX, or axis X, until it reaches the plane YAZ in D′. These three parallels determine the position of M in space relatively to the three planes; for the same three parallels cannot simultaneously belong to any other point. Now, the two parallels MK′, MH determine the position of a plane parallel to ZAY, which produced, will cut the axis X, and let K be the point of intersection. In like manner the parallels MK′, MD′ and the parallels MD′, MH determine the positions of planes respectively parallel to XAY and XAZ, and each of them produced will cut the axes Z and Y, the first say in A′, the second in D. Connecting now K with K′ and H, A′ with K′ and D′, and D with D′ and H, we obtain a parallelopipedon, and consequently MK′ = HK and MD′ = AK. Therefore to determine the position of M relatively to the three planes, it is enough to draw the parallel MH, or co-ordinate z, and from H, HK or co-ordinate y parallel to AY, and take in connection with them the segment AK or co-ordinate x on the axis

AX. These three co-ordinates determine the position of M relatively to the origin A of the axes. These co-ordinates will be regarded as positive or as negative according as their directions are toward X, Y, and Z, or toward the opposite sides. Varying the values of these co-ordinates x, y, z from $-\infty$ to $+\infty$ we can evidently obtain the position of any point in space. The axes AX, AY, AZ are commonly taken at right angles, in which case the co-ordinates are said to be *orthogonal*.

In this supposition join A with M, A being taken as pole, AM is the radius-vector ρ of M. Call θ the angle which ρ makes with AZ, and ω the angle which AH makes with AX, which is the angle formed by the planes ZAH, ZAX. These three elements ρ, θ, ω, determine the position of the point M in space relatively to the centre A, and to the axes, for three given values of these elements cannot belong simultaneously to more than one point, and taking ρ from 0 to $+\infty$, θ from $0°$ to $180°$ or π, ω from $0°$ to $360°$ or 2π, the position of every point in space can be determined by means of them. These are the *polar co-ordinates* of points in space. We may from these co-ordinates obtain the rectilinear co-ordinates of the same points, or *vice versa*. In fact, we have from the right-angled triangles AKH, AHM,

$$AK = AH \cos \omega = \rho \cos (90° - \theta) \cos \omega = \rho \sin \theta \cos \omega,$$
$$HK = AH \sin \omega = \rho \sin \theta \sin \omega,$$
$$MH = \rho \cos \theta.$$

Now AK, HK, MH are the rectilinear co-ordinates x, y, z of M; hence

$$x = \rho \sin \theta \cos \omega, \quad y = \rho \sin \theta \sin \omega, \quad z = \rho \cos \theta.$$

From the same right-angled triangles,

$$\overline{AM}^2 = \overline{AH}^2 + \overline{MH}^2 = \overline{AK}^2 + \overline{KH}^2 + \overline{MH}^2,$$

$$MH = AM \cos MAZ; \text{ hence } \cos MAZ = \frac{MH}{\sqrt{AK^2 + KM^2 + MH^2}},$$

$$HK : AK :: \sin HAK : \cos HAK; \text{ hence } tg\, HAK = \frac{HK}{AK}.$$

Substituting the corresponding values, the same equations become

$$\rho = \sqrt{x^2 + y^2 + z^2},\ \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}},\ tg\,\omega = \frac{y}{x}.$$

XXIX. *Equation of the plane.*

Let (Fig.16) AX, AY, AZ be three orthogonal axes, and BCD an indefinite plane in space, which meets the axis AZ in C and cuts the planes ZAX, ZAY along the straight lines BCF, DCI. Let MH, HK, KA be the co-ordinates z, y, x of any point M of the plane. Since the plane determined by MH, HK is parallel to ZAY, the intersections UI′, TI′ of MHK produced, with the plane of the axis AX, AY, and with the given plane BCD, are respectively parallel to YI, DI. In like manner the intersection KV of the same plane MHK with the plane of the axes AZ, AX is parallel to AZ. Hence UKV = YAZ = 90° and UI′T = YID. Call q the segment AC of the axis AZ between the origin and the point met by the plane in space, m the tangent of the angle XFB, and n the tangent of the angle YID. Concerning the straight line I′T, referred to the axes KU, KV, we shall have (3) III., MH = $n \cdot$ KH + KE; i. e.,

$$z = ny + \text{KE};$$

but with regard to F B referred to the axes AX, AZ we have

$$\text{KE} = mx + q;$$

hence

$$z = mx + ny + q,$$

an equation between the constants m, n, q and the co-ordinates x, y, z of the point M of the plane; but M is any point of the given plane; hence the last equation is the equation of the plane, and for any values taken at pleasure for x and y, we may obtain through it the value of the third co-ordinate z.

Should the plane pass through the origin of the axes, then $q = 0$, and the equation becomes in this case

$$z = mx + ny.$$

Representing m by $-\frac{A}{C}$, n by $-\frac{B}{C}$, and q by $-\frac{D}{C}$, the last and the preceding equations are easily changed into

$$Ax + By + Cz + D = 0,$$
$$Ax + By + Cz = 0;$$

the first of which represents any plane at all; the second, any plane passing through the origin of the axes.

XXX. *Equations of the straight line in space.*

Let (Fig. 17) any straight line RR′ in space be referred to the orthogonal axes AX, AY, AZ. Draw from any point M of RR′, MH perpendicular to the plane of the axes AX, AY, and MN perpendicular to the plane of the axes AY, AZ, and let PP′ be the intersection between the plane of the axes AX, AY and the plane determined by the line RR′ in space and the perpendicular MH. Let also QQ′ be the intersection between the plane of the axes AZ, AY and the plane determined by the line RR′ in space and the perpendicular MN. These two intersections PP′, QQ′ are called *projections* of the straight line in space, the first on the plane XAY, the second on the plane ZAY. Let now AK, KH, HM be the co-ordinates x, y, z of the point M. The first two x, y belong also to the point H of the projection PP′ referred to the axes AX, AY, and the two y, z belong also to the point N of the projection QQ′ referred to the axes AZ, AY. Now, (3) III., let

$$y = ax + b, \quad y = a'z + b'$$

be the equations of the projections each referred to the axes of its own plane. By means of them the straight line RR′ in space may be also represented. For taking the value of any of the three co-ordinates, by means of the two equations we obtain the other two, and all the three co-ordinates belong to one of the points of the line in space.

To conclude these sketches of analytical geometry, we may remark that as a plane surface in space, so also a curved surface may be referred to the orthogonal axes. Its equation, however, would be found of a degree higher than the first. And as a straight line in space can be referred to the same axes and be represented by the equations of the projections of the line on two of the planes formed by the same axes, so likewise a curved line in space can be referred to these axes, and represented by the equations of the projections of the curve in space upon two of the planes of the axes.

PRINCIPLES OF

INFINITESIMAL CALCULUS.

PART I.

DIFFERENTIAL CALCULUS.

I. *Infinitesimal quantities; different orders and expressions of the same.*

WE call that quantity *infinitesimal* which is conceived to be less than any given quantity of the same kind, however small.

Representing by α an infinitesimal quantity, which we shall call of the *first order*, the powers α^2, α^3, α^4, . . . α^n of the same quantity will be infinitesimals of the *second*, of the *third* . . . of the nth *order*; inasmuch as in the series of these quantities each must be regarded as infinitely less than the preceding and infinitely greater than the following.

Thus if any quantity β divided by the infinitesimal α gives the finite quotient κ, β must be regarded as an infinitesimal of the first order; and if

$$\frac{\beta}{\alpha} = \frac{\gamma}{\alpha^2} = \frac{\delta}{\alpha^3} = \cdots \frac{\omega}{\alpha^n} = \kappa,$$

γ, δ, . . . ω must be regarded as infinitesimals of the 2d, 3d, . . . nth orders.

Hence the infinitesimals of different orders can be expressed as follows:

$$\beta = \kappa\alpha,\ \gamma = \kappa\alpha^2,\ \delta = \kappa\alpha^3,\ . \ . \ . \ \omega = \kappa\alpha^n;$$

that is,

The product of a finite quantity by an infinitesimal of any order is an infinitesimal quantity of the same order.

We must add to these preliminaries the following principle, generally admitted in the analysis of infinitesimal calculus, and found to be correct in all its physical applications, i. e., *Infinitesimal quantities disappear when compared with finite quantities, or when compared with infinitesimal quantities of a lower order*, which comes to the same as to say, two finite quantities which differ from each other by an infinitesimal one, or two infinitesimal quantities which differ from one another by an infinitesimal of a higher order, are or may be considered as identical. In fact, it follows from the definition of the infinitesimal quantity, that the difference in both cases is less than any quantity which can be assigned.

II. *Functions.*

A quantity is called *constant* or *variable* according as it has a fixed or variable value.

Variable quantities may and do frequently depend on each other, and then they are said to be *functions* of one another. Thus, for example, if by changing the value of the quantity x the value of another quantity y is also varied; y is called a function of x, and *vice versa*. That y is a function of x is expressed by the equation

$$y = f(x).$$

In this equation x is called an *independent* variable, inasmuch as we make y depend on any value *arbitrarily* given to x. If we should make x vary according to the arbitrary values given to y, then y would be the independent variable, in which case the dependence of x on y would be expressed by the equation

$$x = \mathrm{F}(y),$$

the capital letter F being used instead of f, to signify the different form of the function, as φ, χ . . . would be used for other functions differing from the preceding.

These are called *explicit* functions, to distinguish them from the *implicit*, in which the function is not immediately given by the independent variable; as, for instance, in the equation

$$y^2 - 2xy - a = 0,$$

in which y is a function of x, but not immediately given by x. If the equation be resolved, we then have

$$y = x \pm \sqrt{a + x^2};$$

i. e., y given immediately by x; and representing $x \pm \sqrt{a + x^2}$ by $\mathrm{F}x$, $y = \mathrm{F}x$ is the explicit function of x deduced from the implicit one by means of the resolution of the given equation.

III. *Differentials.*

Let

$$y = ax,\ y = \frac{b}{x}$$

be two equations between x and y, which also represent two different functions of x. By increasing x, the function y increases in the first and decreases in the second equation, and *vice versa,* whatever the increase of x may be. In every case, however, an infinitesimal change of x is necessarily attended by a change of y equally infinitesimal. Representing thus by dx, dy the infinitesimal variations of x and of y, we deduce from the preceding equations the two following:

$$y \pm dy = a\,(x \pm dx),\ \ y \pm dy = \frac{b}{x \pm dx};$$

and supposing, as is often done, that dx represents the increase as well as the diminution of the variable x, and dy the increase and the diminution of the function; the above equations are more simply written as follows:

$$y + dy = a\,(x + dx),\ \ y + dy = \frac{b}{x + dx};$$

dx is called the *differential* of x, and dy the differential of y or of the function of x. From the last and the given equations it is easy to see that

$$dy = a\,(x + dx) - ax, \text{ or } dy = \frac{b}{x + dx} - \frac{b}{x}.$$

And generally supposing the given equation to be $y = f(x)$, we shall obtain

$$dy = f(x + dx) - f(x);$$

that is,

The differential of any function of x *is equal to the difference between the first and the second state of the function, the second state being the result of an infinitesimal change of* x *in the given function.*

Differential calculus has for its object to determine the differentials of given functions. But before we proceed to give the principal and most common rules of this calculus, the following theorems require to be demonstrated.

IV. *Preliminary theorems.*

Let n be a positive whole number increasing indefinitely and having for its limit *infinity*, i. e., a value superior to any assignable value, and take the binomial $\left(1 + \frac{1}{n}\right)^n$, in which, when n has reached its limit, $\frac{1}{n}$ is necessarily infinitesimal. We know from algebra (see Treat. § 69) that

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{2\cdot 3}\frac{1}{n^3}$$
$$+ \cdots + \frac{n(n-1)(n-2)\cdots(n-(n-1))}{2\cdot 3\cdots n}\frac{1}{n^n}.$$

The second member of this equation can be easily transformed into the following:

$$2 + \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{2\cdot 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$
$$+ \frac{1}{2\cdot 3\cdots n}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right),$$

in which all the factors $1-\frac{1}{n}$, $1-\frac{2}{n}, \cdots 1-\frac{n-1}{n}$, are positive, and their number and values increase with n; hence also $\left(1+\frac{1}{n}\right)^n$ increases in value with n. But whatever n may be, each one of the same factors is less than 1; hence

$$\frac{1}{2}\left(1-\frac{1}{n}\right)<\frac{1}{2}, \frac{1}{2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)<\frac{1}{2\cdot 3}, \text{ etc.,}$$

and consequently

$$\left(1+\frac{1}{n}\right)^n<2+\frac{1}{2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2\cdot 3\ldots n},$$

and with greater reason

$$\left(1+\frac{1}{n}\right)^n<2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}.$$

Now $\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}\cdots+\frac{1}{2^{n-2}}\right)$,

and (Treat. § 63. ex)

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{(n-2)}}=2-\frac{1}{2^{n-2}}<2;$$

therefore we have at once,

$$\left(1+\frac{1}{n}\right)^n>2, \text{ and } \left(1+\frac{1}{n}\right)^n<3;$$

and although by increasing n, $\left(1+\frac{1}{n}\right)^n$ increases also, still it cannot increase so much as to become $=3$. The value, therefore, of $\left(1+\frac{1}{n}\right)^n$ is represented by a number between 2 and 3. The letter e is used to represent this number, and its approximate value obtained by substituting in the above formulas, for n, positive numbers greater and greater, is 2,7182818; that is,

$$e=2{,}7182818\ldots$$

Hence, however great the value of n may be, even if it be infinite (infinity is represented by the sign ∞), in which case $\frac{1}{n}$ is infinitesimal, $\left(1+\frac{1}{n}\right)^n$ will be equal to e; nay, then only does it acquire the exact value of e when $n=\infty$. Thus we have

$$\left(1+\frac{1}{\infty}\right)^{\infty}=e,$$

or, representing the infinitesimal $\frac{1}{\infty}$ by ω,

$$(1+\omega)^{\frac{1}{\omega}}=e.$$

1st. Therefore, *The binomial* $(1+\omega)$, *in which* ω *is infinitesimal, raised to the infinite power* $\frac{1}{\omega}$, *gives for result* $e=2{,}7182818\ldots$

Take now with $(1-\omega)^{-\frac{1}{\omega}}$, an infinitesimal quantity α, such that $1-\omega=\frac{1}{1+\alpha}$, and consequently $-\omega=-\frac{\alpha}{1+\alpha}$; we shall have

$$(1-\omega)^{-\frac{1}{\omega}}=\left(\frac{1}{1+\alpha}\right)^{-\frac{1+\alpha}{\alpha}}=\frac{1}{\left(\frac{1}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}}}=(1+\alpha)^{\frac{1+\alpha}{\alpha}};$$

but $\quad (1+\alpha)^{\frac{1+\alpha}{\alpha}}=(1+\alpha)^{\frac{1}{\alpha}+\frac{\alpha}{\alpha}}=(1+\alpha)\,(1+\alpha)^{\frac{1}{\alpha}},$

and $(1+\alpha)^{\frac{1}{\alpha}}=e$, therefore $(1-\omega)^{-\frac{1}{\omega}}=e+\alpha e$, and neglecting the infinitesimal,

$$(1-\omega)^{-\frac{1}{\omega}}=e.$$

2d. That is, *The binomial* $(1-\omega)$, *in which* ω is *infinitesimal, raised to the infinite negative power* $-\frac{1}{\omega}$, *gives for result* $e=2{,}7182818\ldots$

We admit the circle to be the limit of an inscribed polygon the number of whose sides increases indefinitely; we must consequently admit also the circle to coincide with the inscribed polygon of an infinite number of sides. But the sides of this polygon are necessarily infinitesimal. Therefore an infinitesimal chord in the circle (the same may be said of any other curve) coincides with the arc. Let us now represent by 2β the infinitesimal arc coinciding with this chord. The chord being equal to $2 \sin \beta$, we shall have $2 \sin \beta = 2\,\beta$, and consequently $\frac{\sin \beta}{\beta} = 1$.

3d. That is, *The ratio between an infinitesimal arc and its sine is equal to* 1.

V. *Differentials of algebraic functions.*

Let $f(z)$ represent any of the following functions:

$$\text{I. } a \pm z, \quad \text{II. } az, \quad \text{III. } \frac{a}{z}, \quad \text{IV. } z.^a$$

According to the definition (III.) we shall have

$$d(a \pm z) = a \pm z \pm dz - (a \pm z) = \pm dz;$$

that is,

I. *The differential of* $a \pm z$ *is the same as that of* $\pm z$*;* and since supposing $z = \varphi(x)$, we infer $d\,(a \pm \varphi(x)) = \pm d\,\varphi(x)$,

The differential of $a \pm \varphi(x)$ *is the same as that of* $\varphi(x)$.

From the second we have

$$daz = az + adz - az = adz; \text{ i. e.,}$$

II. *The differential of the product of a variable by a constant is the product of the constant by the differential of the variable;* and since making $z = \varphi(x)$ we obtain

$$da\,\varphi(x) = ad\,\varphi(x).$$

So also *the differential of the product of a function of* x *by a constant, equals the product of the same constant by the differential of the function.*

We have from the third

$$d\frac{a}{z} = \frac{a}{z + dz} - \frac{a}{z} = -\frac{adz}{z^2}; \text{ i. e.,}$$

III. *The differential of a constant divided by a variable equals the negative product of the constant by the differential of the variable divided by the square of the same variable;* and taking $z = \varphi(x)$,

The differential of a constant divided by a function of x *equals the negative product of the same constant by the differential of the function, divided by the square of the function.*

Lastly, from z^a we obtain, in the supposition that a is a whole number,

$$dz^a = (z + dz)^a - z^a = z^a + az^{a-1}\,dz + \frac{a\,(a-1)}{2}\,z^{a-2}\,dz^{-2}$$
$$+ \ldots - z^a.$$

Or, neglecting the terms multiplied by the differentials of the orders superior to dz,

$$dz^a = az^{a-1}\,dz\,;$$

from which, regarding z as a simple independent variable or as a function of x, we infer that,

IV. *The differential of the power of a variable* x *or of the power of a function of* x *equals the product of the index by the given power diminished by one unit, and all multiplied by the differential of the variable or of the function.* The same rule is applicable to the case of a being any number whatever. From the first equation, $dz^a = (z + dz)^a - z^a$, we deduce the following:

$$dz^a = z^a\left(1 + \frac{dz}{z}\right)^a - z^a = z^a\left[\left(1 + \frac{dz}{z}\right)^a - 1\right].$$

Now $\left(1 + \frac{dz}{z}\right)^a - 1$ is an infinitesimal which can be represented by θ, and thus

$$\left(1 + \frac{dz}{z}\right)^a = 1 + \theta.$$

Applying logarithms to this last equation, we have

$$al\left(1 + \frac{dz}{z}\right) = l\,(1 + \theta)$$

and
$$a \frac{l\left(1 + \frac{dz}{z}\right)}{(l(1+\theta)} = 1.$$

But from the first equation we have also

$$dz^a = z^a\,\theta = z^a\,\theta \times a\,\frac{l\left(1+\frac{dz}{z}\right)}{l(1+\theta)},$$

from which

$$\frac{dz^a}{dz} = \frac{z^a}{dz}\,\theta\, a\,\frac{l\left(1+\frac{dz}{z}\right)}{l(1+\theta)} = \frac{\frac{z}{dz}\,l\left(1+\frac{dz}{z}\right)}{\frac{1}{\theta}\,l(1+\theta)}\,az^{a-1}.$$

Now $\frac{z}{dz}\,l\left(1+\frac{dz}{z}\right) = l\left(1+\frac{dz}{z}\right)^{\frac{z}{dz}}$, and $\frac{1}{\theta}\,l(1+\theta) = l(1+\theta)^{\frac{1}{\theta}}$,

and dz being infinitesimal, and consequently $\frac{z}{dz}$ infinite as well as $\frac{1}{\theta}$, it follows from the 1st theorem of the preceding number that

$$\left(1+\frac{dz}{z}\right)^{\frac{z}{dz}} = (1+\theta)^{\frac{1}{\theta}} = e;$$

hence
$$\frac{dz^a}{dz} = az^{a-1},$$

and consequently
$$dz^a = az^{a-1}\,dz,$$

whatever be the numerical value of the constant a; therefore *The differential*, etc.

The functions whose differentials we have found embrace all the cases of algebraic functions. Let us now find the

VI. *Differentials of transcendental functions.*

Let $f(z)$ represent any of the following functions:

I. lz, II. a^z, III. $\sin z$, IV. $\cos z$, V. $tg\, z$, VI. $\cot z$, VII. arc $(\sin = z)$, VIII. arc $(\cos = z)$, IX. arc $(tg = z)$, X. arc $(\cot = z)$.

Arc $(\sin = z)$ signifies an arc whose sine is z, and in like manner the last three functions signify an arc whose cosine or tangent or cotangent is z.

Taking now the differentials, we have from the first

$dlz = l(z + dz) - lz$; but $l(z + dz) - lz = l\left(\frac{z + dz}{z}\right)$; hence

$$dlz = l\left(1 + \frac{dz}{z}\right),$$

and
$$\frac{dlz}{dz} = \frac{1}{dz}\, l\left(1 + \frac{dz}{z}\right);$$

$$\text{but } \frac{1}{dz}\, l\left(1 + \frac{dz}{z}\right) = \frac{1}{z}\,\frac{z}{dz}\, l\left(1 + \frac{dz}{z}\right) = \frac{1}{z}\, l\left(1 + \frac{dz}{z}\right)^{\frac{z}{dz}};$$

hence
$$dlz = \frac{le}{z} dz;$$

and taking e for the base of the logarithms, as is commonly done,

$$dlz = \frac{dz}{z};$$

hence regarding, as usual, z as a simple variable or as a function of another variable x,

I. *The differential of the logarithm of a variable or function of a variable is obtained by dividing its differential by the same variable or function.*

From a^z we have

$$da^z = a^{z + dz} - a^z = a^z(a^{dz} - 1).$$

Now $a^{dz} - 1$ is infinitesimal, and may be represented by θ, so that $a^{dz} = 1 + \theta$; and taking the logarithms

$$dz = \frac{l(1+\theta)}{la};$$

hence $$da^z = a^z \cdot \theta = a^z \cdot \theta \frac{la}{l(1+\theta)} dz;$$

but $$\theta \frac{la}{l(1+\theta)} = \frac{l(a)}{\frac{1}{\theta} l(1+\theta)} = \frac{l(a)}{l(1+\theta)^{\frac{1}{\theta}}} = la;$$

therefore

$$da^z = a^z la\, dz; \text{ i. e.,}$$

II. *The differential of the exponential quantity* a^z *is the product of the same quantity by the logarithm of the root* a *multiplied by the differential of the exponent, whether it be an independent variable or function of another variable.*

We may come to the same conclusion by a more speedy process.

Make $a^z = y$, apply logarithms and take the differential, we shall have

$$d \log a^z = d \log y = \frac{dy}{y} = \frac{da^z}{a^z};$$

hence

$$da^z = a^z d \log a^z = a^z d \cdot z \log a = a^z\, la\, dz.$$

From the third and fourth functions we have

$$d \sin z = \sin(z + dz) - \sin z,$$

$$d \cos z = \cos(z + dz) - \cos z.$$

And (Trig., p. 252)

$$\sin(z + dz) - \sin z = 2 \cos \tfrac{1}{2}(2z + dz) \sin \tfrac{1}{2}dz,$$

$$\cos(z + dz) - \cos z = -2 \sin \tfrac{1}{2}(2z + dz) \sin \tfrac{1}{2}dz.$$

But from theorem 3d (IV) we have $\sin \frac{1}{2}dz = \frac{1}{2}dz$, and $2z + dz$ may be regarded as equal to $2z$; hence

$$d \sin z = \cos z\, dz, \quad d \cos z = -\sin z\, dz; \text{ i. e.,}$$

III. *The differential of the sine is equal to the product of the cosine by the differential of the arc.*

IV. *The differential of the cosine is equal to the negative product of the sine by the differential of the arc.*

From the fifth and sixth functions we have

$$d \operatorname{tg} z = d\frac{\sin z}{\cos z} = \frac{\sin (z + dz)}{\cos (z + dz)} - \frac{\sin z}{\cos z}$$

$$d \cot z = d\frac{\cos z}{\sin z} = \frac{\cos (z + dz)}{\sin (z + dz)} - \frac{\cos z}{\sin z};$$

hence (Trig., p. 253)

$$d \operatorname{tg} z = \frac{\sin (z + dz) \cos z - \cos (z + dz) \sin z}{\cos (z + dz) \cos z} = \frac{\sin (z + dz - z)}{\cos (z + dz) \cos z}$$

$$= \frac{\sin dz}{\cos (z + dz) \cos z}.$$

But $\sin dz = dz$, $z + dz$ is equivalent to z; hence

$$d \operatorname{tg} z = \frac{dz}{\cos^2 z}.$$

By a like process we obtain

$$d \cot z = -\frac{dz}{\sin^2 z}; \text{ i. e.,}$$

V. *The differential of the tangent equals the differential of the arc divided by the square of the cosine.*

VI. *The differential of the cotangent equals the negative differential of the arc divided by the square of the sine.*

Calling y the arc whose sine or cosine is z, with the equations $y = \text{arc} (\sin = z)$, $y = \text{arc} (\cos = z)$, we shall have the two following:

$$z = \sin y, \quad z = \cos y,$$

and $$dz = \cos y \, . \, dy, \quad dz = - \sin y \, . \, dy;$$

hence dy or

$$d \text{ arc} (\sin = z) = \frac{dz}{\cos y}, \quad d \text{ arc} (\cos = z) = -\frac{dz}{\sin y}.$$

Now $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - z^2}$, $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - z^2}$,

therefore

$$d \cdot \text{arc} (\sin = z) = \frac{dz}{\sqrt{1 - z^2}}, d \cdot \text{arc} (\cos = z) = -\frac{dz}{\sqrt{1 - z^2}}; \text{ i. e.,}$$

VII. *The differential of the arc whose sine is the variable* z, *is equal to the differential of the variable divided by the square root of* $(1 - z^2)$.

VIII. *The differential of the arc whose cosine is the variable* z, *is equal to the negative differential of the variable divided by the square root of* $(1 - z^2)$.

Calling now y the arc whose tangent or cotangent is z, with the equations $y = \text{arc}\,(\text{tg} = z)$, $y = \text{arc}\,(\cot = z)$, we shall have the two following:

$$z = \text{tg}\, y,\ z = \cot y,$$

and (V., VI.)

$$dz = \frac{dy}{\cos^2 y},\ dz = -\frac{dy}{\sin^2 y};$$

hence dy or

$$d \cdot \text{arc}\,(\text{tg} = z) = \cos^2 y \cdot dz = \frac{1}{\sec^2 y}\, dz = \frac{dz}{1 + \text{tg}^2 y} = \frac{dz}{1 + z^2},$$

$$d\, \text{arc}\,(\cot = z) = -\sin^2 y \cdot dz = -\frac{1}{\text{cosec}^2 y}\, dz = -\frac{dz}{1 + \cot^2 y}$$

$$= -\frac{dz}{1 + z^2};\ \text{i. e.,}$$

IX. *The differential of the arc whose tangent is the variable* z, *is equal to the differential of the variable divided by* $(1 + z^2)$.

X. *The differential of the arc whose cotangent is the variable* z, *is equal to the negative differential of the variable divided by* $(1 + z^2)$.

VII. *Differentials of the sum, product, and quotient of different functions of the same variable* x.

Let now the following functions of the same variable x be given,

$$u = \text{F}\,(x), \quad y = f(x), \quad z = \varphi\,(x),$$

and let s be their sum; i. e.,

$$s = \text{F}\,(x) + f\,(x) + \varphi\,(x),$$

we shall have

$$du = \mathrm{F}(x + dx) - \mathrm{F}(x), \quad dy = f(x + dx) - f(x),$$
$$dz = \varphi(x + dx) - \varphi(x),$$

and also

$$ds = \mathrm{F}(x + dx) - \mathrm{F}(x) + f(x + dx) - f(x) + \varphi(x + dx) - \varphi(x).$$

Therefore

$$ds = du + dy + dz; \text{ i. e.,}$$

I. *The differential of the algebraic sum of different functions of the same variable is equal to the sum of the differentials of each function.*

Let now p be the product of y by z; i. e., let

$$p = y \cdot z = f(x) \times \varphi(x),$$

and consequently,

$$p^2 = y^2 \cdot z^2 = [f(x)]^2 \cdot [\varphi(x)]^2,$$

now

$$lp^2 = ly^2 + lz^2;$$

also (VI. I.)

$$dlp^2 = \frac{dp^2}{p^2}, \quad dly^2 = \frac{dy^2}{y^2}, \quad dlz^2 = \frac{dz^2}{z^2},$$

and (V. IV.)

$$dp^2 = 2p\,dp, \quad dy^2 = 2y\,dy, \quad dz^2 = 2z\,dz;$$

therefore
$$\frac{2p\,dp}{p^2} = \frac{2y\,dy}{y^2} + \frac{2z\,dz}{z^2},$$

or
$$\frac{dp}{p} = \frac{dy}{y} + \frac{dz}{z};$$

that is, since $p = y \cdot z$,

$$d(y \cdot z) = zdy + ydz,$$

or
$$d[f(x)\,\varphi(x)] = \varphi(x)\,df(x) + f(x)\,d\,\varphi(x); \text{ i. e.,}$$

II. *The differential of the product of the two functions of the same variable is obtained by multiplying each function by the differential of the other and adding together the two products.*

It is known from algebra that a positive quantity raised to a power indicated by any exponent, either positive or negative, gives always a positive result. Now the base of logarithms is positive in every system; hence negative quantities admit of no logarithms but imaginary ones. To avoid the inconvenience of these imaginary logarithms, the equation $p = y \cdot z$ has been squared in the process of the preceding theorem. To give an example of the same theorem, let $y = x^3$ and $z = \sin x$. We shall have

$p = x^3 \cdot \sin x$ and $dp = d\,(x^3 \cdot \sin x) = 3x^2 \sin x\, dx + x^3 \cos x\, dx.$

Let, lastly, q be the quotient $\frac{y}{z} = \frac{f(x)}{\varphi(x)}$. From $q = \frac{y}{z}$ we infer $y = q \cdot z$ and

$$dy = zdq + qdz;$$

hence

$$zdq = dy - qdz = dy - \frac{y}{z}\, dz = \frac{zdy - ydz}{z},$$

and therefore

$$dq = \frac{zdy - ydz}{z^2},$$

or

$$d\frac{f(x)}{\varphi(x)} = \frac{\varphi(x)\, df(x) - f(x)\, d\,\varphi(x)}{[\varphi(x)]^2}; \text{ i. e.,}$$

III. *The differential of the quotient of two functions of the same variable is obtained by taking the difference between the product of the denominator by the differential of the numerator, and the product of the numerator by the differential of the denominator, and dividing this difference by the square of the denominator.*

Let, for example, the quotient be $\frac{2x^3}{\log(x)}$, we shall have

$$d\,\frac{2x^3}{\log(x)} = \frac{6x^2 \log(x)\, dx - 2x^2\, dx}{\log^2(x)} = \frac{2x^2\,(3 \log(x) - 1)\, dx}{\log^2 x}.$$

VIII. *Successive differentials and their orders.*

Let, for instance, $y = x^n$,
we shall have (V. IV.)

$$dy = nx^{n-1}\, dx.$$

The differential dx of the independent variable being taken as a constant and always the same in the succeeding differentials; from the above differential (which is another function of x) again differentiated, we shall have (V. II. IV.)

$$d\,(dy) = n\,(n - 1)\,x^{n-2}\,dx^2.$$

$d\,(dy)$ is represented by d^2y and the following differentials by d^3y, d^4y, etc. Thus following the same process, the first and the succeeding differentials of $y = x^n$ are given as follows:

$$\text{I.}\quad \begin{cases} dy = dx^n = nx^{n-1}\,dx, \\ d^2y = d^2x^n = n\,(n - 1)\,x^{n-2}\,\overline{dx}^2, \\ d^3y = d^3x^n = n\,(n - 1)\,(n - 2)\,x^{n-3}\,\overline{dx}^3, \\ \cdots\cdots\cdots\cdots \\ d^ny = d^nx^n = n\,(n - 1)\,(n - 2)\ldots(n - (n - 1))\,\overline{dx}^n. \end{cases}$$

These successive differentials are called also differentials of various orders, 1st, 2d, 3d, . . . nth. We may remark that in the above example the last differential is constant, and consequently $d^{n+1}\,y = 0$.

The line placed above dx in the differentials of the second and following orders is to distinguish the power of dx from the differential of x raised to a power. Thus, whereas $\overline{dx}^a$, for example, represents dx raised to the power a, dx^a signifies the differential of x raised to the power a.

Let, secondly, $y = a^x$,
we shall have (VI. II.)

$$dy = a^x\,ladx,$$

and consequently, $ladx$ being constant,

$$d^2y = a^xl^2a\overline{dx}^2,\ \text{etc.}, \ldots \text{i. e.},$$

$$\text{II.}\quad \begin{cases} dy = da^x = a^xla\,dx, \\ d^2y = d^2a^x = a^xl^2a\,\overline{dx}^2, \\ \cdots\cdots\cdots\cdots \\ d^ny = d^na^x = a^xl^na\,\overline{dx}^n. \end{cases}$$

Let, thirdly, $y = \sin x$, we shall obtain (VI. III. IV.)

$$\text{III.}\quad \begin{cases} dy = d\sin x = \cos x\,dx, \\ d^2y = d^2\sin x = -\sin x\,\overline{dx}^2, \\ d^3y = d^3\sin x = -\cos x\,\overline{dx}^2, \\ d^4y = d^4\sin x = \sin x\,\overline{dx}^2, \ \ldots \end{cases}$$

Let, lastly, $y = \log(x)$. We shall have (VI. I.)

$$dy = \frac{dx}{x} = \frac{1}{x}\,dx;$$

hence (V. III. IV.)

$$d^2y = -\frac{1}{x^2}\,\overline{dx}^2, \text{ and } d^3y = \frac{\overline{dx}^2}{x^4}\,\overline{dx} = \frac{2x\,\overline{dx}^3}{x^4} = \frac{2}{x^3}\overline{dx}^3;$$

hence

$$\text{IV.}\quad \begin{cases} dy = d\cdot\log(x) = \dfrac{dx}{x}, \\ d^2y = d^2\cdot\log(x) = -\dfrac{\overline{dx}^2}{x^2}, \\ d^3y = d^3\log(x) = \dfrac{1\cdot 2\,\overline{dx}^3}{x^3}, \\ d^4y = d^4\log(x) = -\dfrac{1\cdot 2\cdot 3\,dx^4}{x^4}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ d^ny = d^n\log(x) = \pm\dfrac{1\cdot 2\cdot 3\ldots(n-1)\,\overline{dx}^n}{x^n}. \end{cases}$$

In the last of which formulas the negative sign occurs when n is an even number.

IX. *Derivative functions and their orders.*

Let us resume the formula $y = x^n$ and its successive differentials (VIII. I.) We shall easily deduce from them the following equations:

$$\frac{dy}{dx} = nx^{n-1}, \ \frac{d^2y}{\overline{dx}^2} = n(n-1)\,x^{n-2},$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)\,x^{n-3}\ldots\frac{d^ny}{dx^n} = n(n-1)\ldots(n-(n-1)),$$

all of which, the last excepted, are functions of x, and being derived from the original function $y = x^n$, they are called *derivatives* of that function.

Let now $y = f(x)$ be any function of x. The first, second, and following derivatives of this function are represented by $f'(x)$, $f''(x)$, $\ldots f^{(n)}(x)$; i. e.,

$$\frac{dy}{dx} = f'(x), \ \frac{d^2y}{\overline{dx}^2} = f''(x) \ldots \frac{d^ny}{\overline{dx}^n} = f^{(n)}(x),$$

and are called derivatives of the 1st, of the 2d, . . . of the nth order.

Thus, *The derivative function of any order is given by the ratio between the corresponding differential of the primitive function and the corresponding power of the independent variable.*

But from the last equations we obtain

$$dy = f'(x)\,dx, \ d^2y = f''(x)\,\overline{dx}^2, \ \ldots \ d^ny = f^{(n)}(x)\,\overline{dx}^n;$$

hence

The differential of any order of a given function is given by the product of the derivative of the same order by the corresponding power of the differential of the independent variable.

Since in these last equations $f'(x), f''(x), \ldots$ or the equivalent ratios $\frac{dy}{dx}, \frac{d^2y}{\overline{dx}^2}, \ldots$ perform the office of coefficients; they are also called *differential coefficients* of various orders. It is plain, from what precedes, that the derivative functions are obtained by finding the successive differentials and omitting in them the differential dx of the independent variable and its powers.

X. *Maclaurin's formula.*

Suppose the function $f(x)$ to be capable of being developed into a series arranged according to the increasing powers of x as follows:

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots$$

in which A_0 and the coefficients $A_1, A_2, \ldots$, which are independent of x and constant, are unknown. The object of Maclaurin's theorem or formula, as it is called, is to find these coefficients.

It follows from the definition of the differential (III.) that the differential of a constant quantity is equal to 0;

hence $$dA_0 = dA_1 = dA_2 = \dots = 0,$$

and $$d2A_2 = d2 \cdot 3A_3 = \dots = 0;$$

hence, from what has been said in the preceding number and from the rules II. and IV. of No. V., we shall have with the primitive function

$$f(x) = A_0 + A_1 x + A_2 x^2 + \dots$$

the derivatives

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \dots$$

$$f''(x) = 2A_2 + 2 \cdot 3A_3 x + 3 \cdot 4A_4 x^2 + \dots$$

$$f'''(x) = 2 \cdot 3A_3 + 2 \cdot 3 \cdot 4A_4 x + \dots \text{ etc.}$$

In all these equations x may have any value, without affecting the constants A_0, A_1, . . . , but making $x = 0$ in the primitive and derivative functions we have

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2A_2,\ f'''(0) = 2 \cdot 3A_3, \dots$$

hence the constants A_0, A_1, . . . are given by the primitive function and by the derivatives, making in them $x = 0$. Substituting the values of the constants thus obtained in the primitive, we shall have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{2 \cdot 3} f'''(0) + \dots$$

which is Stirling's formula, more commonly known as Maclaurin's. It answers the purpose of developing functions into series according to the increasing powers of the variable.

Let us see some applications, and let, first, $f(x) = e^x$. We then have (VIII. II.) $f'(x) = e^x$, and consequently also $f''(x) = f'''(x) = \dots = e^x$; and therefore, taking $x = 0$, $f(0) = f'(0) = f''(0) = \dots = 1$; hence

$$\text{(I.)} \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \dots$$

and making in this formula $x = 1$,

$$e = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots$$

by means of which the value of e (IV.) can easily be obtained as nearly as desirable by increasing the number of terms of the series.

Let, secondly, $f(x) = \sin x$. We have (VIII. III.)

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x,$$
$$f''''(x) = \sin x, \dots$$

and making $x = 0$,

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1,$$
$$f''''(0) = 0, \quad f^{(v)}(0) = 1 \dots$$

Therefore

$$\text{(II.)} \quad \sin x = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \dots$$

Let, thirdly, $f(x) = \cos x$, we shall have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x,$$
$$f^{(IV)}(x) = \cos x, \dots$$

hence

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0,$$
$$f^{(IV)}(0) = 1, \dots$$

and consequently

$$\text{(III.)} \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} - \dots$$

XI. *Taylor's formula.*

Let us take the function $f(x + h)$, in which h represents an addition made to x, either positive or negative. Considering this addition as variable and the undetermined quantities, B_0, B_1, B_2, B_3, . . . independent of h in the supposition that $f(x + h)$ is capable of being developed into a series arranged according to the increasing powers of h, we may, as in the preceding paragraph, represent $f(x+h)$ by the series $B_0 + B_1h + B_2h^2 + B_3h^3 + \ldots$, in which B_0, B_1, B_2, ... necessarily depend on x. Supposing now h variable, and taking the successive derivatives of $f(x+h)$ relatively to h alone, making then in the primitive and in the derivatives $h=0$, we shall find, as for Maclaurin's formula, the equations

$$B_0 = f(x),\ B_1 = f'(x),\ B_2 = \frac{1}{2}f''(x),\ B_3 = \frac{1}{2\cdot 3}f'''(x), \ldots$$

and consequently

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{2\cdot 3}f'''(x) + \ldots$$

which is Taylor's formula, by means of which we obtain the second state of a function developed into a series of terms, arranged according to the increasing powers of the addition h made to the variable in the function $f(x)$.

1st. Let, for instance, $f(x) = \sqrt{x}$, we shall have (V. IV.)

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},\ f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x^3}},$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} = \frac{3}{8\sqrt{x^5}}, \ldots$$

hence, if in the given function $\sqrt{x}$ we change x into $x+h$, Taylor's formula will give us

$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8\sqrt{x^3}} + \frac{h^3}{16\sqrt{x^5}} - \ldots$$

2d. Let also $f(x) = \frac{1}{\sqrt{x}}$, we shall have (V. III. and IV.)

$$f'(x) = -\frac{1}{2\sqrt{x^3}},\ f''(x) = \frac{3}{4\sqrt{x^5}},\ f'''(x) = -\frac{3\cdot 5}{2\cdot 4\sqrt{x^7}},$$

$$f^{(\text{IV})}(x) = \frac{3\cdot 5\cdot 7}{2\cdot 8\sqrt{x^9}}, \ldots$$

Therefore, according to Taylor's formula,

$$\frac{1}{\sqrt{x+h}} = \frac{1}{\sqrt{x}} - \frac{1}{2}\frac{h}{\sqrt{x^3}} + \frac{3}{2\cdot4}\frac{h^2}{\sqrt{x^5}} - \frac{3\cdot5}{2\cdot4\cdot6}\frac{h^3}{\sqrt{x^7}} + \frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\frac{h^4}{\sqrt{x^9}} - \ldots$$

making now $x = 1$, and $h = -z^2$,

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2}z^2 + \frac{3}{2\cdot4}z^4 + \frac{3\cdot5}{2\cdot4\cdot6}z^6 + \frac{3\cdot5\cdot7}{2\cdot4\cdot6.8}z^8 + \ldots$$

as we would obtain by applying to $\frac{1}{\sqrt{1-z^2}}$, or to its equal $(1-z^2)^{-\frac{1}{2}}$, the development of the Newtonian formula.

Taylor's formula rests on the fact that $f(x+h)$ is capable of being developed into a convergent series having $f(x+h)$ for limit of its convergency. If this fact be not verified, the formula must necessarily fail to give the value of the function represented by an unlimited series. This is precisely the case in the last example if z be supposed greater than 1. For whereas $\frac{1}{\sqrt{1-z^2}}$, for any finite value of z has a finite and fixed value, the series increases in value in proportion as it increases in the number of terms. But if $z < 1$, then (Alg. § 47) the series $1 + z^2 + z^4 + z^6 + \ldots$ indefinitely protracted, has for its value $\frac{1}{1-z^2}$; it is, therefore, a convergent series. With greater reason, therefore, the second member of the last equation is a convergent series, whose terms are the same as $1 + z^2 + z^4 + \ldots$, all multiplied, except the first, by a constantly diminishing fraction. For the coefficient of the third term is the coefficient of the second multiplied by a fraction, the coefficient of the fourth term is the coefficient of the third multiplied by another fraction, etc.

Thus, in the supposition of $z < 1$, the function $\frac{1}{\sqrt{1-z^2}}$ is exactly represented by the series of the second member of the above equation indefinitely protracted.

XII. *Maxima and minima of functions of a single variable.*

Any real function $f(x)$ can always be represented by the ordinates of a curve CC′ (Fig. 18) corresponding to abscissas representing the different values of x. Let now h be a positive and infinitesimal quantity, and let the value x_m of x be represented by the abscissa AK. From K take KK′ = KK″ = h. If we find that the ordinate KM $= f(x_m)$ is greater than the preceding K′M′ and the following K″M″, it is called a *maximum* of $f(x)$; if, on the contrary, KM be found less than K′M′ and K″M″, it is a *minimum* of the same function; i. e., any function $f(x)$ will be a maximum or a minimum for a particular value x_m of x according as we shall have

$$f(x_m) > f(x_m \pm h),$$

or

$$f(x_m) < f(x_m \pm h),$$

or in other terms, according as we shall have

$$(\text{I.}) \quad \begin{cases} f(x_m \pm h) - f(x_m) < 0, \\ \text{or} \\ f(x_m \pm h) - f(x_m) > 0. \end{cases}$$

Taylor's formula enables us to find whether and when these conditions are verified; for in the case of h being positive we have from this formula

$$(\text{II.}) \quad \begin{cases} f(x_m + h) - f(x_m) = hf'(x_m) + \frac{h^2}{2}f''(x_m) + \\ \qquad \frac{h^3}{2\cdot 3}f'''(x_m) + \frac{h^4}{2\cdot 3\cdot 4}f^{\text{IV}}(x_m) + \dots \\ \text{and when } h \text{ is negative} \\ f(x_m - h) - f(x_m) = -hf'(x_m) + \frac{h^2}{2}f''(x_m) \\ \qquad -\frac{h^3}{2\cdot 3}f'''(x_m) + \frac{h^4}{2\cdot 3\cdot 4}f^{\text{IV}}(x_m) - \dots \end{cases}$$

Now the sum of infinitesimal quantities of different orders can have no other sign but that by which the infinitesimal of the lowest order is affected; hence the first members of the equations (II.) cannot have the same sign unless $f'(x_m) = 0$; but if $f''(x_m)$ does not vanish with $f'(x_m)$, then the first members of the two equations will be affected with the same sign, positive or negative, according as $f''(x_m) > 0$ or < 0, and $f(x_m)$ will be a minimum in the first case, and a maximum in the second. In case that with $f'(x_m) = 0$, $f''(x_m)$ also should be $= 0$; then, in order that $f(x_m)$ be a maximum or a minimum, $f'''(x_m)$ also must vanish; and supposing that $f^{\text{IV}}(x_m)$ does not vanish with the preceding derivative, $f(x_m)$ will be a maximum when $f^{\text{IV}}(x_m) < 0$, and a minimum when $f^{\text{IV}}(x_m) > 0$, etc. In general, let $f^{(n)}(x_m)$ be the first derivative which does not vanish. If n be an odd number, $f(x_m)$ is neither a maximum nor a minimum. If n be an even number, then $f(x_m)$ is a maximum when $f^{(n)}(x_m) < 0$, and a minimum when $f^{(n)}(x_m) > 0$. But the same value x_m which, in this case, makes a maximum or a minimum of the primitive function $f(x_m)$, fulfils the equation $f'(x_m) = f''(x_m) = \ldots = f^{(n-1)}(x_m) = 0$, therefore, taking only the first and last member. The values x_m of x which can render $f(x)$ a maximum or a minimum must be looked for among the roots of the equation

$$f'(x) \left(= \frac{dy}{dx}\right) = 0,$$

which equation must always be verified whatever be the index (n) of the derivatives of a higher order than the first, which does not vanish.

Let, for instance,

$$f(x) = x^3 - 6x^2 + 9x - 3.$$

To see if this function admits of a maximum or a minimum let us make the first derivative equal to zero; i. e., $f'(x) = 3x^2 - 12x + 9 = 0$. If there is any value capable of making a maximum or a minimum of $f(x)$ it must certainly be among the roots of this equation, which is reducible to the following:

$$x^2 - 4x + 3 = 0,$$

and which resolved gives us

$$x_m = 1, \text{ and } x_m = 3.$$

Now the second derivative is, in our case, $f''(x) = 6x - 12$, which does not vanish by substituting in it for x, $x_m = 1$, $x_m = 3$. It is, besides, negative for the first of these two values, and positive for the second; therefore the function of x

$$x^3 - 6x^2 + 9x - 3$$

acquires a maximum value when $x = 1$, and a minimum when $x = 3$. With the first of these values substituted we have $f(x_m) = 1$, with the second $f(x_m) = -3$.

XIII. *Values of functions which assume an undetermined form.*

The ratio $F(x) = \dfrac{f(x)}{\varphi(x)}$ may assume the undetermined form $\dfrac{0}{0}$, when for a particular value of x both functions $f(x)$ and $\varphi(x)$ become zero. We may ask if such a form can correspond to a definite value, and how this value can be known. The finding of this value will be a reply to both questions.

From the given equation we have $F(x)\varphi(x) - f(x) = 0$; hence (VII. I. II.) the derivative

$$\varphi(x) F'(x) + F(x) \varphi'(x) - f'(x) = 0.$$

But there is, by supposition, some value of x which makes $\varphi(x) = 0$. Substituting in the above derivative this particular value of x, we shall have, in this case, $F(x)\varphi'(x) = f'(x)$, and consequently for the same value

$$F(x) \left(= \frac{f(x)}{\varphi(x)} = \frac{0}{0}\right) = \frac{f'(x)}{\varphi'(x)}.$$

But if the derivative $f'(x)$, $\varphi'(x)$ should also become equal to zero for the same value of x, and so likewise the following derivatives till the nth order exclusively, we would then have

$$F(x) = \frac{f(x)}{\varphi(x)} = \frac{0}{0} = \frac{f^{(n)}(x)}{\varphi^{(n)}(x)}; \text{ i. e.,}$$

I. *The true value of the ratio between two functions which assume the undetermined form $\frac{0}{0}$ when a particular value of* x *is placed to them, is given by the ratio between the derivatives of the same order which are the first of those which do not vanish simultaneously when the same particular value of* x *is placed in them.*

Let, for example, $f(x) = 1 - \cos x$, $\varphi(x) = x^2$, we shall have

$$f'(x) = \sin x, \quad \varphi'(x) = 2x, \quad f''(x) = \cos x, \quad \varphi''(x) = 2,$$

and consequently, first, $F(x) = \frac{1 - \cos x}{x^2}$, which becomes $\frac{0}{0}$ when $x = 0$. Secondly, $F(x) = \frac{\sin x}{2x}$, which also becomes $\frac{0}{0}$ when $x = 0$. Lastly, $F(x) = \frac{\cos x}{2} = \frac{1}{2}$ when $x = 0$; hence when $x = 0$

$$F(x) = \frac{1 - \cos x}{x^2} = \frac{0}{0} = \frac{1}{2}.$$

The ratio $F(x) = \frac{f(x)}{\varphi(x)}$ may also assume the undetermined form $\frac{\infty}{\infty}$ for some particular value of x. In this case we shall have $\frac{1}{f(x)} = \frac{1}{\varphi(x)} = 0$, and also $\frac{1}{f(x)} : \frac{1}{\varphi(x)} = \frac{0}{0}$. Now the derivatives of $\frac{1}{f(x)}$ and $\frac{1}{\varphi(x)}$ are (V. III.) $-\frac{f'(x)}{f^2(x)}$ and $-\frac{\varphi'(x)}{\varphi^2(x)}$; therefore, from the preceding theorem,

$$\frac{1}{f(x)} : \frac{1}{\varphi(x)} \left(= \frac{0}{0}\right) = \frac{f'(x)}{f^2(x)} : \frac{\varphi'(x)}{\varphi^2(x)},$$

and multiplying each member of the equation by $\frac{f^2(x)}{\varphi^2(x)}$,

$$\frac{f(x)}{\varphi(x)}\left(=\frac{\infty}{\infty}\right)=\frac{f'(x)}{\varphi'(x)}.$$

Therefore, if for the same value of x the derivatives of $f(x)$ and $\varphi(x)$ should become infinite till the nth order exclusively, the ratio of the given functions will be determined as in the preceding case by the formula

$$\mathrm{F}(x)=\frac{f(x)}{\varphi(x)}=\frac{\infty}{\infty}=\frac{f^{(n)}(x)}{\varphi^{(n)}(x)};\ \text{i. e.,}$$

II. *The value of the ratio of two functions which for a particular value of* x *becomes* $\frac{\infty}{\infty}$ *is given by the ratio of the two first derivatives of the same order which neither vanish together nor together become infinite with the same value of* x.

Let, for example, $f(x)=\log(x)$ and $\varphi(x)=\cot(x)$. Supposing $x=0$, we have $\log(x)=-\infty$, $\cot x=\infty$; thus for the particular value 0 of x, $\frac{f(x)}{\varphi(x)}=\frac{\log(x)}{\cot(x)}=-\frac{\infty}{\infty}$. Taking now the derivatives, we obtain (VI. I. VI.) $f'(x)=\frac{1}{x}$ and $\varphi'(x)=-\frac{1}{\sin^2 x}$; hence $\frac{f'(x)}{\varphi'(x)}=-\frac{\sin^2 x}{x}$ equal to $\frac{0}{0}$, for $x=0$; but the derivative of $-\sin^2 x=-2\sin x\cos x$ and the derivative of x is 1; hence $\frac{f'(x)}{\varphi'(x)}=\frac{0}{0}=\frac{2\sin x\cos x}{1}=0$, or $\mathrm{F}(x)=\frac{\log(x)}{\cot x}=-\frac{\infty}{\infty}=0$ when $x=0$.

It may also happen that one of the functions becomes 0 for a certain value of x, and the other becomes ∞ for the same value. It is easy to see what, in this case, would be the determined value of $\frac{f(x)}{\varphi(x)}$, but the value of the product $\mathrm{F}(x)=f(x)\,\varphi(x)$ would have the undetermined form $0\cdot\infty$. Now $f(x)\,\varphi(x)=f(x):\frac{1}{\varphi(x)}=\frac{0}{0}=\varphi(x):\frac{1}{f(x)}=\frac{\infty}{\infty}$. Thus the present case is reducible to one or other of the two preceding, and

III. *The product of two functions which for a particular value of* x *becomes* $0 \cdot \infty$, *is obtained from the ratio between the derivative of one of the factors and the quotient resulting from unity divided by the other factor.*

Thus, let $f(x) = x$ and $\varphi(x) = \log(x)$. The product $x \log(x)$ becomes $-0 \cdot \infty$ when $x = 0$. Now the derivative of $\log(x)$ is $\frac{1}{x}$ and the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, therefore when $x = 0$

$$f(x) \cdot \varphi(x) = x \log x = -0 \cdot \infty = \frac{1}{x} : -\frac{1}{x^2} = -x = -0.$$

Besides the preceding, three more undetermined expressions deserve our attention, i. e., 0^0, ∞^0, 1^∞, which the function

$$F(x) = [f(x)]^{\varphi(x)},$$

assumes, when, for certain values of x both functions $f(x)$ and $\varphi(x)$ become $= 0$, or $f(x) = \infty$ and $\varphi(x) = 0$, or, finally $f(x) = 1$ and $\varphi(x) = \infty$. It might also happen that with $f(x) = 0$ we would have $\varphi(x) = \infty$, from which would result another undetermined expression 0^∞. But let it be observed that from the given formula we deduce $l\,F(x) = \varphi(x)\, l[f(x)] = \dfrac{\log[f(x)]}{\frac{1}{\varphi(x)}}$; hence $F(x) = e^{\frac{\log[f(x)]}{\frac{1}{\varphi(x)}}}$; for from this as well as from the given formula we deduce $l\,F(x) = \varphi(x)\, l[f(x)]$.

Thus $$F(x) = [f(x)]^{\varphi(x)} = e^{\frac{\log[f(x)]}{\frac{1}{\varphi(x)}}}.$$

Commencing with the last case, we have

$$F(x) = 0^\infty = e^{-\frac{\infty}{0}} = 0 \text{ in every supposition.}$$

In the first case $F(x) = 0^0 = e^{-\frac{\infty}{\infty}}$ for $\dfrac{\log[f(x)]}{\frac{1}{\varphi(x)}} = -\dfrac{\infty}{\infty}$.

In the second, $\mathrm{F}(x) = \infty^0 = e^{\frac{\infty}{\infty}}$ for $\dfrac{\log[f(x)]}{\dfrac{1}{\varphi(x)}} = + \dfrac{\infty}{\infty}$.

In the third, $\mathrm{F}(x) = 1^\infty = e^{\frac{0}{0}}$ " " " " $\dfrac{0}{0}$.

Therefore the determination of the expressions 0^0, ∞^0, 1^∞ is obtained by the same expedient by which the preceding undetermined expressions have been determined. To give an example of each of the last three cases, let,

1st, $f(x) = \varphi(x) = x$. We shall have

$$\mathrm{F}(x) = x^x = e^{\frac{\log(x)}{\frac{1}{x}}} = 0^0 = e^{-\frac{\infty}{\infty}} \text{ when } x = 0.$$

But $\dfrac{1}{x}$ is the derivative of $\lg x$, and $-\dfrac{1}{x^2}$ the derivative of $\dfrac{1}{x}$; hence $\dfrac{\log(x)}{\dfrac{1}{x}} = -\dfrac{\infty}{\infty} = -\dfrac{1}{x} : \dfrac{1}{x^2} = -x = -0$ when x $= 0$ and $e^{-0} = \dfrac{1}{e^0} = 1$; i. e., when $x = 0$

$$\mathrm{F}(x) = x^x = 0^0 = 1.$$

2d. Let $f(x) = x$, $\varphi(x) = \dfrac{1}{x}$, and suppose $x = \infty$, we shall have $\mathrm{F}(x) = x^{\frac{1}{x}} = \infty^0 = e^{\frac{\log x}{x}} = e^{\frac{\infty}{\infty}}$ when $x = \infty$. But $\dfrac{1}{x}$ is the derivative of $\log(x)$, and 1 the derivative of x; hence $\dfrac{\log(x)}{x} = \dfrac{\infty}{\infty} = \dfrac{1}{x} = 0$ when $x = \infty$. But $e^0 = 1$. Therefore when $x = \infty$

$$\mathrm{F}(x) = x^{\frac{1}{x}} = \infty^0 = 1.$$

3d. Let $f(x) = x$, $\varphi(x) = \dfrac{1}{1 - x}$. Taking $x = 1$ we obtain

$F(x) = x^{\frac{1}{1-x}} = 1^{\infty} = e^{\frac{\log(x)}{1-x}} = e^{\frac{0}{0}}$. Now $\frac{1}{x}$ is the derivative of lg (x), and -1 the derivative of $1-x$; hence $\frac{\log(x)}{1-x} = \frac{0}{0}$ $= -\frac{1}{x} = -1$ when $x = 1$; i. e., when $x = 1$

$$F(x) = x^{\frac{1}{1-x}} = 1^{\infty} = e^{-1} = \frac{1}{e}.$$

XIV. *Chord of an infinitesimal arc of a continual curve.*

Let DABD′ (Fig. 19) be any continued curve. Divide the arc into three parts at pleasure, DA, AB, BD, and draw the corresponding chords, producing the first to T, and the last until it reaches the first in C. From this construction we have TCB = CAB + CBA. Calling ε this angle, and designating by a, b, c the sides CB, CA, AB of the triangle CAB, we have (Trig. §§ 20 (e_4) and 12)

$$c^2 = a^2 + b^2 + 2ab \cos \varepsilon.$$

And since (Tr. § 20) $\cos \varepsilon = 1 - 2 \sin^2 \frac{1}{2} \varepsilon$ also,

$$c^2 = (a + b)^2 - 4ab \sin^2 \tfrac{1}{2}\varepsilon,$$

easily reduced to the following:

$$\frac{c^2}{(a + b)^2} = 1 - \frac{4ab}{(a + b)^2} \sin^2 \tfrac{1}{2}\varepsilon,$$

in which the coefficient $\frac{4ab}{(a + b)^2}$ is equal to $1 - \left(\frac{a-b}{a+b}\right)^2$; i.e., less than unity.

Now, in a continual curve the smaller the arc DABD′ becomes, the smaller also become the angles CAB, CBA, and when the arc becomes infinitesimal the angles also become infinitesimal, and ε likewise, which is equal to their sum. Therefore, in this supposition, $\sin^2 \frac{1}{2}\varepsilon$ is an infinitesimal of the second order, which, in the last equation, being besides multiplied by a coefficient < 1, can be suppressed, giving us thereby

$\frac{c^2}{(a+b)^2} = 1$, or $c = a + b$. But if, when the arc $DAmBD'$ is infinitesimal, the chord AB does not differ from the sum of the sides AC, CB, it differs much less from the arc AmB subtended by it. Hence in any curve *An infinitesimal continual arc and its chord coincide with each other*, or *their ratio is equal to unity.*

It follows from this theorem that *The chord may be taken instead of the corresponding infinitesimal arc ;* and since, in this case, the chord necessarily coincides with the tangent, it follows also that *Any curve may be regarded as a polygon of an infinite number of infinitesimal sides which produced will be as many tangents of the different points of the curve.*

XV. *Tangent, subtangent, normal and subnormal of any plane curve.*

Let CC′ (Fig. 20) represent any plane curve referred to the orthogonal axes AX, AY, and let $y = f(x)$ be its equation. Let also TT′ be the indefinite tangent of any point M of the curve, and MP a perpendicular to the tangent from the point M of contact, the co-ordinates x and y of which are AK, KM. The segment MT of the indefinite or geometrical tangent of M, contained between the point of contact and the axis of abscissas, is called *tangent* of the point M. In like manner the segment MN of the perpendicular MP contained between the same point of contact and the axis of abscissas, is called *normal* of the point M. The tangent is represented by t, the normal by n. Of the two segments TK, KN of the axis of abscissas, measured from the points met by the tangent and by the normal to the ordinate of the point of contact, the first is called *subtangent* and the second *subnormal* of the point M, and are respectively represented by $t_,$ and $n_,$.

To determine the length of these four functions, observe, first, that calling X, Y the co-ordinates of the tangent referred to the axes of the curve, and X′, Y′ the co-ordinates of the normal, and (tx) the angle MTX, since both lines pass through the point M, and one is perpendicular to the other, we have (A. G. III. 2d, 3d) for the equations of these lines

$$Y - y = (X - x) \operatorname{tg}(tx), \quad Y' - y = -(X' - x)\frac{1}{\operatorname{tg}(tx)}.$$

Take now from M the arc MM′ infinitesimal, whose coordinates AK′, K′M′ will be respectively represented by $x + dx$, $y + dy$, and drawing from M on M′K′, MD parallel to AX, we shall have also $MD = dx$, $DM' = dy$. The arc MD, being infinitesimal, may be regarded as rectilinear and coincident with MT′, therefore (IX.)

$$\operatorname{tg}(tx) = \operatorname{tg} M'MD = \frac{dy}{dx} = f'(x);$$

hence from the preceding equations

$$Y - y = (X - x) f'(x), \quad Y' - y = -(X' - x)\frac{1}{f'(x)}.$$

The abscissa X corresponding to $Y = 0$ is $-$AT, and the abscissa X′ corresponding to $Y' = 0$ is AN; hence, making in the last equations Y and $Y' = 0$, we shall obtain

$$AT + x = \frac{y}{f'(x)}, \quad AN - x = yf'(x);$$

i. e.,
$$t_{,} = \frac{y}{f'(x)}, \quad n_{,} = yf'(x);$$

and since from the right-angled triangles MKT, MKN

$$MT = \sqrt{\overline{KM}^2 + \overline{TK}^2}, \quad MN = \sqrt{\overline{KM}^2 + \overline{KN}^2},$$

so also, for the values of the tangent and of the normal,

$$t = \sqrt{y^2 + t_{,}^2}, \quad n = \sqrt{y^2 + n_{,}^2},$$

or
$$t = y\sqrt{1 + \frac{1}{f'^2(x)}}, \quad n = y\sqrt{1 + f'^2(x)}.$$

These and the preceding formulas, being altogether general, can be applied to the lines of the second order.

1st. *Functions of the parabola.* Commencing with the parabola, whose equation is (A. G. V.)

$$y = \sqrt{2px},$$

i. e., $f(x) = \sqrt{2px}$, we shall have (V. IV. and IX.) $f'(x) = \frac{p}{\sqrt{2px}} = \frac{p}{y}$; hence, since in the parabola (L. c.) $\rho = x + \frac{1}{2}p$ designates the distance of the focus from the point (x, y) of the curve, we shall have

$$t_{,} = \frac{y^2}{p} = 2x, \quad n_{,} = p;$$

i. e., the *subtangent of any point of the parabola is equal to the double of the abscissa of that point*, as we have already found (A. G. VIII.) with a different process, and *the subnormal is constant and equal to the semiparameter.* Concerning the tangent and the normal, we have from the preceding general equations

$$t = y\sqrt{1 + \frac{y^2}{p^2}} = \sqrt{2px + 4x^2} = \sqrt{4x(x + \frac{1}{2}p)} = \sqrt{4x\rho},$$

$$n = y\sqrt{1 + \frac{p^2}{y^2}} = \sqrt{2px + p^2} = \sqrt{2p(x + \frac{1}{2}p)} = \sqrt{2p\rho};$$

i. e., *The tangent of any point of the parabola is mean geometrical proportional between the focal distance of the point of contact and the quadruple of the abscissa of the same point. The normal of any point of the parabola is mean geometrical proportional between the focal distance of the same point and the parameter.*

2d. *Functions of the ellipse.* The ellipse referred to its own axes is represented (A. G. XI.) by the following equation:

$$y = \frac{b}{a}\sqrt{a^2 - x^2};$$

hence, in this case, $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$ and (V. IV.) $f'(x) = -\frac{bx}{a\sqrt{a^2 - x^2}}$; and from the equation (2) (A. G. XI.)

$a^2 - b^2 = a^2e^2$. From the above general formulas we obtain for the ellipse

$$t_{,} = -\frac{a^2 - x^2}{x} = -\frac{a^2}{x} + x, \quad n_{,} = -\frac{b^2}{a^2}x;$$

i. e., taking into account only the absolute value, *The subtangent in the ellipse is equal to the difference between the abscissa of the point of contact and the square of the transverse semiaxis divided by the same abscissa.* Hence the subtangent is independent of the conjugate axis. *The subnormal is equal to the product of the abscissa of the point of contact by the square of the conjugate divided by the square of the transverse semiaxis.* Consequently the ratio of the subnormal and the abscissa of the point of contact is constant in the ellipse.

The general formulas of the tangent and normal become for the ellipse

$$t = y\sqrt{1 + \frac{a^2(a^2 - x^2)}{b^2x^2}} = \frac{y}{bx}\sqrt{b^2x^2 + a^4 - a^2x^2} =$$
$$\frac{y}{bx}\sqrt{a^4 - (a^2 - b^2)x^2};$$

but $a^2 - b^2 = a^2e^2$, hence

$$t = \frac{a}{b}\cdot\frac{y}{x}\sqrt{a^2 - e^2x^2},$$

$$n = y\sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}} = \frac{y}{a\sqrt{a^2 - x^2}}\sqrt{a^4 - (a^2 - b^2)x^2}$$
$$= \frac{y}{\sqrt{a^2 - x^2}}\sqrt{a^2 - e^2x^2};$$

but $y = \frac{b}{a}(\sqrt{a^2 - x^2})$; hence

$$n = \frac{b}{a}\sqrt{a^2 - e^2x^2};$$

i. e., *The tangent and the normal of any point of the ellipse are given by formulas analogous to the equation of the curve, chang-*

ing in the latter for both of them x^2 *into* e^2x^2, *and for the tangent also the coefficient* $\frac{b}{a}$ *into* $\frac{a}{b} \cdot \frac{y}{x}$.

3d. *Functions of the hyperbola.* The equation of the hyperbola referred to its own axes is (A. G. (3) XIX.)

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

hence $f'(x) = \frac{bx}{a\sqrt{x^2 - a^2}}$, and observing that from the equation (2) (A. G. XIX.) $a^2 + b^2 = a^2e^2$, following the same process as for the ellipse, we shall find

$$t_{,} = \frac{x^2 - a^2}{x} = x - \frac{a^2}{x}, \quad n_{,} = \frac{b^2}{a^2}x,$$

and

$$t = \frac{a}{b} \cdot \frac{y}{x} \sqrt{e^2x^2 - a^2}, \quad n = \frac{b}{a}\sqrt{e^2x^2 - a^2},$$

from which follow exactly the same inferences as for the ellipse; hence the preceding conclusions with regard to the functions of the ellipse are applicable to those of the hyperbola.

XVI. *Differential total and partial of a function of different independent variables.*

So far, we have supposed functions depending on only one variable. Let us now pass to see how differentials of functions containing more than one independent variable can be obtained. Let, for example, μ be a function of the variables x, y, z, independent of each other, and having the following form:

$$\mu = \frac{x \log(y)}{\sin z}.$$

The *total differential* of μ is the difference between the value of the given function and that which the same function assumes when we change all the variables by an infinitesimal quantity, or (3) the difference between the first and second state of the function when each one of the variables undergoes an infinitesimal change. Thus, representing by $d\mu$ the infinitesimal change of μ, resulting from those of all the variables, we shall have

$$d\mu = \frac{(x + dx) \log (y + dy)}{\sin (z + dz)} - \frac{x \log (y)}{\sin z}.$$

If all the variables are not changed, but only one of them, the change which μ undergoes in consequence of that of the variable is evidently a *partial differential.* It is represented by $d_x\mu$ when x alone is changed into $x + dx$, or by $d_y\mu$, $d_z\mu$ when y or z alone is changed. In this supposition, we shall have

$$d_x\mu = \frac{(x + dx) \log (y)}{\sin z} - \frac{x \log (y)}{\sin z}, \quad d_y\mu = \frac{x \log (y + dy)}{\sin z} - \frac{x \log (y)}{\sin z}, \quad d_z\mu = \frac{x \log (y)}{\sin (z + dz)} - \frac{x \log (y)}{\sin z}.$$

More generally, If the function μ be represented by $f(x, y, z)$, the total and partial differentials will be expressed as follows:

$$d\mu = f(x + dx, y + dy, z + dz) - f(x, y, z),$$

$$d_x\mu = f(x + dx, y, z) - f(x, y, z), \quad d_y\mu = f(x, y + dy, z) - f(x, y, z), \quad d_z\mu = f(x, y, z + dz) - f(x, y, z).$$

Concerning the partial differentials, they are obtained exactly as the differentials of the functions of only one variable, considering the other variables as constant. It remains, then, only to see how the total differential can be obtained. Before we proceed to this, observe that the differential of $y = f(x)$ is expressed by $dy = f'(x)\, dx$. Now, the derivative $f'(x)$ is a function of $f(x)$. Suppose, for example, $f(x) = \sqrt{2px}$, from which $f'(x) = \sqrt{\frac{p}{2x}}$. A change of any of the three factors 2, p, x contained in $f(x)$ will be evidently attended by a corresponding change in $f'(x)$; and if the change be infinitesimal, the change in $f'(x)$ also will be infinitesimal. Thus, supposing, for example, that we make an infinitesimal change in the coefficient $2p$, the derivative will be $f'(x) \pm \delta$, differing from $f'(x)$ by the infinitesimal quantity δ. Hence to the equations

$$y = f(x), \quad dy = f'(x)\,dx$$

we may add another,

$$dy_{,} = [f'(x) \pm \delta]\,dx,$$

corresponding to the same value of the independent x, when in the given function $f(x)$ some other element besides x is submitted to an infinitesimal change. Now, from the last equation we have $dy_{,} = f'(x)\,dx \pm \delta \cdot dx$. But $\delta \cdot dx$ is an infinitesimal of the second order, therefore $dy_{,} = f'(x)\,dx = dy$; i. e., the differential of $y = f(x)$ remains unchanged whether the other elements of the function do not change together with x or be they also submitted to an infinitesimal change.

Let us now apply all this to

$$\mu = f(x, y, z),$$

differentiating first μ with regard to x, we have

$$d_x\mu = f(x + dx, y, z) - f(x, y, z);$$

differentiating then $f(x + dx, y, z)$ by y, we shall obtain the same result as by differentiating the given μ by y; i. e.,

$$d_y\mu = f(x + dx, y + dy, z) - f(x + dx, y, z).$$

Lastly, taking the differential of $f(x + dx, y + dy, z)$ with regard to z, we shall again obtain the same result as by taking the differential of the given μ with regard to z; i. e.,

$$d_z\mu = f(x + dx, y + dy, z + dz) - f(x + dx, y + dy, z);$$

adding now together these three differentials, we obtain

$$d_x\mu + d_y\mu + d_z\mu = f(x + dx, y + dy, z + dz) - f(x, y, z);$$

but the second member of this equation is the total differential $d\mu$; hence

$$d\mu = d_x\mu + d_y\mu + d_z\mu.$$

The same process is applicable to any number of variables, therefore *The total differential of a function of different variables is equal to the sum of the partial differentials of the same functions.*

Applying it to the case of $\mu = \frac{x \log(y)}{\sin z}$, which we have taken above as an example of a function of different variables, we have (V. and VI.)

$$d_x\mu = \frac{\log(y)}{\sin z}\,dx,\; d_y\mu = \frac{x}{y \sin z}\,dy,\; d_z\mu = -\frac{x \log(y) \cos z}{\sin^2 z}\,dz;$$

hence

$$d\,\frac{x \log(y)}{\sin z} = \frac{\log(y)}{\sin z}\,dx + \frac{x}{y \sin z}\,dy - \frac{x \log(y) \cos z}{\sin^2 z}\,dz.$$

XVII. *Derivative functions.*

Recalling to mind what has been said (IX.) concerning derivative functions, it will be easily admitted that $\frac{d_x\mu}{dx}, \frac{d_y\mu}{dy}, \frac{d_z\mu}{dz}$, are the partial derivative functions of $\mu = f(x, y, z)$ with regard to x, y, z. These same functions are also represented by $f'_x(x, y, z)$, $f'_y(x, y, z)$, $f'_z(x, y, z)$. Thus,

$$d_x\mu = f'_x(x, y, z)\,dx,\; d_y\mu = f'_y(x, y, z)\,dy,\; d_z\mu = f'_z(x, y, z)\,dz,$$

and consequently the formula $d\mu = d_x\mu + d_y\mu + d_z\mu$ may be represented also by

$$d\mu = f'_x(x, y, z)\,dx + f'_y(x, y, z)\,dy + f'_z(x, y, z)\,dz,$$

or by

$$d\mu = \frac{d_x\mu}{dx}\,dx + \frac{d_y\mu}{dy}\,dy + \frac{d_z\mu}{dz}\,dz.$$

Nay, since the partial differentials with regard to x, y, z are sufficiently indicated by their respective denominators, the signs x, y, z affixed to d may be and are ordinarily omitted. Representing thus the differential of μ more simply we have

$$d\mu = \frac{d\mu}{dx}\,dx + \frac{d\mu}{dy}\,dy + \frac{d\mu}{dz}\,dz.$$

In like manner, following the analogy of the differentials and derivatives of various orders of the functions of only one variable, we represent by $d^2_x\mu$, $d^2_y\mu$, $d^2_z\mu$, the differentials of the second order of μ relatively to x, y, and z, and by $d_xd_y\mu$, $d_xd_z\mu$, or $d_xd_yd_z\mu$ the differentials of μ, first with regard to y and then with regard to x, or first with regard to z and then with regard to x, or first with regard to z, then with regard to y and then with regard to x, etc., by succeeding differentiations. Also the corresponding derivatives will be represented by

$f''_x(x,y,z), f''_y(x,y,z)$, etc., or by $\frac{d^2\mu}{dx^2}, \frac{d^2\mu}{dy^2}, \frac{d^2\mu}{dz^2}, \frac{d^2\mu}{dxdy}, \ldots$ hence

$$d^2_x\mu = f''_x(x, y, z)\,dx^2 = \frac{d^2\mu}{dx^2}\overline{dx}^2,$$

$$d^2_y\mu = f''_y(x, y, z)\,\overline{dy}^2 = \frac{d^2\mu}{dy^2}\overline{dy}^2,$$

$$d^2_z\mu = f''_z(x, y, z)\,\overline{dz}^2 = \frac{d^2\mu}{dz^2}\overline{dz}^2,$$

and

$$d_xd_y\mu = \frac{d^2\mu}{dxdy}\,dxdy, \quad d_xd_z\mu = \frac{d^2\mu}{dxdz}\,dxdz, \ldots$$

Whatever be the order kept in the successive differentiation of μ, first with regard to x, for example, and then with regard to y, or *vice versa*, the result is the same, for from the differentials

$$d_x\mu = f(x + dx, y, z) - f(x, y, z),$$
$$d_y\mu = f(x, y + dy, z) - f(x, y, z),$$

we obtain

$$d_yd_x\mu = f(x + dx, y + dy, z) - f(x, y + dy, z) - f(x + dx, y, z) + f(x, y, z),$$

$$d_xd_y\mu = f(x + dx, y + dy, z) - f(x + dx, y, z) - f(x, y + dy, z) + f(x, y, z).$$

Now the second members of these equations are identical;

$$d_yd_x\mu = d_xd_y\mu.$$

For example, take again $\mu = \frac{x \log(y)}{\sin z}$. We shall have

$$d_y d_x \mu = d_y \frac{\log(y)\, dx}{\sin z} = \frac{dydx}{y \sin z},$$

$$d_x d_y \mu = d_x \frac{xdy}{y \sin z} = \frac{dxdy}{y \sin z}.$$

With the preceding observations, it is not difficult to find the formulas by means of which we may obtain the successive and total differentials of a function of different variables. Let, for example, $\mu = F(x, y)$ be a function containing two independent variables, x and y. We shall have for the differential of the first order

$$d\mu = d_x\mu + d_y\mu = \frac{d\mu}{dx} dx + \frac{d\mu}{dy} dy,$$

and for the differential of the second order

$$d^2\mu = d_x(d_x\mu + d_y\mu) + d_y(d_x\mu + d_y\mu).$$

Now $d_x(d_x\mu + d_y\mu) = d^2_x\mu + d_x d_y\mu$, and $d_y(d_x\mu + d_y\mu) = d^2_y\mu + d_x d_y\mu$; hence

$$d^2\mu = d^2_x\mu + d^2_y\mu + 2d_x d_y\mu = \frac{d^2\mu}{dx^2} dx^2 + \frac{d^2\mu}{dy^2} dy^2 + 2\frac{d^2\mu}{dxdy} dxdy.$$

If the given function $F(x, y)$ be constantly equal to zero or to any constant quantity C, $d\mu = d^2\mu = 0$; i. e.,

$$\frac{d\mu}{dx} dx + \frac{d\mu}{dy} dy = 0,$$

$$\frac{d^2\mu}{dx^2} dx^2 + \frac{d^2\mu}{dy^2} dy^2 + 2\frac{d^2\mu}{dxdy} dxdy = 0.$$

But in this case one of the variables, y, for example, is function of the other; hence $dy = \frac{dy}{dx} dx$, $d^2y = \frac{d^2y}{dx^2} dx^2$, and the first of the last two equations becomes $\frac{d\mu}{dx} dx + \frac{d\mu}{dy} \cdot \frac{dy}{dx} dx = 0$. The second term of the second equation, or its equivalent $d^2_y\mu = dF'_y(x, y)\, dy$, in which dy is variable with y, is the differential of a product of two functions of y; hence

$$d^2_y\mu = \mathrm{F}''_y(x, y)\, dy^2 + \mathrm{F}'_y(x, y)\, d^2y = \frac{d^2\mu}{dy^2} dy^2 + \frac{d\mu}{dy} d^2y$$
$$= \frac{d^2\mu}{dy^2} \frac{dy^2}{dx^2} dx^2 + \frac{d\mu}{dy} \frac{d^2y}{dx^2} dx^2.$$

This value substituted in the above second equation, together with that of dy in the third term, gives $\frac{d^2\mu}{dx^2} dx^2 + \frac{d^2\mu}{dy^2} \frac{dy^2}{dx^2} dx^2 + \frac{d\mu}{dy} \frac{d^2y}{dx^2} dx^2 + 2 \frac{d^2\mu}{dxdy} \frac{dy}{dx} dx^2 = 0$, in which dx^2 is common factor as dx is common factor of the corresponding preceding equation; hence, in the supposition of $y = f(x)$, we infer from the above equations

$$\frac{d\mu}{dx} + \frac{d\mu}{dy} \cdot \frac{dy}{dx} = 0,$$

$$\frac{d^2\mu}{dx^2} + \frac{d^2\mu}{dy^2} \cdot \frac{dy^2}{dx^2} + \frac{d\mu}{dy} \cdot \frac{d^2y}{dx^2} + 2 \frac{d^2\mu}{dxdy} \frac{dy}{dx} = 0.$$

Now (XV.) $\frac{dy}{dx}$ or $f'(x) = \mathrm{tg}\,(tx)$. Placing this value in the last equations, and F (x, y), or simply F, instead of μ, we obtain from the same

$$\text{(D)} \quad \begin{cases} \dfrac{d\mathrm{F}}{dx} + \dfrac{d\mathrm{F}}{dy} \mathrm{tg}\,(tx) = 0, \\ \dfrac{d^2\mathrm{F}}{dx^2} + \dfrac{d^2\mathrm{F}}{dy^2} \mathrm{tg}^2\,(tx) + \dfrac{d\mathrm{F}}{dy} \cdot \dfrac{d^2y}{dx^2} + 2 \dfrac{d^2\mathrm{F}}{dxdy} \mathrm{tg}\,(tx) = 0. \end{cases}$$

XVIII. *Singular points of plane curves.*

We call *singular* those points of a curve which present some peculiarities inherent to the character of the curve. Such are the *multiple* points *double, triple,* etc., i. e., those through which pass different branches of the curve, each having a different tangent. An example of this kind is represented by Fig. 21. *Points of regress* or *cusps* are likewise singular points. They are those in which a branch of the curve stops to begin, as it were, another branch, both branches having in the same point a common tangent, whether the two branches turn mutually their convexity, as in the first example of Fig. 22, or one of them turns the convexity to the concavity of the other, as in the second example of the same figure. *Isolated*, or *conjugate* points, are also called singular points. They are entirely separated from the branches of the curve, although their co-ordinates fulfil the equation of the same curve. In treating of these classes only of singular points, we shall avail ourselves of the last formulas (D) of the preceding paragraph.

Let $F(x, y) = 0$ be the equation of a curve referred to orthogonal axes, and let (tx) be the angle which the geometrical tangent of the point (x, y) forms with the axis of abscissas. The formulas (D) co-exist with $F(x, y) = 0$, and must be simultaneously verified for each point (x, y) of the curve. Now the first (D) is verified either when both terms are equal but affected with opposite signs, or when each term is separately equal to zero. When the factor $\mathrm{tg}\,(tx)$ admits of different values, as in the case of multiple points, since for the same values of x and y, the derivatives $\frac{dF}{dx}, \frac{dF}{dy}$ do not change, if for one of the values of tg, (tx) the two terms mutually eliminate each other, they will not for another, unless we suppose both derivatives equal to zero. A similar observation is applicable to the case of $\mathrm{tg}\,(tx)$ imaginary, which happens for isolated points; i. e., unless both derivatives be equal to zero, the equation cannot be verified. Thus, if the equation $F(x, y) = 0$ belongs to a curve which contains singular points, the co-ordinates of these points may be found among those which fulfil the equations

$$(D_1) \ldots \frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0.$$

In this supposition, the second (D) becomes

$$(D_2) \ldots \frac{d^2F}{dx^2} + \frac{2d^2F}{dxdy}\,\mathrm{tg}\,(tx) + \frac{d^2F}{dy^2}\,\mathrm{tg}^2\,(tx) = 0,$$

from which, substituting the values of x and y deduced from (D_1), if $\frac{d^2F}{dy^2}$ does not disappear, we obtain two values for tg (tx), either real, and equal or unequal, or two imaginary values. In the first of which cases, the point (x, y) would be a point of regress, in the second, a double point, in the last, an isolated point. Should the terms of (D_2) disappear by the substitution of x and y obtained from $(D_{,})$, in order to find whether the curve admits of any singular point it would be necessary to have recourse to derivatives of higher order. But let us see an example of each of the three cases just mentioned. Let, first, $F(x, y) = y^2 + x^4 - x^2 = 0$, from which $\frac{dF}{dx} = 4x^3 - 2x$, $\frac{dF}{dy} = 2y$. Hence, in the present case, the equations $(D_{,})$ become

$$4x^3 - 2x = 0, \quad 2y = 0,$$

from which $x = 0, y = 0$, which fulfil the equation and belong to the point of the curve passing through the origin of the axes, which may be a singular point. To see if such be the case, let us take the partial derivatives of the second order, which are $\frac{d^2F}{dx^2} = 12x^2 - 2$, $\frac{d^2F}{dxdy} = 0$, $\frac{d^2F}{dy^2} = 2$, and placing in them $x = y = 0$, $\frac{d^2F}{dx^2} = -2$, $\frac{d^2F}{dxdy} = 0$, $\frac{d^2F}{dy^2} = 2$, hence the equation (D_2) is, in this case,

$$-2 + 2\text{tg}^2 (tx) = 0,$$

and, consequently, tg $(tx) = 1$, tg $(tx) = -1$. The origin of the co-ordinates is, therefore, a *double* point, and the branches of the curve have their tangent forming an angle of 45° on each side of the axis of abscissas. The form of the curve is similar to that of Fig. 21.

Let, second, $F(x, y) = ay^2 - x^3 = 0$, from which $\frac{dF}{dx} = -3x^2$, $\frac{dF}{dy} = 2ay$; hence for the equations $(D_{,})$

$$-3x^2 = 0, \quad 2ay = 0,$$

and, consequently, $x = y = 0$. Taking the partial derivatives of the second order, and making in them $x = y = 0$, we find $\frac{d^2F}{dx^2} = -6x = 0$, $\frac{d^2F}{dxdy} = 0$, $\frac{d^2F}{d^2y} = 2a$; hence (D_2)

$$2a \operatorname{tg}^2 (tx) = 0,$$

and, consequently, $\operatorname{tg} (tx) = \pm 0$. Hence the point of the curve corresponding to the origin of the axes is a point of regress, and the branches of the curve have, in that point, the axis of abscissas for common tangent. This curve is called a cubic parabola, and is represented by the first Fig. 22.

Let, third, $F(x, y) = y^2 - x^4 + a^2x^2 = 0$, from which $\frac{dF}{dx} = -4x^3 + 2a^2x$, $\frac{dF}{dy} = 2y$, and, consequently, for the equations (D_1)

$$-4x^3 + 2a^2x = 0, \quad 2y = 0;$$

consequently in this case also $x = y = 0$. Taking now the partial derivatives of the second order and making in them $x = y = 0$, we obtain $\frac{d^2F}{dx^2} = -12x^2 + 2a^2 = 2a^2$, $\frac{d^2F}{dxdy} = 0$, $\frac{d^2F}{dy^2} = 2$. Thus, in this case, we have for (D_2)

$$2a^2 + 2 \operatorname{tg}^2 (tx) = 0;$$

hence $\operatorname{tg} (tx) = \pm a\sqrt{-1}$. The point, therefore, corresponding to the origin of the axes, is an isolated point.

XIX. *Convexity and concavity — points of inflection.*

As the derivatives of a given equation $F(x, y)$ offer a criterion to find out if the corresponding curve admits of any singular point, so the derivatives of $y = f(x)$ offer a criterion to determine whether the corresponding curve turns its convexity or its concavity to the axis of abscissas.

Let (Fig. 23) M be a point of the curve represented by $y = f(x)$ having AK, KM for its co-ordinates x and y, and let

KK′ be an infinitesimal increment dx of x. We shall have $AK' = x + dx$, and the corresponding ordinate $K'M' = f(x + dx)$. Represent by v and u the abscissas and ordinates of the tangent of the point M, and let $u_{,}$ be the ordinate corresponding to the abscissa AK′. It is plain that from M forward, the curve will turn its concavity or its convexity to the axis of abscissas according as M′K — NK′ is negative or positive, that is, according as

$$f(x + dx) - u_{,} < \text{ or } > 0.$$

Now the equation of the tangent is (XV.) $u - y = (v - x) f'(x)$, therefore $u_{,} - y = (x + dx - x) f'(x) = f'(x)\, dx$, or $u_{,} = f(x) + f'(x)\, dx$.

Taylor's formula gives us, besides, (XI.)

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)\, \overline{dx}^2 + \frac{1}{2 \cdot 3} f'''(x)\, \overline{dx}^3 + \ldots$$

Hence, subtracting from this the preceding,

$$f(x + dx) - u_{,} = \tfrac{1}{2} f''(x)\, \overline{dx}^2 + \frac{1}{2 \cdot 3} f'''(x)\, \overline{dx}^3 + \ldots$$

Now the sign of the second number depends on that of the first term, or rather on that of the factor $f''(x)$. Therefore, supposing that $f''(x)$ does not vanish, $f(x + dx) - u_{,}$ will be $<$ or > 0 when $f''(x)$ is $<$ or > 0. Therefore, if the derivative of the second order of $y = f(x)$ is negative, the curve from M forward turns its concavity to the axis of abscissas, and if the same derivative results positive, the curve turns its convexity to that axis. If $f''(x)$ vanishes, the criterion will be taken from $f'''(x)$, and if this also vanishes, from $f^{\text{IV}}(x)$, etc.

Should the branch of the curve from M toward the axis AY, change the bending of its curvature, the point M is then called a point of inflection, and the branch of the curve turns its convexity toward the axis of abscissas on one side of it, and on the other its concavity. The derivatives of the equation of the curve, which we shall continue to represent by $y = f(x)$, will reveal to us if M is a point of inflection, which is one of those that belong to the class of singular points. Let (Fig. 24) M be a point of inflection of M″MM′, and TT′the tangent corresponding to M, which, without ceasing to be a tangent, must necessarily cut the curve in M. Taking KK′ = KK″ = dx, and representing as before by u, v the ordinates and abscissas of the tangent, by $u_{\prime}$ the ordinate corresponding to AK″ or AK′; since the differences M′K′ — N′K′, M″K″ — N″K″ must be necessarily affected by a different sign, the signs also of $f(x + dx) - u$, and $f(x - dx) - u$, must be different from each other. In order to see if and when this condition is verified, and consequently if the curve admits of one or more points of inflection, let us resume the equation already obtained above.

$$f(x + dx) - u_{\prime} = \tfrac{1}{2}f''(x)\,\overline{dx}^2 + \frac{1}{2 \cdot 3}f'''(x)\,\overline{dx}^3 + \ldots$$

from which, changing in it dx into $-dx$,

$$f(x - dx) - u_{\prime} = \tfrac{1}{2}f''(x)\,\overline{dx}^2 - \frac{1}{1 \cdot 3}f'''(x)\,dx^3 + \ldots$$

These equations show that the differences cannot be affected with opposite signs unless $f''(x) = 0$. Hence, if the curve admits of any point of inflection, the co-ordinates of that point must fulfil the equation $f''(x) = 0$, and, consequently, *vice versa,* the co-ordinates of the points of inflection must be found among those which fulfil the same equation. It is not sufficient, however, that real co-ordinates x_m, y_m fulfil the equation $f''(x) = 0$ to enable us to infer that the curve admits of points of inflection, if the same co-ordinates x_m, y_m annul the derivatives of higher orders, except when the first derivative which is not annulled is of an uneven order, 3d, 5th, etc. When, therefore, the equation $y = f(x)$ is such that for certain real co-ordinates x_m, y_m, the derivative $f''(x)$ becomes zero, and the first of the subsequent derivatives which does not vanish is of an uneven order, the curve admits of as many points of inflection as there are pairs of co-ordinates x_m, y_m for which the said conditions are verified.

Let us take, for example, the transcendental curve represented by the equation $y = f(x) = \sin x$, from which (IX. and VIII. III.) $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{\text{IV}}(x) = \sin x$, etc. The first condition to be verified is that $f''(x) = 0$ be resolved with real values x_m of x. Now $x_m = 0, = \pi, = 2\pi, = 4\pi, = \ldots$ are all real roots of $f''(x) = 0$. The second condition is that $f'''(x)$ or $f^{\text{V}}(x)$, etc., is the first of the derivatives which does not vanish when x_m is placed in them. But, in our case, $f'''(x) = -\cos x = -1, +1, -1, +1, -$ etc., when $x = 0, = \pi, = 2\pi, = \ldots$ Hence the curve represented by $y = \sin x$ admits of an infinite number of points of inflection. The form of this curve is partly indicated by Fig. 25. Its tangent, at the origin of the axis, bisects the angle formed by the same axis: for tg $(tx) = f'(x) = \cos x$, and with $x = 0$, $\cos x = 1$; hence tg $(tx) = 1 =$ tg $(45°)$.

PART II.

INTEGRAL CALCULUS.

XX. *Indefinite Integrals.*

THE object of integral calculus is opposite to that of differential. It consists in finding the function from which a given differential has been obtained. Integral and differential are correlative. Thus, as adx is the differential of ax, so ax is the integral of adx. The integral of a given differential is designated by affixing to it the symbol $\int$, which signifies *sum*, as the letter d, adopted in differential calculus, signifies difference. $\int adx$ signifying the same thing as ax, we may write the equation

$$(1) \ldots \int adx = ax,$$

the first member of which indicates, the second expresses, the integral of adx, and the equation is read *Integral of* adx *is equal to* ax.

Let us here observe two things: first, that as (IX.) $f'(x)\,dx$ represents the differential of any function of x, $f(x)$, and as $df(x)$ and $f'(x)\,dx$ signify the same thing, we shall have $\int df(x) = \int f'(x)\,dx = f(x)$; i. e., the integral of a differential only indicated, is obtained by suppressing both signs $\int$ and d. Secondly, that as, C being a constant, $d(f(x) + C) =$ (V.) $df(x)$ equal, in both cases, $f'(x)\,dx$; $\int f'(x)\,dx$ may be given by $f(x) + C$, as well as simply by $f(x)$; i. e.,

$$\text{II.} \quad \begin{cases} \int f'(x)\,dx = f(x) + C, \\ \int f'(x)\,dx = f(x), \end{cases}$$

$f(x) + C$ is called *complete*, and $f(x)$ *incomplete*, integral of $f(x)\,dx$. In both cases, however, the integral is called *indefinite*, for the reason to be given in a following number.

XXI. *General theorems.*

Resuming again $f(x)$ and its differential $f'(x)\,dx$, we shall have besides $df(x) = f'(x)\,dx$ and $\int f'(x)\,dx = \int df(x) = f(x)$, also, representing by a a constant,

$$a\int f'(x)\,dx = a\int df(x) = af(x);$$

but from $df(x) = f'(x)\,dx$ we have also $adf(x) = af'(x)\,dx$ and $adf(x) = daf(x)$; hence

$$\int af'(x)\,dx = \int daf(x) = af(x).$$

Therefore $\int af'(x)\,dx = a\int f'(x)\,dx$; i. e.,

1st. *Constant factors of a given differential to be integrated may be placed outside the sign of integration.*

We know (VII. I.) that $d[(\mathrm{F}(x) + f(x) + \varphi(x)] = d\mathrm{F}(x) + df(x) + d\varphi(x)$; hence also $\int[d\mathrm{F}(x) + df(x) + d\varphi(x)] = \int d[\mathrm{F}(x) + f(x) + \varphi(x)]$; but $\int d[\mathrm{F}(x) + f(x) + \varphi(x)] = \mathrm{F}(x) + f(x) + \varphi(x)$ and $\int d\mathrm{F}(x) = \mathrm{F}(x), \int df(x) = f(x)$, $\int d\varphi(x) = \varphi(x)$, therefore

$$\int[d\mathrm{F}(x) + df(x) + d\varphi(x)] = \int d\mathrm{F}(x) + \int df(x) + \int d\varphi(x);$$

i. e.,

2d. *The integral of the sum of different differentials is equal to the sum of the integrals of each term.* Thus (V. II., and VI. III. and I.)

$$\int\left(adx - b\cos xdx + c\frac{dx}{x}\right) = ax - b\sin x + c\log(x).$$

We have (V. IV.) $dx^{m+1} = (m+1)\,x^m dx$. Therefore $\int(m+1)\,x^m dx = \int dx^{m+1} = x^{m+1}$. Now $\int(m+1)\,x^m dx = (m+1)\int x^m dx$, therefore

$$\int x^m dx = \frac{x^{m+1}}{m+1};\text{ i. e.,}$$

3d. *The integral of* $\mathrm{x^m dx}$ *is obtained by suppressing* dx, *adding* 1 *to the exponent* m, *and dividing* $\mathrm{x^{m+1}}$ *by* $\mathrm{m} + 1$. Thus, for example,

$$\int 3x^2dx = 3\int x^2dx = x^3,$$

$$\int \frac{adx}{x^2} = a \int x^{-2}dx = -\frac{a}{x},$$

$$\int \sqrt[3]{x^2 \cdot dx} = \int x^{\frac{2}{3}} dx = \frac{3}{5} \sqrt[3]{x^5}.$$

XXII. *Immediate integration; integration by substitution; integration by parts.*

These different methods of integration are used, now one, now another, according as circumstances may suggest.

I. When the given differential is such as to show immediately the function from which it has been obtained, as in the cases examined, Nos. V. and VI., the integration is obtained immediately without having recourse to any rule, and on this account, the integration is called *immediate*. Thus, for example, we know (VI. IV.) that $\sin x dx$ is the differential of $-\cos x$; hence we conclude immediately

$$\int \sin x dx = -\cos x.$$

In like manner (VI. II.) we obtain immediately

$$\int a^x l(a)\, dx = a^x;$$

and since $a^x dx = \frac{1}{l(a)} a^x l(a)\, dx$, also

$$\int a^x dx = \frac{1}{l(a)} a^x.$$

Also (VI. VII., VIII., IX.)

$$\int \frac{dx}{\sqrt{1-x^2}} = \text{arc}(\sin = x), \quad \int -\frac{dx}{\sqrt{1-x^2}} = \text{arc}(\cos = x),$$

$$\int \frac{dx}{1+x^2} = \text{arc}(\text{tg} = x).$$

II. When the given differential does not show from its form the function from which it has been obtained, it is then by means of substitution so modified as to take one of the known forms, and the integral is then found. This method is accordingly called method *by substitution*. Let, for example, $x(\sqrt{a^2 - x^2})\, dx$ be a given differential. Make $a^2 - x^2 = z$, and consequently $dx = -\frac{dz}{2x}$. Thus $\int x\,(\sqrt{a^2 - x^2})\,dx =$

$-\int x\,(\sqrt{z})\frac{dz}{2x} = -\frac{1}{2}\int z^{\frac{1}{2}}dz = -\frac{1}{2}\frac{z^{\frac{3}{2}}}{\frac{3}{2}} = \frac{z^{\frac{3}{2}}}{3}$. Therefore

$$\int x\,(\sqrt{a^2 - x^2})\,dx = \tfrac{1}{3}\,(a^2 - x^2)^{\frac{3}{2}}.$$

But let $\pm\frac{dx}{(\sqrt{a^2 - x^2})}$ be the given differential. Make $\frac{x}{a} = z$; i. e., $x = az$, consequently $dx = adz$, we shall have $\int \pm\frac{dx}{\sqrt{a^2 - x^2}}$ $= \int \pm\frac{adz}{\sqrt{a^2 - a^2z^2}} = \int \pm\frac{dz}{\sqrt{1 - z^2}}$. Now (VI. VII.) $+\frac{dz}{\sqrt{1 - z^2}}$ $= d\,\text{arc}\,(\sin = z)$ and $-\frac{dz}{\sqrt{1 - z^2}} = d\,\text{arc}\,(\cos = z)$. Therefore

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{arc}\left(\sin = \frac{x}{a}\right),$$

$$\int -\frac{dx}{\sqrt{a^2 - x^2}} = \text{arc}\left(\cos = \frac{x}{a}\right).$$

Let finally the given differentials be $\cos x \sin^2 x dx$ and $\sin x \cos^2 x dx$. Make for the first of them $\sin x = z$, and consequently $\cos x dx = dz$, for the second $\cos x = z$, and consequently $-\sin x dx = dz$; we shall have

$$\int \cos x \sin^2 x dx = \int z^2 dz = \tfrac{1}{3}z^3 = \frac{\sin^3 x}{3},$$

$$\int \sin x \cos^2 x dx = \int - z^2 dz = -\tfrac{1}{3}z^3 = -\frac{\cos^3 x}{3}.$$

Changing in these formulas x into $\frac{1}{2}x$,

$$\int \cos\tfrac{1}{2}x \sin^2\tfrac{1}{2}x\, d\tfrac{1}{2}x = \frac{\sin^3\frac{1}{2}x}{3},\ \int \sin\tfrac{1}{2}x \cos^2\tfrac{1}{2}x\, d\tfrac{1}{2}x = -\frac{\cos^3\frac{1}{2}x}{3}.$$

But $d\frac{1}{2}x = \frac{1}{2}dx$, and the constant coefficient $\frac{1}{2}$ being brought out of the integral sign, the same equations will be changed into the following.

$$\int \cos\tfrac{1}{2}x \sin^2\tfrac{1}{2}x dx = \tfrac{2}{3}\sin^3\tfrac{1}{2}x,\ \int \sin\tfrac{1}{2}x \cos^2\tfrac{1}{2}x dx = -\tfrac{2}{3}\cos^3\tfrac{1}{2}x.$$

III. Integration by parts is effected when, instead of integrating the given function, the same is resolved into two parts, each one of which is integrated separately. This method is based on a formula which we establish as follows: Representing by y and z two functions of the same variable x, we have (VII. II.) $d(y \cdot z) = zdy + ydz$; hence

$$ydz = d(y \cdot z) - zdy;$$

i. e., the product ydz of a finite by a differential function of x is resolvable into two differential terms, the integration of which, if more conveniently obtained, may be taken instead of that of ydz, to which it is equal. Now the last equation gives us

$$\int ydz = y \cdot z - \int zdy; \text{ i. e.,}$$

The integral of the product of two functions of the same variable, one finite and the other differential, is obtained by taking from the product of the first function by the integral of the second, the integral of the product of the second integrated by the differential of the first.

Let, for example, $\sqrt{a^2 - x^2} \times dx$ or its equal $\left(\sqrt{\frac{a^2}{x^2} - 1}\right) xdx$ be a given function to be integrated; we shall have

$$\int \sqrt{a^2 - x^2} \times dx = \int \left(\sqrt{\frac{a^2}{x^2} - 1}\right) xdx = \text{— (V. IV.)}$$

$$\int \left(\sqrt{\frac{a^2}{x^2} - 1}\right) d\frac{x^2}{2}.$$

The latter member is represented by the above general formula, hence

$$\int \left(\sqrt{\frac{a^2}{x^2} - 1}\right) d\frac{x^2}{2} = \left(\sqrt{\frac{a^2}{x^2} - 1}\right) \frac{x^2}{2} - \int \frac{x^2}{2} d\left(\frac{a^2}{x^2} - 1\right)^{\frac{1}{2}}$$

$$= \left(\sqrt{\frac{a^2}{x^2} - 1}\right) \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{2} \left(\frac{a^2}{x^2} - 1\right)^{-\frac{1}{2}} d\frac{a^2}{x^2}.$$

Now (V. III.) $d\frac{a^2}{x^2} = -\frac{a^2}{x^4} dx^2 = -\frac{2a^2x}{x^4} dx = -\frac{2a^2dx}{x^3}$. Hence

$$\int\left(\sqrt{\frac{a^2}{x^2}-1}\right)d\frac{x^2}{2}=\left(\sqrt{\frac{a^2}{x^2}-1}\right)\frac{x^2}{2}+\int\frac{a^2dx}{2x\sqrt{\frac{a^2}{x^2}-1}}$$

$$=\tfrac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2-x^2}};$$

but we have found above $\int\frac{dx}{\sqrt{a^2-x^2}}=\text{arc}\left(\sin=\frac{x}{a}\right)$, therefore

$$\int(\sqrt{a^2-x^2})\,dx=\tfrac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\,\text{arc}\left(\sin=\frac{x}{a}\right)\right].$$

Let, for another example, $(a^2-x^2)^{\frac{3}{2}}dx$ be the given function to be integrated. Observe first that $(a^2-x^2)^{\frac{3}{2}}=(a^2-x^2)\times(\sqrt{a^2-x^2})=a^2\sqrt{(a^2-x^2)}-x^2\sqrt{a^2-x^2}$, also $x^2\sqrt{a^2-x^2}=x\times x\sqrt{a^2-x^2}$; hence

$$(a^2-x^2)^{\frac{3}{2}}dx=a^2\sqrt{(a^2-x^2)}\,dx-x\times x\sqrt{(a^2-x^2)}\,dx.$$

Now $x\sqrt{(a^2-x^2)}\,dx=-d\tfrac{1}{3}(a^2-x^2)^{\frac{3}{2}}$; therefore

$$\int(a^2-x^2)^{\frac{3}{2}}dx=a^2\int\sqrt{(a^2-x^2)}\,dx+\tfrac{1}{3}\int x\cdot d(a^2-x^2)^{\frac{3}{2}};$$

the last integral of this equation is again represented by $\int ydx$, and consequently $=x(a^2-x^2)^{\frac{3}{2}}-\int(a^2-x^2)^{\frac{3}{2}}dx$. Therefore,

$$\int(a^2-x^2)^{\frac{3}{2}}dx=a^2\int\sqrt{(a^2-x^2)}\,dx+\tfrac{1}{3}x(a^2-x^2)^{\frac{3}{2}}-$$
$$\tfrac{1}{3}\int(a^2-x^2)^{\frac{3}{2}}dx,$$

from which, taking the common factor $\int(a^2-x^2)^{\frac{3}{2}}dx$ of the first and last term, alone in the first member,

$$\int(a^2-x^2)^{\frac{3}{2}}dx=\tfrac{3}{4}a^2\int\sqrt{(a^2-x^2)}\,dx+\tfrac{1}{4}x(a^2-x^2)^{\frac{3}{2}};$$

but, from the preceding example,

$$\int\sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}\left[x\sqrt{(a^2-x^2)}+a^2\,\text{arc}\cdot\sin\left(=\frac{x}{a}\right)\right];$$

hence, finally,

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$$\int (a^2 - x^2)^{\frac{3}{2}} dx = \tfrac{3}{8}a^2x \sqrt{(a^2 - x^2)} + \tfrac{3}{8}a^4 \text{ arc} \cdot \sin \left(= \frac{x}{a}\right)$$
$$+ \tfrac{1}{4} x (a^2 - x^2)^{\frac{3}{2}}.$$

The same method of integration by parts is applicable immediately to the differentials $x \cdot \sin xdx$, $x \cdot \cos xdx$; i. e., observing that (VI. III. and IV.) $\sin xdx = d - \cos x$, $\cos xdx = d \sin x$,

$$\text{(1st.)} \begin{cases} \int x \cdot \sin xdx = \int x \cdot d - \cos x = - x \cos x - \\ \qquad \int - \cos xdx = \sin x - x \cos x, \\ \int x \cdot \cos xdx = \int x \cdot d \sin x = x \sin x - \int \sin xdx \\ \qquad = \cos x + x \sin x. \end{cases}$$

From these two equations we infer the following:

$$\int 2x \cdot \sin 2xd2x = \sin 2x - 2x \cos 2x, \int \tfrac{1}{2}x \sin \tfrac{1}{2}xd\tfrac{1}{2}x = \sin \tfrac{1}{2}x$$
$$- \tfrac{1}{2}x \cos \tfrac{1}{2}x,$$
$$\int 2x \cdot \cos 2xd2x = \cos 2x + 2x \sin 2x, \int \tfrac{1}{2}x \cos \tfrac{1}{2}xd\tfrac{1}{2}x = \cos \tfrac{1}{2}x$$
$$+ \tfrac{1}{2}x \sin \tfrac{1}{2}x,$$

which, bringing out of the differential and integral signs the constant coefficients 2 and $\frac{1}{2}$, are easily changed into

$$\text{(2d.)} \begin{cases} \int x \sin 2xdx = \tfrac{1}{4} \sin 2x - \tfrac{1}{2}x \cos 2x, \\ \int x \cos 2xdx = \tfrac{1}{4} \cos 2x + \tfrac{1}{2}x \sin 2x, \\ \int x \sin \tfrac{1}{2}xdx = 4 \sin \tfrac{1}{2}x - 2x \cos \tfrac{1}{2}x, \\ \int x \cos \tfrac{1}{2}xdx = 4 \cos \tfrac{1}{2}x + 2x \sin \tfrac{1}{2}x. \end{cases}$$

Let, for a last example, $\sin^3 xdx$, $\cos^3 xdx$ be the differentials to be integrated. First, they may be resolved into two factors as follows: $\sin^2 x \cdot \sin xdx$, $\cos^2 x \cdot \cos xdx$; but $\sin xdx = d - \cos x$, and $\cos xdx = d \sin x$. Thus

$$\int \sin^3 xdx = \int \sin^2 x \cdot d - \cos x = - \sin^2 x \cos x - \int - \cos x \, d \sin^2 x,$$
$$= - \sin^2 x \cos x + 2 \int \cos x \sin x$$
$$d \sin x),$$
$$= - \sin^2 x \cos x + 2 \int \cos^2 x \sin xdx,$$
$$= - \sin^2 x \cos x +$$
$$2 \int (1 - \sin^2 x) \sin xdx.$$

Now $\int (1 - \sin^2 x) \sin xdx = \int (\sin xdx - \sin^3 xdx) =$

$-\cos x - \int \sin^3 x dx$; therefore $\int \sin^3 x dx = -\sin^2 x \cos x - 2\cos x - 2\int \sin^3 x dx$. Hence

$$\int \sin^3 x dx = -\tfrac{1}{3}\sin^2 x \cos x - \tfrac{2}{3}\cos x.$$

In like manner we shall have

$$\int \cos^3 x dx = \int \cos^2 x \cdot d \sin x = \cos^2 x \sin x - \int \sin x \cdot d \cos^2 x,$$

and with a process altogether like the preceding we find

$$\int \cos^3 x dx = \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x.$$

These and the preceding integrals become completed by adding any arbitrary constant, C, to them.

XXIII. *Definite or limited integrals.*

Let C be any arbitrary constant and $y = f(x) + C$ be any continual function of x, we shall have $dy = f'(x) dx$. Now $f(x)$ may be taken, as it is in reality, for the sum of infinitesimal elements dy as many in number as there are infinitesimal elements dx in the x of $f(x)$. Calling now x_0 the particular value of x which makes $f(x) = -C$, we shall have $f(x_0) = y_0 - = C$, and consequently

$$y = f(x) - f(x_0),$$

in which y, which is the integral of $f'(x) dx$, represents the sum of as many infinitesimal elements dy as are the infinitesimals dx contained in the difference $x - x_0$; for the sum of those elements which would make $f(x) = f(x_0)$ is destroyed by $f(x_0)$. But so long as x has no fixed value, y or $\int f'(x) dx$ remains undetermined. It will become *determined* when x receives a particular value, for instance, x_m. Now to designate that the y corresponding to the integral of $f'(x) dx$ represents the sum corresponding to $x - x_0$ or to $x_m - x_0$, we affix to the sign $\int$ the two values of the variable as follows: $\int_{x_0}^{x}$ or $\int_{x_0}^{x_m}$, and we consequently write:

$$\text{(I.)} \int_{x_0}^{x} f'(x) dx = f(x) - f(x_0),$$

$$\text{(II.)} \int_{x_0}^{x_m} f'(x) dx = f(x_m) - f(x_0),$$

in both of which we must remember $f(x_0)$ replaces any arbitrary constant C.

Further to illustrate what has been said above, make $x_0 + dx = x_,$, $x_, + dx = x_2$, $x_2 + dx = x_3$, ... $x_{m-1} + dx = x_m$, we shall have

$$f(x_0 + dx) - f(x_0) = f'(x_0)\,dx = dy_0 = f(x_,) - f(x_0),$$
$$f(x_, + dx) - f(x_,) = f'(x_,)\,dx = dy_, = f(x_2) - f(x_,),$$
$$f(x_2 + dx) - f(x_2) = f'(x_2)\,dx = dy_2 = f(x_3) - f(x_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(x_{m-1} + dx) - f(x_{m-1}) = f'(x_{m-1})\,dx = dy_{m-1} = f(x_m) - f(x_{m-1});$$

hence

$$dy_0 + dy_, + \ldots + dy_{m-1} = f(x_m) - f(x_0);$$

but $dy_0 + dy_, + \ldots + dy_{m-1}$ is the sum of all the elements of y from $y = y_0$ to $y = y_m$, corresponding to those of x from $x = x_0$ to $x = x_m$; y_0, as stated above, is $= 0$; hence the sum of all the elements is y_m or the difference $f(x_m) - f(x_0)$. Now $y_m = \int f'(x_m)\,dx$, which is represented by $\int_{x_0}^{x_m} f(x)\,dx$; therefore the equation (II.), in which x_m represents any determined value of x, which changed into another x_n will give $\int_{x_0}^{x_n} f'(x)\,dx = f(x_n) - f(x_0)$, and consequently

$$\int_{x_0}^{x_m} f'(x)\,dx - \int_{x_0}^{x_n} f'(x)\,dx = f(x_m) - f(x_n).$$

But $f(x_m) = y_m$, $f(x_n) = y_n$; hence $f(x_m) - f(x_n) = y_m - y_n$, which contains all the infinitesimal elements dy corresponding to those of x in the difference $x_m - x_n$. Hence $f(x_m) - f(x_n)$ is the limited integral of $f'(x)\,dx$ between x_m and x_n, and accordingly represented by $\int_{x_n}^{x_m} f'(x)\,dx$; i. e.,

$$\text{(III.)} \quad \int_{x_n}^{x_m} f'(x)\,dx = f(x_m) - f(x_n).$$

We infer, therefore, from what precedes:

1st. *The integral of a differential expression* F (x) dx *taken* *between two determined limits, is the sum of all the values which* F (x) dx *receives by the infinitesimal variations of* x *from one limit to the other.*

2d. *The integral of the same expression is given by the difference of the values which its indefinite integral, abstraction being made from the constant, receives when we substitute in it the values of* x *corresponding to the two limits.*

To illustrate these theorems, let, for example, RR′ (Fig. 26) be a parallel to the axis of abscissas OX of the orthogonal system YOX, and consequently perpendicular to OY. The ordinates of the different points of RR′ are necessarily all equal to the constant segment OR, which we shall call h; y therefore does not vary with x, but the area limited by OR, RR′, OX and the ordinate of a point of RR′, varies with the abscissa of that point. This area, therefore, is a function of x, which we may represent by y. Now the same area is the product of the constant ordinate h by the variable abscissa. Hence

$$y = hx;$$

and since, in our present supposition with $x = 0$, y also is equal to zero, no constant is added to hx, or the constant, in this case, is zero.

Let now KK′ $= dx$, we shall have $dy =$ KM′, and the integral of KM′ ($= hdx$) is the area KR. But if in the equation $y = hx$, instead of $x =$ OK we would take $x =$ OB, the integral of hdx would be the area BR; in each case, however, the area is evidently equal to the sum of as many infinitesimal areas KM′ $= dy$ as there are infinitesimal elements dx in $x =$ OK or $=$ OB. Now OK and OB are any two abscissas, and to distinguish one from the other we may call the first x_m and the second x_n. $x_m - x_n$ is then the segment BK of the axis of abscissas between the ordinates y_m, y_n, and the area KA or $(x_m - x_n)h$ is $= x_m h - x_n h$, which is the sum of as many elements dy as there are infinitesimal elements dx in the segment $x_m - x_n$ = BK, i. e. the integral of KM′ no less than KR, with this difference, that the latter area is taken from $x = 0$ to $x = n$, and accordingly designated by $\int_0^{x_m} hdx$; the former is taken from $x = x_n$ to $x = x_m$, and consequently expressed by $\int_{x_n}^{x_m} hdx$; and as in the first case $\int_0^{x_m} hdx = hx_m$, so in the second

$$\int_{x_n}^{x_m} hdx = hx_m - hx_n,$$

which corresponds exactly with the preceding theorems.

We may now proceed to apply the last theorem to some particular functions. But first let us observe that an altogether indefinite integral may be expressed by the integral limited by one term. For taking the integral of $f'(x)\,dx$ from x_0 to x, we have $\int_{x_0}^{x} f'(x)\,dx = f(x) - f(x_0)$ and consequently

$$d\int_{x_0}^{x} f'(x)\,dx = d f(x) - d f(x_0) = f'(x)\,dx,$$

and

$$\int f'(x)\,dx = \int \cdot\, d\int_{x_0}^{x} f'(x)\,dx = \int_{x_0}^{x} f'(x)\,dx\,;$$

but

$$\int f'(x)\,dx = f(x) + \text{C and} \int_{x_0}^{x} f'(x)\,dx = f(x) - f(x_0),$$

in which $-f(x_0)$ represents any arbitrary quantity C; hence

$$\int f'(x)\,dx = \int_{x_0}^{x} f'(x)\,dx,$$

x_0 being that particular value of x which renders $f(x) = -$ C.

Coming now to the applications, let

$$\frac{1}{x}\,dx,\ \frac{1}{a^2 + x^2}\,dx,\ x^m\,dx$$

be the given differential functions. $\frac{1}{x}\,dx$ or $\frac{dx}{x}$ is (VI. I.) the differential of $l(x)$, therefore the indefinite integral of $\frac{1}{x}\,dx$ is $l(x) + \mathrm{C}$; hence the integral of the same function from $x = x_0$ to any value of x of this variable is

$$\int_{x_0}^{x} \frac{1}{x}\,dx = \log(x) - \log(x_0).$$

The second function, or its equal $\frac{dx}{a^2 - x^2}$, is (VI. IX.) the differential of $\frac{1}{a}$ arc $\left(\text{tg } = \frac{x}{a}\right)$. Therefore its indefinite integral is $\frac{1}{a}$ arc $\left(\text{tg} = \frac{x}{a}\right) + \mathrm{C}$; hence the integral of the same function from $x = 0$ to $x = a$ is

$$\int_0^a \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\text{ arc (tg} = 1) - \frac{1}{a}\text{ arc (tg} = 0).$$

Now arc (tg = 1) = 45° $= \frac{\pi}{4}$, arc (tg = 0) = 0; therefore

$$\int_0^a \frac{1}{a^2 - x^2}\,dx = \frac{\pi}{4a}.$$

The last function is (V. IV.) the differential of $\frac{x^{m+1}}{m+1}$; hence its indefinite integral is $\frac{x^{m+1}}{m+1} + \mathrm{C}$, and consequently its definite integral from $x = 0$ to $x = 1$ is

$$\int_0^1 x^m dx = \frac{1}{m+1}.$$

XXIV. *Differentials of an arc, of an area terminated by an arc, of a sector, and their corresponding integrals.*

I. Let (Fig. 27) the plane curve CC′ be referred to the orthogonal axes OX, OY, and let σ be an arc of it taken from A, a determined point whose co-ordinates are x_0, y_0, to another point M variable and whose co-ordinates are any two x and y. Let also $y = f(x)$ be the equation of the curve and KK′ an infinitesimal increment dx of x. Drawing the ordinate K′M′, and from M, MD parallel to OX, the sides MD, DM′ of the infinitesimal right-angled triangle MM′D will respectively be equal to dx and dy. The infinitesimal arc MM′ $= d\sigma$, which may be regarded as coinciding with the chord, will be equal to $\sqrt{dx^2 + dy^2} = \left(\sqrt{1 + \frac{dy^2}{dx^2}}\right) dx$, and since $\frac{dy}{dx} = f'(x)$, the differential of the arc σ is

$$d\sigma = (\sqrt{1 + f'^2(x)})\, dx,$$

and consequently the integral

$$\int_{x_0}^{x} (\sqrt{1 + f'^2(x)})\, dx = \sigma$$

taken from the abscissa $x = x_0$ to any other x, gives the rectilinear measure of σ for any plane curve; hence the last formula answers the purpose of rectifying plane curves.

Let, for example, AM (Fig. 28) be the cycloidal arc σ taken from the vertex where is the origin of the axes, to the point M whose co-ordinates are x and y. The equation of the cycloid referred to the rectangular axes and as already determined (XXVII., A. G.) is $y = c \cdot \text{arc}\left(\sin = \frac{\sqrt{2cx - x^2}}{c}\right) + \sqrt{2cx - x^2}$, in which $2c$ represents the diameter aa' of the generating circle. Taking the differential of this equation, and to simplify the operation make $\sqrt{2cx - x^2} = z$, we shall have, first, $y = c \cdot \text{arc}\left(\sin = \frac{z}{c}\right) + z$. Therefore (VI.vii.) $dy = c \frac{d\frac{z}{c}}{\sqrt{1 - \frac{z^2}{c^2}}} + dz = \left(\frac{c}{\sqrt{c^2 - z^2}} + 1\right) dz$. But $z = \sqrt{2cx - x^2}$; hence $z^2 = 2cx - x^2$ and $dz = \frac{(c - x)\, dx}{\sqrt{2cx - x^2}}$. Therefore

$$dy = \left(\frac{c}{\sqrt{c^2 - 2cx + x^2}} + 1\right) \frac{(c - x)}{\sqrt{2cx - x^2}}\, dx$$

$$= \left(\frac{c}{c-x}+1\right)\frac{(c-x)}{\sqrt{2cx-x^2}}\,dx$$

$$= \frac{2c-x}{\sqrt{2cx-x^2}}\,dx = \frac{2c-x}{\sqrt{x}\sqrt{2c-x}}\,dx\,;$$

finally,

$$dy = \sqrt{\frac{2c-x}{x}}\,dx,$$

and consequently $f'(x)$ or

$$\frac{dy}{dx} = \sqrt{\frac{2c-x}{x}}.$$

Hence for the cycloid the arc σ is given by

$$\int_{x_0}^{x}\left(\sqrt{1+f'^2(x)}\right)dx = \int_{x_0}^{x}\left(\sqrt{1+\frac{2c-x}{x}}\right)dx =$$

$$\int_{x_0}^{x}\sqrt{\frac{2c}{x}}\,.\,dx.$$

But $\sqrt{\frac{2c}{x}}\,dx = \sqrt{2c}\cdot\frac{dx}{\sqrt{x}}$, and $\frac{dx}{\sqrt{x}} = d2\sqrt{x}$; hence

$$\int\sqrt{\frac{2c}{x}}\cdot dx = \int\sqrt{2c}\cdot 2d\sqrt{x} = 2\sqrt{2c}\int d\cdot\sqrt{x} = 2\sqrt{2c\cdot x}\,;$$

therefore, from the preceding number,

$$\int_{x_0}^{x}\sqrt{\frac{2c}{x}}\,dx = 2\sqrt{2c\cdot x} - 2\sqrt{2c\cdot x_0}.$$

Taking $x_0 = 0$ or the abscissa of the origin A of the axes, and for x the abscissa AK corresponding to the point M, so that $\sigma =$ AM, we shall obtain

$$AM = 2\sqrt{2\cdot c\cdot AK}.$$

Now $2c$ is the diameter of the generating circle, and the chord AC which joins the extremity A of the diameter with the point C of the circle met by the ordinate KM, is mean geometrical proportional between the diameter and AK; therefore $\sqrt{2\cdot c\cdot AK} = AC$; hence the arc AM of the cycloid is the double of the chord of the generating circle joining the origin of the axes with the point met by the ordinate of M. But if AK should become equal to the diameter $2c$, we would obtain $AB = 4c$; i. e., the rectilinear length of the cycloidal arc from the origin of the axes to the base is twice the diameter of the generating circle; hence the whole length of the cycloidal line is four times the same diameter.

II. Let, secondly, α be the area of ABKM (Fig. 29), terminated by an arc σ of the plane curve CC′, by the ordinates y_0, y of the extreme points of the same arc, and by the difference $x - x_0$ of the corresponding abscissas. Taking, as before, KK′ $= dx$, the infinitesimal area between the ordinates of M and M′, dx and $d\sigma$, is the differential of α, and consequently MKK′M′$=$ $d\alpha$, MM′, as infinitesimal, coincides with the chord; hence $d\alpha$ may be regarded as a trapezoid whose height is dx, and y, $y + dy$ the parallel bases, therefore $d\alpha = \frac{1}{2}(y + y + dy)\,dx = ydx + \frac{1}{2}dydx$. Neglecting the second term as an infinitesimal of the second order, we shall have

$$d\alpha = ydx,$$

and representing as usually by $y = f(x)$ the equation of the curve referred to the orthogonal axes OX · OY, also

$$d\alpha = f(x)\,dx.$$

Now the integral of this function, taken from x_0 to x, gives us α; i. e.,

$$\alpha = \int_{x_0}^{x} f(x)\,dx.$$

Let, for example, the given curve be the ellipse referred to the axes of the curve, and whose equation is (XII., A. G.) $y = \frac{b}{a}\sqrt{a^2 - x^2}$, we shall have

$$\alpha = \int_{x_0}^{x} \frac{b}{a}\sqrt{a^2 - x^2} \cdot dx = \frac{b}{a}\int_{x_0}^{x} \sqrt{a^2 - x^2}\,dx.$$

Now (XXII. III.)

$$\int(\sqrt{a^2 - x^2}) \cdot dx = \tfrac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \cdot \text{arc}\left(\sin = \frac{x}{a}\right)\right].$$

Taking $x_0 = 0$ this integral becomes zero, and taking $x = a$ it becomes $\frac{1}{2}a^2 \cdot \frac{\pi}{2}$; hence

$$\alpha = \int_{0}^{a} \frac{b}{a}\sqrt{a^2 - x^2} = \tfrac{1}{4}\pi ab.$$

But the area between the limits $x = 0$, $x = a$, is the fourth part of the area of the ellipse; hence, calling A the area of the whole ellipse, πab will represent this area; or, since (XI. (2), A. G.) $ab = a^2\sqrt{1 - e^2}$,

$$A = \pi ab = \pi a^2\sqrt{1 - e^2}.$$

Let another example be taken from the cycloid, which (XXVII., A. G.) is represented by the equations

$$x = c(1 - \cos\omega), \quad y = c(\omega + \sin\omega);$$

from which $dx = c\sin\omega d\omega$ and $ydx = c^2\omega\sin\omega d\omega + c^2\sin^2\omega d\omega$ $=$ (Trig. § 18 (g′)) $c^2\omega\sin\omega d\omega + c^2\left(\frac{1-\cos 2\omega}{2}\right)d\omega$. But ydx is the differential $d\alpha$ of the area; hence the area of the semicycloid, which corresponds to the definite integral of ydx taken from $x = 0$ to $x = 2c$, is given by

$$\alpha = \int_0^{2c} ydx = \int_0^{2c}\left[c^2\omega\sin\omega d\omega + \frac{c^2}{2}d\omega - \frac{c^2\cos 2\omega}{2}d\omega\right];$$

but (XXI. II.) the last member is resolvable as follows:

$$c^2\int_0^{2c}\omega\sin\omega d\omega + \frac{c^2}{2}\int_0^{2c}d\omega - \frac{c^2}{2}\int_0^{2c}\cos 2\omega d\omega.$$

Now with $x = 0$ also $\omega = 0$; with $x = 2c$, $\omega = \pi$; hence

$$\int_0^{2c}\omega\sin\omega d\omega = \int_0^{\pi}\omega\sin\omega d\omega = \text{(XXII. III. 1st.)}\ \pi:$$

$$\int_0^{2c}d\omega = \int_0^{\pi}d\omega = \pi, \int_0^{2c}\cos 2\omega d\omega = \int_0^{\pi}\tfrac{1}{2}\cos 2\omega d\,2\omega =$$

$$\tfrac{1}{2}\int_0^{\pi}\cos 2\omega\, d\,2\omega = \text{(VI. III.)}\ \tfrac{1}{2}\sin 2\pi = 0;$$

therefore

$$\alpha = c^2(\pi + \tfrac{1}{2}\pi) = \frac{3}{2}\pi c^2.$$

Now α represents the area of the semicycloid. The area, therefore, of the whole cycloid is

$$2\alpha = 3\pi c^2;$$

i. e., three times the area of the generating circle.

If the axes, instead of being orthogonal, form any angle θ, and the area AEHM $= \alpha$ be terminated by the oblique co-ordinates AE, MH ($= y$); the infinitesimal increment MHH′M′ $= d\alpha$, corresponding to the infinitesimal increment dx of the abscissa OH, may be considered as a parallelogram, neglecting the triangle MM′D as an infinitesimal of the second order. Now the area of this parallelogram is HH′ · MK; and HH′ $= dx$, MK = MH · $\sin\theta = y \cdot \sin\theta$; therefore

$$d\alpha = y \cdot \sin\theta \cdot dx.$$

Thus
$$\alpha = \sin\theta \int_{x_0}^{x} y dx;$$

which is a formula more general than the preceding to obtain the area of a plane surface terminated by a curved line.

III. Let O (Fig. 30) be the origin of the axes OX, OY, to which the plane curve CC′ is referred, and the pole of the polar co-ordinates, having OX for polar axis. Taking A for an invariable point, let x_0, y_0 be the rectilinear co-ordinates OB, BA of that point, and ρ_0, ω_0 the polar co-ordinates OA and AOX of the same point. Let also x, y, and ρ, ω be the rectilinear and polar co-ordinates of the movable point M. The curvilinear sector AOM, which we shall represent by α', increases by diminishing ω. The same sector is also equal to AOB + ABKM — OMK $= \frac{x_0 y_0}{2} + \alpha - \frac{xy}{2}$; hence (VII. I. and II.) $d\alpha' = d\frac{x_0 y_0}{2} + d\alpha - d\frac{xy}{2} = d\alpha - \frac{ydx + xdy}{2}$. But $d\alpha = ydx$; hence $d\alpha' = \frac{ydx - xdy}{2}$. But if we suppose α and ω to increase together, taking namely the variations in a retrograde order, the signs then of dx and dy will be changed, thus

$$d\alpha' = \pm \frac{ydx - xdy}{2};$$

in which the upper sign is taken in the first, and the lower sign in the second supposition.

To express the same differential by means of polar co-ordinates, let us take the well-known formulas $x = \rho \cos \omega$, $y = \rho \sin \omega$, from which $dx = \cos \omega \, d\rho - \rho \sin \omega d\omega$, $dy = \sin \omega d\rho + \rho \cos \omega d\omega$. Hence

$$xdy = \rho \cos \omega \sin \omega \, d\rho + \rho^2 \cos^2 \omega d\omega,$$

$$ydx = \rho \sin \omega \cos \omega d\rho - \rho^2 \sin^2 \omega d\omega,$$

therefore

$$ydx - xdy = - \rho^2 (\sin^2 \omega + \cos^2 \omega) \, d\omega = - \rho^2 \, d\omega;$$

hence

$$d\alpha' = \pm \tfrac{1}{2}\rho^2 \, d\omega;$$

in which the upper sign is taken when α and ω increase or diminish together. The area of the sector or α' is therefore given by the second, or by the third member of the following equation:

$$\alpha' = \pm \tfrac{1}{2} \int_{x_0}^{x} ydx - xdy = \pm \tfrac{1}{2} \int_{\omega_0}^{\omega} \rho^2 \, d\omega.$$

Let, for example, the circle be the curve in which the sector is taken from $\omega = 0°$, to $\omega = 360°$; ρ in this case becomes a constant if the pole be taken in the centre, as we suppose it to be, and equal to the radius r; hence

$$\alpha' = \tfrac{1}{2} r^2 \int_{0}^{2\pi} d\omega = \pi r^2,$$

as we know from geometry.

XXV. *Circular curvature; osculatory circle and radius of curvature of a plane curve.*

Representing by r the radius of a circle in contact with a straight line, the same circle approaches to, or recedes from, the tangent according as the radius r increases or decreases. In other words, the curvature of the circle varies with the radius, but reciprocally; it increases, namely, or decreases with the ratio $\frac{1}{r}$, which ratio may consequently be taken to represent the curvature of the circle having r for radius. The curvature of the circle being the same everywhere, it will be represented everywhere by $\frac{1}{r}$. Now any plane curve may be considered as composed of infinitesimal circular elements, and the circle corresponding to each element is called the *osculatory* circle of that element; and consequently, r being the radius of the circle, $\frac{1}{r}$ gives the measure of the curvature of the same element. But the curvature of plane curves different from the circle is different at different points, consequently $\frac{1}{r}$ is a variable quantity for these curves.

Let (Fig. 31) CAC′ be any plane curve whose equation is $y = f(x)$, and let MAM′ be one of its infinitesimal elements bisected in A. To determine the radius of the osculatory circle of this element, observe, first, that the tangent of the curve corresponding to the middle point of AM coincides with the element AM; and the tangent corresponding to the middle point of AM′ coincides with the element AM′. Let Tt, T′t' be the two tangents, and m, m' the points of contact. Call (tx), $(t'x)$ the angles which the same tangents form with the axis of abscissas, and (tt') the infinitesimal angle which they form one with the other. Now the perpendiculars mD, m'D to the tangents meet in the centre of the osculatory circle, corresponding to MAM′, and MD is consequently the radius of this circle. Representing by $d\sigma$, AM = AM′ = mm', the quadrilateral mDm'A gives us

$$m\text{D}m' = 180° - m\text{A}m' = (tt').$$

Therefore, taking the arc corresponding to (tt'), in the circle having 1 for radius, and calling the arc also (tt'), we have

$$(tt') : d\sigma :: 1 : r;$$

hence
$$r = \frac{d\sigma}{(tt')}.$$

But $(tt') = (tx) - (t'x) = -[(t'x) - (tx)] = -d\,(tx)$; and since (XV.) $\text{tg}\,(tx) = f'(x)$, $tx = \text{arc}\,(\text{tg} = f'(x))$; also (XXIV. I.) $d\sigma = [1 + f'^2(x)]^{\frac{1}{2}}\,dx$; therefore

$$r = -\frac{[1 + f'^2(x)]^{\frac{1}{2}}\,dx}{d\,\text{arc}\,(\text{tg} = f'(x))} = (\text{VI. IX.}) - \frac{[1 + f'^2(x)]^{\frac{3}{2}}}{f''(x)}.$$

This equation gives the length of the radius of the osculatory circle, or radius of curvature corresponding to any point (x, y) of any curve, in a form easily applicable to particular cases.

Let us take, for example, the parabola for which we have $f(x) = \sqrt{2px}$, and consequently $f'(x) = \frac{p}{\sqrt{2px}} = \frac{p}{f(x)} = \frac{p}{y}$, $f''(x) = -\frac{p\,f'(x)}{f^2(x)} = -\frac{p^2}{y^3}$. Hence the radius of curvature for the parabola is

$$r = \frac{-\left(1 + \frac{p^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{p^2}{y^3}} = \frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2} = \frac{(2px + p^2)^{\frac{3}{2}}}{p^2}.$$

Now (XV. 1st) $\sqrt{2px + p^2} = n$, which is the normal of the point (x, y) of the parabola. Therefore for the parabola $r = \frac{n^3}{p^2}$; i. e., *The radius of curvature of any point* (x, y) *of the parabola is equal to the cube of the corresponding normal divided by the square of the semiparameter.* Now, p being constant, r varies directly as n^3; and n, which has the minimum value $= p$, when $x = 0$, increases with x indefinitely. Hence in the parabola, the greatest curvature is at the vertex of the curve for which the radius of the osculatory circle is equal to the semiparameter; and the curvature of the branches diminishes continually as the branches recede from the vertex.

XXVI. *Evolutes and involutes.*

Conceive a thread, flexible and inextensible, applied over the curve CDB (Fig. 32) so as perfectly to coincide with it from C to B. If this thread be gradually removed from the curve, so that the removed portion be rectilinear, on the same plane as the curve and tangent to the curve, as DM, for instance; the extreme point M of the thread thus evolved traces out another curve, which is called the *involute*, as the one from which the thread is evolved is called the *evolute*. From the same evolute different involutes may be obtained, taking threads of different lengths, as, for instance, CDL, which, when applied to the curve, goes beyond B to A, BA being tangent to the curve in B, and the involute corresponding to this thread being ALL′A′. Now whatever be the involute obtained by the evolution of the thread, the centres of the osculatory circles of the involute are all along the evolute, each and all of whose points are centres of these circles. Let, in fact, the extremity M or L describe, by the evolution of the thread, the infinitesimal arc MM′ or LL′. The length of the evolved thread, in passing from the first to the second position, varies only by an infinitesimal quantity; hence the arc LL′ will coincide with that of a circle described by a radius having its centre in D, between the contacts of the first and second tangent, and LD or MD for length. Now what we say of the infinitesimal element LL′ or MM′ is applicable to each and all the other elements of the involutes. Hence all the centres of the radii of curvature of the same curves are along BDC, of which each point is one of them. We come to the same conclusion by a different process. Let AA′ be any portion of curve, whose curvature diminishes from A to A′. Supposing that AA′ turns its concavity toward the axis AX, the tangents of its different points will form angles with AX, constantly diminishing from A to A′. Now the radius of the osculatory circle of each point of the curve is perpendicular to the tangent corresponding to that point; hence the radii of the osculatory circles, corresponding to the points L and L′, will form an angle with each other, and representing by LM, L′M′ these two perpendiculars, they will meet somewhere at a greater distance from L than D, the centre of the osculatory circle corresponding to L, on account of the diminishing curvature of AA′ toward A′. Now if we take LL′ infinitesimal, the prolongation of LD, from D to the point of intersection with the normal from L′, is also infinitesimal; hence the same point is the centre of the osculatory circle of L′. Observe, in fact, that a circle may be described which passes through L and L′, whatever the arc LL′ may be, having its centre somewhere on the prolongation of LD. But when LL′ becomes infinitesimal the prolongation of LD also becomes infinitesimal, the arc of the circle coincides with the element LL′ of the curve, and the point of intersection of the two normals from L and L′ is the centre of this circle, and particularly the centre of the osculatory circle corresponding to L′. Following the same process for succeeding points, we obtain a polygon of infinitesimal sides, the points of concurrence of which are centres of osculatory circles of the different points of the curve AA′. But a polygon of infinitesimal sides coincides with a curve, and each infinitesimal side coincides with a tangent to this curve. Again, this same side, produced, forms the radius of the osculatory circle of the point of the given curve met by it. Hence the centres of the osculatory circles of AA′, which represents any curve different from the circle, form another curve BDC, and the radii are the tangents of this curve taken from the points of contact to the points of AA′ met by them, the points of contact being the centres.

The law with which the curvature varies is different for different curves; hence each curve has its own evolute. Let us see two examples in the evolute of the parabola and in that of the cycloid.

I. We have seen in the preceding paragraph (XXV.) that the radius of the osculatory circle of the vertex of the parabola equals the semiparameter p. Now the axis AX (Fig. 33) of the parabola is perpendicular to the tangent corresponding to the vertex A. Taking, therefore, on the axis, $AB = p$, B is the centre of the osculatory circle of the vertex A, and one of the points of the evolute; the axis AX is besides a tangent of the evolute in B. Taking now any point L of the upper branch, whose co-ordinates are $x = AH$, $y = HL$, let LN be the normal corresponding to that point, and (XXV.) taking $LD = \frac{(2px + p^2)^{\frac{3}{2}}}{p^2}$, D is the centre of the osculatory circle corresponding to L.

To find the equation of the evolute let us refer it to the orthogonal axes having their origin in B, and the axis of abscissas coinciding with the axis AX of the parabola. Representing by $x_{,}$, $y_{,}$ the co-ordinates of the evolute, drawing from D, DH′ perpendicular to AX, we shall have for the point D, $x_{,} = BH'$, $y_{,} = H'D$. Draw from D, DD, parallel to AX, and produce LH until it meets this parallel in $D_{,}$. Let, lastly, N be the point of the axis AX, met by the radius LD. LN is the normal of the point L which (XV. 1st) is equal to $(2px + p^2)^{\frac{1}{2}}$, HN is the subnormal and (*ibi*) $= p$. Now from the similar triangles LHN, LDD, we have LN : LD :: NH : $DD_{,}$, LN : LD :: LH : $LD_{,}$; i. e.,

$$(2px + p^2)^{\frac{1}{2}} : \frac{(2px + p^2)^{\frac{3}{2}}}{p^2} :: p : HH';$$

$$(2px + p^2)^{\frac{1}{2}} : \frac{(2px + p^2)^{\frac{3}{2}}}{p^2} :: (2px)^{\frac{1}{2}} : LD_{,};$$

from which

$$HH' = 2x + p,\ LD_{,} = \sqrt{2px} + \frac{2x}{p}\sqrt{2px};$$

and consequently, since $BH' (= x_{,}) = HH' - HB = HH' + AH - AB$, and $DH' (= y_{,}) = LD_{,} - LH$;

$$x_{,} = 3x,\ y_{,} = \frac{2x}{p}\sqrt{2px}.$$

Substituting in the second of these equations the value of x taken from the first, and squaring the members of the resulting equation, we obtain

$$y_{,}^2 = \frac{8}{27p}x_{,}^3,$$

the equation of the evolute of the parabola of the second order, which is itself called a parabola, but of the third order or cubic. It has two symmetrical branches, one on each side of the axis AX from B toward X, the branches turning their convexity to the axis, and ending in a cusp at B, where AX is a tangent to both branches.

II. We have (XXIV. I.) for the cycloid, $\frac{dy}{dx} = \sqrt{\frac{2c - x}{x}}$, taking the origin of the orthogonal axes in the vertex, or extremity, A of the axis AA′ (Fig. 34). But let us take the origin of the orthogonal axes at the extremity B of the base BB′, taken as axis of abscissas, BY′, perpendicular to the base, being the axis of ordinates. Let now M be any point of the cycloid, whose co-ordinates x', y', with reference to the new system, are BK′, K′M. Now BA′ $= \pi c$, c being the radius of the generating circle; hence MK, or the ordinate y of M, referred to the axes AX, AY, is equal to A′B $- x' = c\pi - x'$: the abscissa of M referred to the same axes or $x =$ AA′ $-$ A′K $= 2c - y'$. Hence $dx = - dy'$, $dy = - dx'$. Therefore, substituting these values in the above equation, we obtain

$$\frac{dy'}{dx'} = \sqrt{\frac{2c - y'}{y'}}.$$

Now the first of the two formulas $y\sqrt{1 + f'^2(x)}$, $\frac{[1 + f'^2(x)]^{\frac{3}{2}}}{f''(x)}$, gives (XV.) the value of the normal n, and the second gives (XXV.) the value of the radius r of the osculatory circle of any point of a plane curve. In our case $f'^2(x') = (\frac{dy'}{dx'})^2 = \frac{2c - y'}{y'} = -1 + \frac{2c}{y'}$, and this equation differentiated gives $2f'(x')f''(x)\,dx' = -\frac{2c}{y'^2}\frac{dy'}{dx'}\,dx' = -\frac{2c}{y'^2}f'(x')\,dx'$; hence $f''(x') = -\frac{c}{y'^2}$. Therefore (XV)

$$n = \mathrm{MN} = y' \sqrt{1 + f'^2(x')} = \sqrt{2cy'}$$

$$r = \mathrm{MN} = -\frac{[1 + f'^2(x')]^{\frac{3}{2}}}{f''(x')} = \frac{\left(\frac{2c}{y'}\right)^{\frac{3}{2}}}{\frac{c}{y'^2}} = 2\sqrt{2cy'};$$

hence $r = 2n$; i. e., the radius of curvature of any point of the cycloid is the double of the normal of the same point. Now the normal corresponding to the origin B is $= 0$, and the normal corresponding to the vertex A is $= 2c$; therefore the radius of curvature corresponding to the origin is $= 0$, and the radius of curvature corresponding to the vertex is $= 4c$. Producing therefore AA′ to A, so as to have A′A, $= 2c$, and producing the normal MN of the point M to D, so as to have ND = MN, the evolute of the semicycloid BMA is a curve which passes through the points B, D, A,. To determine the quality of this curve, let the arc BD $= \sigma$, and referring the evolute to BA′ taken for axis of ordinates, and to BX,, taken for axis of abscissas, and representing by x_1, y_1 the co-ordinates, we shall have, with regard to the point D, $x_1 =$ BK, = DD,, and $y_1 =$ DK,. Now, since ND = MN, the triangles NMK′, NDD, are equal; hence MK′ = DD,, i. e., $y' = x_,$; but the arc BD = MD, and MD $= 2\sqrt{2cy'}$, therefore

$$\sigma = 2\sqrt{2cx_,}.$$

Now this value belongs (XXIV. I.) to a cycloidal arc having for axis BB′ $= 2c =$ AA′. Therefore the evolute BDA, of the semicycloid BMA, is another evolute equal to the latter, but inverted. In like manner, the other semicycloid B′A has the corresponding evolute B′A′, which is again another semicycloid equal to B′A.

XXVII. *Integration by series.*

When a differential $f'(x)\,dx$ cannot be accurately integrated, the integration may be obtained by means of a series as near as desirable to its exact value, provided the conditions, which we here subjoin, be verified.

Let X_1, X_2, X_3 . . . represent different functions of the variable x, and let $f'(x)$, from $x = x_0$ to $x = x_m$, be capable of being developed into a converging series $X_1 + X_2 + X_3 + \ldots$, i. e., into a series the terms of which diminish in such a manner that, by increasing indefinitely their number, the sum of all approaches ever more to the fixed and determined limit $f'(x)$. In this supposition we shall have

$$\int_{x_0}^{x_m} f'(x)\,dx = \int_{x_0}^{x_m} X_1\,dx + \int_{x_0}^{x_m} X_2\,dx + \int_{x_0}^{x_m} X_3\,dx + \ldots$$

To simplify the case, let $X_1 = A$, $X_2 = Bx$, $X_3 = Cx^2$, etc. The preceding formula will be changed (XXIII. (I.), (II.),) into the following:

$$\int_{x_0}^{x_m} f'(x)\,dx = A(x_m - x_0) + \tfrac{1}{2}B(x_m^2 - x_0^2) + \tfrac{1}{3}C(x_m^3 - x_0^3) + \ldots$$

Consequently, making $x_0 = 0$, and taking x for x_m,

$$\int_0^x f'(x)\,dx = Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{3}Cx^3 + \ldots$$

And this series represents the definite integral of $f'(x)\,dx$ between $x = 0$, and x, approaching more and more to its exact value the greater is the number of terms that are taken.

By this method of integration we may develop into series those functions which are expressed by definite integrals. Let us see it exemplified in the following cases; and

1st. Let $\log(1 + x)$ be a given function of x. From the first case (XXIII.) we have $\log(1 + x) - \log(1 + x_0) = \int_{x_0}^{x} \frac{dx}{1 + x}$; and consequently, with $x_0 = 0$,

$$\log(1 + x) = \int_0^x \frac{dx}{1 + x} = \int_0^x \frac{1}{1 + x}\,dx.$$

Now (see Alg. § 67), supposing $x < 1$, $\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots$; the series of the second member being unlimited, the condition therefore to be verified, in order to have $\int_0^x \frac{1}{1 + x}\,dx$

expressed by a convergent series, is verified, provided $x < 1$; i. e., in this supposition,

$$\int_0^x \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots;$$

therefore within the same limits,

$$l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

2d. Let the given function be arc ($\sin = x$). Now (VI. VII.) d arc $(\sin = x) = \frac{dx}{\sqrt{1-x^2}}$, therefore

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \text{arc}\,(\sin = x).$$

But (XI. 2d,) if $x < 1$,

$$\frac{1}{\sqrt{1-x^2}} = 1 + \tfrac{1}{2}x^2 + \frac{3}{2\cdot 4}x^4 + \frac{3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots$$

Hence

$$\text{arc}\,(\sin = x) = \int_0^x \left(1 + \tfrac{1}{2}x^2 + \frac{3}{2\cdot 4}x^4 + \frac{3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots\right) dx$$

$$= x + \frac{1}{2\cdot 3}x^3 + \frac{3}{2\cdot 4\cdot 5}x^5 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots$$

By means of this series we may find the value of the semiperiphery π of the circle having 1 for radius as nearly as desirable. For. take $x = \frac{1}{2}$, the corresponding arc is $\frac{\pi}{6}$; hence

$$\pi = 6\left(\tfrac{1}{2} + \frac{1}{2\cdot 3\cdot 2^3} + \frac{3}{2\cdot 4\cdot 5\cdot 2^5} + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 2^7} + \ldots\right.$$

$$= 3,1415926\ldots$$

3d. Let also the given function be arc ($\text{tg} = x$). We have (VI. IX.) $d \cdot$ arc $(\text{tg} = x) = \frac{dx}{1+x^2}$; hence

$$\int_0^x \frac{dx}{1+x^2} = \text{arc}\,(\text{tg} = x).$$

But if $x < 1$, $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$

Now the arc of a tangent < 1 is necessarily $< \frac{\pi}{4}$. Therefore, for the positive arcs from 0^0 to $\frac{\pi}{4}$, we have

$$\text{arc}\,(\text{tg} = x) = \int_0^x (1 - x^2 + x^4 - x^6 + x^8 - \ldots)\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

The complement of arc $(\text{tg} = x)$, when $x < 1$, is an arc between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Representing by z its tangent, the value of this tangent ranges from $z = 1$ to $z = \infty$; hence $\frac{1}{z} < 1$. Also arc $(\text{tg} = z) = \frac{\pi}{2} - \text{arc}\,(\text{tg} = x)$, and since $\text{tg}\,a = \frac{1}{\text{tg}\,(90^\circ - a)}$, $x = \frac{1}{z}$; consequently

$$\text{arc}\,(\text{tg} = z) = \frac{\pi}{2} - \text{arc}\left(\text{tg} = \frac{1}{z}\right);$$

and $$\text{arc}\left(\text{tg} = \frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3z^3} + \frac{1}{5z^5} - \frac{1}{7z^7} + \ldots$$

Therefore for the positive arcs from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ we have

$$\text{arc}\,(\text{tg} = z) = \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \frac{1}{7z^7} - \ldots$$

XXVIII. *Integration of differential equations of the first order and degree, and between two variables.*

An equation between two variables is an equation in which enter only two variables and their differentials. To integrate differential equations means to find a finite equation between the variables of which the differential is a result. The order of the equation is the same as that of the differential of the variable taken as function of the other. For instance, the variables being y and x, if y be considered as function of x, if the differential of y entering in the equation, be of the second or third order, the equation also would be said to be of the same order; and since the order of the derivative follows that of the differential, so also the order of the equation is taken likewise from that of the derivative of the same function. The degree of the differential equation is taken from the power of the differentials, either dx or dy, or both. We limit our discussion on differential equations between two variables, to those only of the first order and degree, such as

$$(g) \quad \varphi(x, y)\, dx + \chi(x, y)\, dy = 0.$$

Now the equation (g) may be the result of the differentiation, for instance, of $f(x, y)$, which we shall represent by μ, or the elements contained in (g) may be combined together otherwise. In the first of these cases (g) is said to be an *exact* or total differential; in the second, it is simply called a differential equation, or, to distinguish better this case from the other, we may call it inexact differential. We shall consider the two cases separately, and, commencing with the *exact differential* first, since (XVI.) $d\mu = d_x\mu + d_y\mu$ or $= \frac{d\mu}{dx}dx + \frac{d\mu}{dy}dy$, and the differential of μ is by supposition the first member of (g), we shall have

$$\varphi(x, y) = \frac{d\mu}{dx}, \quad \chi(x, y) = \frac{d\mu}{dy};$$

but from these equations we obtain

$$\frac{d\,\varphi(x, y)}{dy} = \frac{d^2\mu}{dxdy}, \quad \frac{d\,\chi(x, y)}{dx} = \frac{d^2\mu}{dydx},$$

and (XVII.) $\frac{d^2\mu}{dxdy} = \frac{d^2\mu}{dydx}$; hence, also,

$$(g_{,}) \quad \frac{d\,\varphi(x, y)}{dy} = \frac{d\,\chi(x, y)}{dx}.$$

Therefore, in order that (g) may represent an *exact* differential, the equation $(g_{,})$ must be verified.

Now, to find out the original function $f(x, y)$: Since the first term of (g) is the partial differential of $f(x, y)$, relatively to x; by integrating this term as a differential function of x only, we shall obtain $f(x, y)$. But an indefinite integral may be expressed (XXIII. I.) by an integral, which begins by a particular value of the variable, with an arbitrary constant added to it. Designating then by Y an arbitrary function of y, which in the present case is regarded as constant, we shall have

$$\mu = \int_{x_0}^{x} \varphi(x, y)\, dx + \mathrm{Y}.$$

To determine Y, let us differentiate this equation relatively to y. We shall have

$$d_y\mu = \int_{x_0}^{x} \varphi(x, y + dy)\, dx - \int_{x_0}^{x} (x, y)\, dx + d\mathrm{Y},$$

and (XXI. 2d)

$$d_y\mu = \int_{x_0}^{x} \frac{d\,\varphi(x, y)}{dy}\, dydx + d\mathrm{Y}.$$

But $d_y\mu = \chi(x, y)\, dy$, and, since from $(g_,)$ $\dfrac{d\,\varphi(x, y)}{dy} = \dfrac{d\chi(x, y)}{dx}$, therefore

$$\chi(x, y)\, dy = \int_{x_0}^{x} \frac{d\chi(x, y)}{dx}\, dxdy + d\mathrm{Y}$$

$$\int_{x_0}^{x} \frac{d\chi(x, y)}{dx}\, dxdy + d\mathrm{Y}$$

$$= \chi(x, y)\, dy - \chi(x_0, y)\, dy + d\mathrm{Y};$$

and consequently,

$$d\mathrm{Y} = \chi(x_0, y)\, dy,\ \ \mathrm{Y} = \int_{y_0}^{y} \chi(x_0, y)\, dy + c.$$

Substituting now this last value of Y in the first of the above equations, we shall obtain

$$(g_2)\ \ \mu = \int_{x_0}^{x} \varphi(x, y)\, dx + \int_{y_0}^{y} \chi(x_0, y)\, dy + c,$$

in which c represents, as usual, an arbitrary constant.

Let, for example,

$$(6xy - y^2)\,dx + (3x^2 - 2xy)\,dy = 0,$$

in which $\varphi(x, y) = 6xy - y^2$, $\chi(x, y) = 3x^2 - 2xy$, and consequently,

$$\frac{d\,\varphi(x, y)}{dy} = 6x - 2y = \frac{d\,\chi(x, y)}{dx}.$$

The condition (g_1) is thus verified, and the given equation is an exact differential to which the resolution expressed by (g_2) may be applied. Now $\int_{x_0}^{x} (6xy - y^2)\,dx = 3x^2y - 3x_0^2y - y^2x + y^2x_0$, $\int_{y_0}^{y} (3x_0^2 - 2x_0y)\,dy = 3x_0^2y - 3x_0^2y_0 - x_0y^2 + x_0y_0^2$; hence

$$\mu = 3x^2y - y^2x - (3x_0^2y_0 - x_0y_0^2) + c.$$

The terms $3x_0^2y_0 - x_0y_0^2$ are constant, and, observing that the given equation has 0 for the second member, its integral μ also must be a constant. Representing now by C the sum of all these constants, we shall have

$$3x^2y - y^2x = C,$$

which, differentiated, reproduces the given equation.

Inexact differential; resolution by multiplication. Let us now take (g) as representing any differential equation of the first order and degree between two variables. Its integral may be obtained by different methods, one of which is to find a factor M, by which multiplying (g), the product will be an exact differential. This factor, however, is not easily found, except in the two instances which alone we shall examine here.

First case. The first of these two cases occurs when the factor M, which renders (g) an exact differential, is a function of x only or of y only. But if (g) multiplied by M becomes an exact differential, the condition expressed by (g_1) must be verified about this product; i. e., we must have

$$\frac{d\,[M\,\varphi(x, y)]}{dy} = \frac{d\,[M\,\chi(x, y)]}{dx}.$$

Now (VII. II.)

$$\frac{d\,[\mathrm{M}\,\varphi(x,y)]}{dy} = \mathrm{M}\frac{d\,\varphi(x,y)}{dy} + \varphi(x,y)\frac{d\mathrm{M}}{dy},$$

$$\frac{d\,[\mathrm{M}\,\chi(x,y)]}{dx} = \mathrm{M}\frac{d\,\chi(x,y)}{dx} + \chi(x,y)\frac{d\mathrm{M}}{dx}.$$

Therefore

$$\mathrm{M}\frac{d\,\varphi(x,y)}{dy} + \varphi(x,y)\frac{d\mathrm{M}}{dy} = \mathrm{M}\frac{d\,\chi(x,y)}{dx} + \chi(x,y)\frac{d\mathrm{M}}{dx}.$$

But in the supposition of M being function of the only variable x, $\frac{d\mathrm{M}}{dy} = 0$, as in the supposition of M being function of the only variable y, $\frac{d\mathrm{M}}{dx} = 0$. Hence in the first of these suppositions the condition of integrability of $(g) \times \mathrm{M}$ is, that

$$\frac{1}{\mathrm{M}}\frac{d\mathrm{M}}{dx} = \frac{1}{\chi(x,y)}\left(\frac{d\,\varphi(x,y)}{dy} - \frac{d\,\chi(x,y)}{dx}\right);$$

in the second,

$$\frac{1}{\mathrm{M}}\frac{d\mathrm{M}}{dy} = \frac{1}{\varphi(x,y)}\left(\frac{d\,\chi(x,y)}{dx} - \frac{d\,\varphi(x,y)}{dy}\right).$$

Now $\frac{1}{\mathrm{M}}\cdot\frac{d\mathrm{M}}{dx}$ is a function of the only variable x, and $\frac{1}{\mathrm{M}}\cdot\frac{d\mathrm{M}}{dy}$ is a function of the only variable y; hence M being a function of the only variable x, $(g) \times \mathrm{M}$ cannot be an exact differential, unless

$$\frac{1}{\chi(x,y)}\left(\frac{d\,\phi(x,y)}{dy} - \frac{d\,\chi(x,y)}{dx}\right)$$

be a function of the only variable x. And M being a function of the only variable y, $(g) \times \mathrm{M}$ cannot be an exact differential, unless

$$\frac{1}{\varphi(x,y)}\left(\frac{d\,\chi(x,y)}{dx} - \frac{d\,\varphi(x,y)}{dy}\right)$$

be a function of the only variable y. Supposing now that the one or the other of these conditions is verified, the factor M remains to be found.

Calling, for brevity's sake, χ, φ the functions $\chi(x, y)$, $\varphi(x, y)$, from the last two equations, we have

$$\frac{dM}{M} = \frac{1}{\chi}\left(\frac{d\varphi}{dy} - \frac{d\chi}{dx}\right) dx, \quad \frac{dM}{M} = \frac{1}{\varphi}\left(\frac{d\chi}{dx} - \frac{d\varphi}{dy}\right) dy.$$

Now (VI. I.) $\frac{dM}{M} = d\log(M)$. Therefore

$$\log(M) = \int \frac{1}{\chi}\left(\frac{d\varphi}{dy} - \frac{d\chi}{dx}\right) dx$$

or

$$\log(M) = \int \frac{1}{\varphi}\left(\frac{d\chi}{dx} - \frac{d\varphi}{dy}\right) dy$$

i. e., in the first case,

$$M = e^{\int \frac{1}{\chi}\left(\frac{d\phi}{dy} - \frac{d\chi}{dx}\right) dx};$$

in the second,

$$M = e^{\int \frac{1}{\phi}\left(\frac{d\chi}{dx} - \frac{d\phi}{dy}\right) dy}.$$

Let, for example,

$$\frac{2}{y}\, dx - \frac{x}{y^2}\, dy = 0$$

be the given equation, i. e., let $\varphi = \frac{2}{y}$, $\chi = -\frac{x}{y^2}$; with these elements,

$$\frac{1}{\chi}\left(\frac{d\varphi}{dy} - \frac{d\chi}{dx}\right) = -\frac{y^2}{x}\left(-\frac{2}{y^2} + \frac{1}{y^2}\right) = \frac{1}{x};$$

that is, $\frac{1}{\chi}\left(\frac{d\varphi}{dy} - \frac{d\chi}{dx}\right)$ is a function of the only variable x

And the factor $M = e^{\int \frac{1}{\chi}\left(\frac{d\phi}{dy} - \frac{d\chi}{dx}\right) dx} = e^{\int \frac{dx}{x}}$; but $\frac{dx}{x} = d\log x$, therefore $M = e^{\log(x)} = x$. With this factor the given equation becomes

$$\frac{2x}{y}\, dx - \frac{x^2}{y^2}\, dy = 0,$$

whose first member being an exact differential, its integral is obtained by means of the formula (g_2), from which, making $x_0 = 0$, we obtain

$$\frac{x^2}{y} = C,$$

which, differentiated, reproduces the preceding equation, from which we obtain the given one, dividing it by x.

Second case. The second case in which we propose to find the factor M, is when the equation (g) is homogeneous, and both functions φ and χ of the same degree.

We call homogeneous a function $f(x, y)$, in which the terms are reducible to an integral form and are all of the same dimension, as in the following trinomial:

$$6x^5 + y^3x^2 - 2y^4x,$$

in which the sum of the exponents of the two variables is of the same dimension or degree, 5 in each term.

Supposing n to be the degree, and multiplying each variable by any factor u, we shall evidently have $f(ux, uy) = u^n f(x, y)$. Considering now u as a variable, and, differentiating the last equation with regard to u alone, we shall obtain

$$nu^{n-1} f(x, y)\, du = f'_{ux}(ux, uy)\, dux + f'_{uy}(ux, uy)\, duy,$$
$$= xf'_{ux}(ux, uy)\, du + yf'_{uy}(ux, uy)\, du;$$

from which, making $u = 1$,

$$nf(x, y) = xf'_x(x, y) + yf'_y(x, y);$$

i. e., *The product of the homogeneous function f* (x, y), *by its degree* n, *is equal to the sum of the products of the partial derivatives, by the variable to which the derivatives are referred.*

Let now M be another homogeneous function of x and y, and such, that multiplying (g) by it, the product be an exact differential of μ, and μ itself be a homogeneous function of the nth degree of the same variables. We shall have

$$M\varphi\, dx + M\chi\, dy = d\mu;$$

and from the above theorem,

$$Mx\varphi + My\chi = n\mu.$$

Calling k the degree of M, and h that of φ and χ, the degree of μ must necessarily be $k + h + 1$; i. e., $n = k + h + 1$.

Divide now the first by the second of the last two equations, we shall obtain

$$\frac{\varphi dx + \chi dy}{x\varphi + y\chi} = \frac{1}{n} \cdot \frac{d\mu}{\mu}.$$

Now (VI. I.) $\frac{1}{n} \cdot \frac{d\mu}{\mu} = \frac{1}{n}\ dl(\mu)$; hence the second member of this equation is an exact differential, and consequently the first also; but the first member is the product of (g) by $M = \frac{1}{x\varphi + y\chi}$; hence whenever φ and χ are homogeneous functions of the same degree as x and y, the equation $\varphi dx + \chi dy = 0$ becomes integrable by being multiplied by $M = \frac{1}{x\varphi + y\chi}$.

Let, for example,

$$(xy + y^2)\, dx - x^2 dy = 0$$

be one of the equations represented by (g) in our present supposition. The factor M will be $\frac{1}{x\,(xy + y^2) - yx^2} = \frac{1}{xy^2}$, by which multiplying the given equation, we obtain

$$\left(\frac{1}{y} + \frac{1}{x}\right) dx - \frac{x}{y^2}\, dy = 0;$$

the first member of which being an exact differential, it may be integrated by means of the formula (g_2), from which, taking $x_0 = 0$, we obtain

$$\frac{x}{y} + \log(x) = C$$

for the integral corresponding to the given equation.

Resolution by separation. Besides the method of multiplication, the integral of an incomplete differential may be obtained by separating, when possible, the variables of the functions, so that, representing by X and Y two functions, the first of the only variable x, the second of the only variable y, the equation (g) may be reduced to the form $\mathrm{X}dx + \mathrm{Y}dy = 0$. An equation of this form can be integrated with the rules ordinarily applicable to differential expressions. Now this separation can be obtained easily in the two cases which we propose to examine here.

First case. The first of these cases occurs when (g) is homogeneous; for, making $y = zx$, and substituting this value in (g), which we shall suppose of the nth degree, φ and χ will each have x^n for common factor, and (g) will thus become a function of the only variable z; for, from $\varphi dx + \chi dy = 0$, we infer $\frac{\varphi}{\chi} dx + dy = 0$, in which the ratio $\frac{\varphi}{\chi}$ (we shall call it Z) is a function of the only variable z. Now, from $y = zx$ we have $dy = zdx + xdz$; hence

$$\mathrm{Z}dx + zdx + xdz = 0;$$

and therefore

$$\frac{dx}{x} + \frac{dz}{\mathrm{Z} + z} = 0,$$

which is an equation with separate variables, and whose integral is

$$\log x + \int \frac{dz}{\mathrm{Z} + z} = \mathrm{C},$$

in which substituting $\frac{y}{x}$ for z, we shall obtain the finite equation between x and y corresponding to the given (g).

Let us see an example in the following homogeneous equation of the second degree:

$$(xy - y^2)\, dx - (xy + x^2)\, dy = 0.$$

Making in it $y = zx$, and substituting $zdx + xdz$ for dy, it will become

$$(z - z^2)\, dx - (z + 1)(zdx + xdz) = 0,$$

or

$$2z^2dx + x(z + 1)\, dz = 0,$$

easily reduced to the following:

$$\frac{dx}{x} + \frac{z+1}{2z^2}\,dz = 0,$$

with separate variables; which, integrated, gives us

$$\log(x) + \tfrac{1}{2}\log(z) - \frac{1}{2z} = C,$$

and consequently the finite equation

$$\log(x) + \tfrac{1}{2}\log\left(\frac{y}{x}\right) - \frac{x}{2y} = C,$$

corresponding to the given differential.

Second case. The other case, in which the variables of (g) can easily be separated, is when φ and χ are such functions of x and y, that the ratio $\frac{\varphi}{\chi}$ results equal to a product $X \cdot Y$, in which X is a function of the only variable x, and Y a function of the only variable y; for (g), or its equivalent $\frac{\varphi}{\chi}dx + dy = 0$, then becomes $X \cdot Y dx + dy = 0$; and from this we obtain

$$X dx + \frac{dy}{Y} = 0,$$

with separate variables.

Let, for example, the differential equation be

$$x\sqrt{y^3} \cdot dx - (x\,y + x^3y)\,dy = 0,$$

for which the condition

$$\frac{\varphi}{\chi} = -\frac{x\sqrt{y^3}}{xy + x^3y} = -\frac{1}{1+x^2} \cdot \sqrt{y} = X \cdot Y$$

is verified. Consequently we have $X dx + \frac{dy}{Y} = 0$, or

$$\frac{dy}{\sqrt{y}} - \frac{dx}{1+x^2} = 0,$$

whose indefinite integral is

$$2\sqrt{y} - \text{arc}\,(\text{tg} = x) = C.$$

In other cases, when the separation of the variables is possible, the resolution is obtained by means of substitutions for which no general rule can be assigned. Some analytical processes also can be employed, with advantage, to the same effect. An example of this kind may be seen in the following number.

XXIX. *Integration of linear differential equations of the first order, containing only two variables.*

We call linear equations those in which the dependent variable y, and its differential dy, do not exceed the first degree, and are not multiplied by each other, whatever the degree of the other variable x may be. Hence, representing by X, X_1, X_2 different functions of the only variable x, the equation $Xdy + X_1ydx + X_2dx = 0$, or its equivalent

$$X\frac{dy}{dx} + X_1y + X_2 = 0,$$

is the general formula of all linear equations of the first order, between the variables x and y. Representing by $f(x)$ and $\varphi(x)$ the ratios $\frac{X_1}{X}$, $\frac{X_2}{X}$, the same formula may be expressed also by

$$\text{(L)} \qquad dy + yf(x)\,dx + \varphi(x)\,dx = 0.$$

The integration of (L) is obtained by the separation of the variables; and this separation by means of the following analytical process.

Let u, z be two indeterminate functions of x; such, however, that we may have $y = u \cdot z$, and consequently $dy = udz + zdu$, which value substituted in L gives us

$$udz + z\,(du + uf(x)\,dx) + \varphi(x)\,dx = 0.$$

Now, u is an arbitrary function which can consequently be determined in such a manner as to render the binomial $du + uf(x)\,dx = 0$. Thus the last equation may be resolved into the two following:

$$du + uf(x)\,dx = 0, \quad udz + \varphi(x)\,dx = 0.$$

The first of these equations divided by u, and integrated, gives

$$\log(u) = -\int f(x)\,dx = \log(e^{-\int f(x)\,dx}),$$

in which we omit the constant, being included in the indefinite integral of $f(x)\,dx$. Now the last formula is equivalent to the following:

$$u = e^{-\int f(x)\,dx},$$

in which u is given by a function of x, and such a function as to verify the condition $du + uf(x)\,dx = 0$. But the general formula, as we have seen above, becomes, in this case, $udz + \varphi(x)\,dx = 0$, from which, substituting in it the value of u just found, we obtain

$$dz = -\varphi(x)\,e^{\int f(x)\,dx}\,dx,$$

with separate variables z and x. This formula, integrated, gives

$$z = \frac{y}{u} = -\int \varphi(x)\,e^{\int f(x)\,dx}\,dx + C,$$

or $\quad (L_{,}) \quad y = e^{-\int f(x)\,dx}\left[C - \int \varphi(x)\,e^{\int f(x)\,dx}\,dx\right],$

which is the integral of the general equation (L).

Let, for example, the following differential equation be given:

$$dy + xydx - xdx = 0;$$

or, comparing it with (L), let $f(x) = x$, $\varphi(x) = -x$, and consequently $\int f(x)\,dx = \int xdx = \frac{x^2}{2} + c$, and from $(L_{,})$

$$y = e^{-\frac{x^2}{2}}\left(Ce^{-c} + \int xe^{\frac{x^2}{2}}\,dx\right) = e^{-\frac{x^2}{2}}\left(C_{,} + \int de^{\frac{x^2}{2}}\right),$$

or $$y = 1 + \frac{C_{,}}{e^{\frac{x^2}{2}}}.$$

XXX. *Integration of linear differential equations of the second order, and between two variables.*

We shall limit our resolution to the equations represented by

$$(E) \ldots \frac{d^2y}{dx^2} + a\frac{dy}{dx} + a'y = \chi(x),$$

in which a, a' are constant coefficients. To obtain the resolution, let us take the auxiliary equation

$$R^2 + aR + a' = 0,$$

and let r, r' be its roots. We shall have (Treat. on Alg. § 99) $a = -(r + r')$, $a' = rr'$, and consequently the given equation may be changed into the following:

$$\frac{d^2y}{dx^2} - (r + r')\frac{dy}{dx} + rr'y = \chi(x),$$

or

$$\frac{d^2y}{dx^2} - r\frac{dy}{dx} - r'\left(\frac{dy}{dx} - ry\right) = \chi(x).$$

Now $\frac{d^2y}{dx^2} - r\frac{dy}{dx} = \frac{d\left(\frac{dy}{dx} - ry\right)}{dx}$; hence, making $\frac{dy}{dx} - ry = y'$, (E) may be furthermore changed into

$$\frac{dy'}{dx} - r'y' = \chi(x),$$

which is a linear equation of the first order. Thus the integration of (E) is obtained by means of the integration of the two

$$(E_1)\quad \begin{cases} \dfrac{dy'}{dx} - r'y' = \chi(x), \\ \dfrac{dy}{dx} - ry = y'; \end{cases}$$

both of the first order, and both easily reducible to the form (L) of the preceding number, as follows:

$$dy' = y'r'dx + \chi(x)\,dx,$$
$$dy = yrdx + y'dx,$$

which, compared with (L), and resolved by means of (L_1), give

$$(E_2)\quad \begin{cases} y' = e^{r'x}\left(C + \int \chi x e^{-r'x}\,dx\right), \\ y = e^{rx}\left(C' + \int y'e^{-rx}\,dx\right). \end{cases}$$

The relation between x and y, resulting from these two equations, belongs to the given differential equation (E), and is consequently its complete integral.

To see an application, let

$$\frac{d^2y}{dx^2} \mp b^2y = 0$$

be a linear equation of the second order to be integrated, and let us take it, first, with the upper sign. Its auxiliary equation will, in this case, be $R^2 - b^2 = 0$; and, consequently, $r = b$, $r' = -b$; also $\chi(x) = 0$. Thus the formulas (E_2) will become

$$y' = e^{-bx} \cdot C,\ y = e^{bx}(C' + \int y' e^{-bx}\, dx),$$
$$= e^{bx}(C' + C\int e^{-2bx}\, dx),$$
$$= e^{bx}\left(C' - \frac{C}{2b}\int e^{-2bx} \log e\, d(-2bx)\right).$$

Now (VI. I.) $e^{-2bx} \log e\, d(-2bx) = d\, e^{-2bx}$; hence

$$y = e^{bx}\left(C' - \frac{C}{2b} e^{-2bx}\right) = C'e^{bx} - C''e^{-bx}$$

is the integral of the given equation relatively to the upper sign. Let us now take the positive sign: We shall have $R^2 + b^2 = 0$. Hence $r = b\sqrt{-1}$, $r' = -b\sqrt{-1}$, $\chi(x) = 0$; i. e., everything as with the negative sign, except the change of b into $b\sqrt{-1}$; hence, regarding the imaginary quantities as real ones, we shall have for the integral of the given equation, taken with the lower sign,

$$y = e^{b \cdot x\sqrt{-1}}\left(C' - \frac{C}{2b\sqrt{-1}} e^{-2b \cdot x\sqrt{-1}}\right),$$

$$= C' e^{bx\sqrt{-1}} - \frac{C}{2b\sqrt{-1}} e^{-bx\sqrt{-1}};$$

and, making $b = 1$ and $\frac{C}{2b\sqrt{-1}} = C_2$,

$$y = C' e^{x\sqrt{-1}} - C_2 e^{-x\sqrt{-1}}.$$

Now from Maclaurin's theorem we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \ldots,$$

$$\sin x = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \ldots,$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} - \ldots$$

Taking $\pm x\sqrt{-1}$ in the first of these series instead of x, from this substitution, and from the other two series we obtain

$$e^{x\sqrt{-1}} = \cos x + \sin x \sqrt{-1},$$

$$e^{-x\sqrt{-1}} = \cos x - \sin x \sqrt{-1};$$

hence

$$y = C' (\cos x + \sin x\sqrt{-1}) - C_2 (\cos x - \sin x \sqrt{-1}),$$

$$= (C' - C_2) \cos x + (C' - C_2) \sqrt{-1} \cdot \sin x.$$

Calling K the difference $C' - C_2$, and $K_{,}$ the product $(C' - C_2)\sqrt{-1}$, we shall finally obtain

$$y = K \cos x + K_{,} \sin x.$$

THE END.

Pl. I.

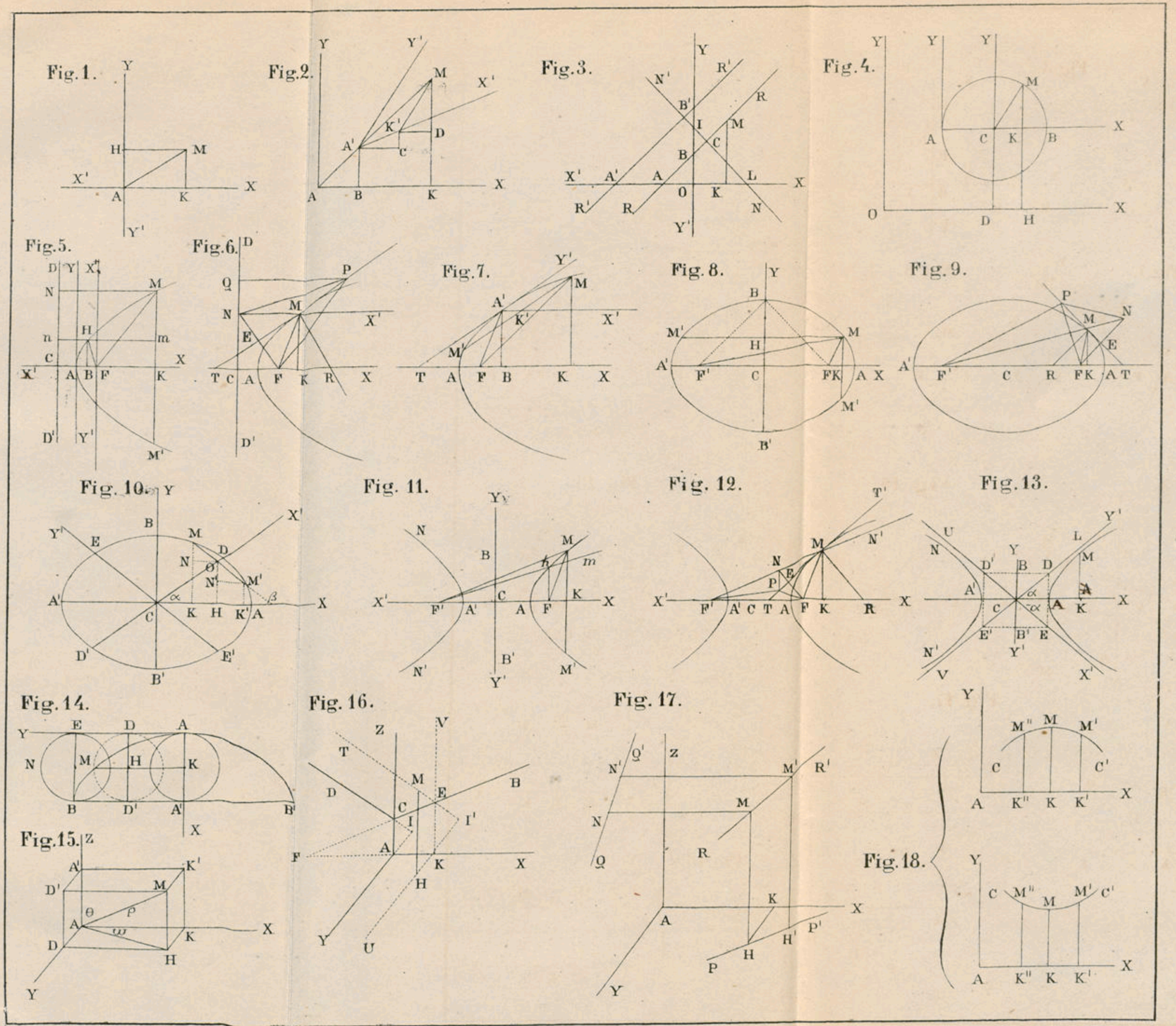

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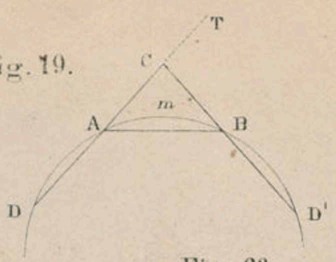
Fig. 19.



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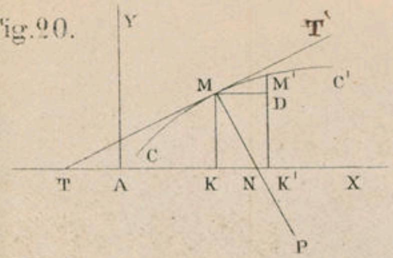

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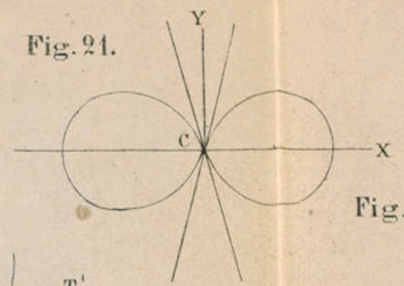

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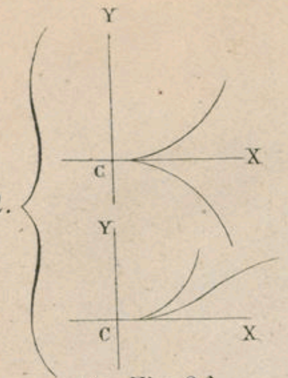

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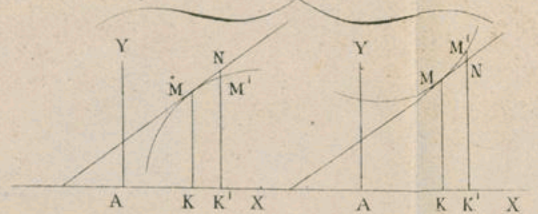

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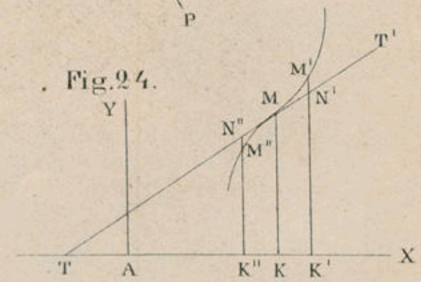

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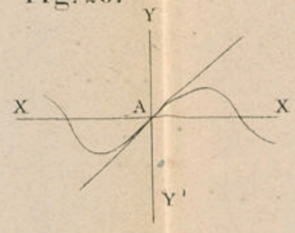

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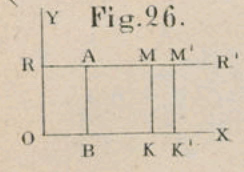

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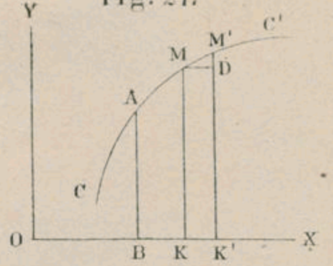

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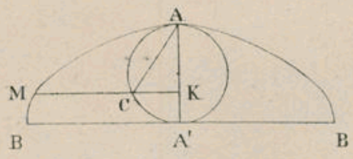

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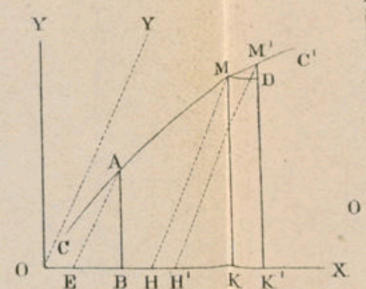

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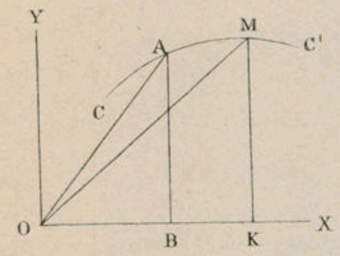

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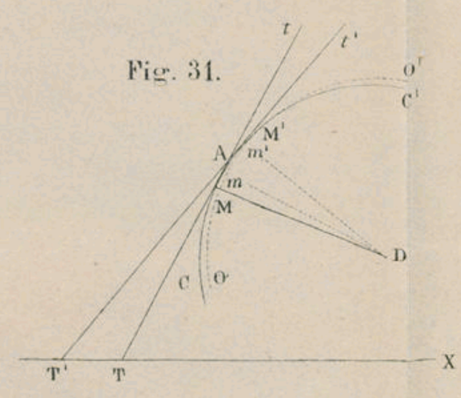

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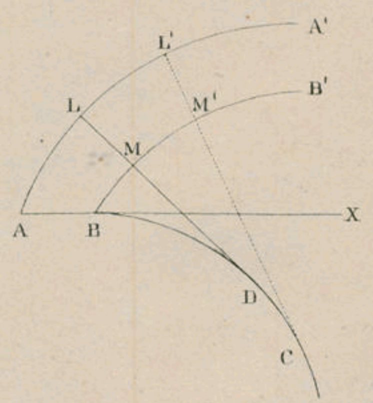

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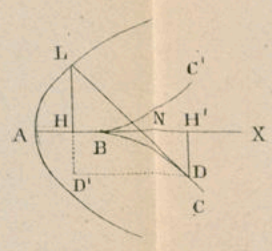

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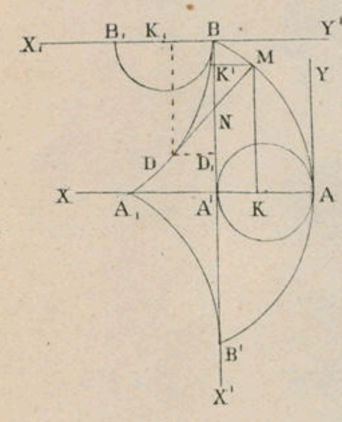

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