ELEMENTS

OF

GEOMETRY AND TRIGONOMETRY.

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PREFACE.

At an early period, when Geometry was the only, or at best the first, branch of mathematical science in which scholastic instruction was given, it was taught by a method which harmonized with the general system of education then prevailing. This had been sanctioned by the practice of many generations, and was upheld by the authority of Euclid himself, whose elementary work on this subject was universally adopted as a text-book.

To question whether our ancestors acted wisely in adopting a plan of education in which the science of mathematics did not hold that large share which is assigned to it in the present system, would be foreign to our purpose. It may, however, be observed that, considering education inasmuch as it is designed to impart to the pupil an aptitude for applying himself to the various professions and arts of civil life, it seems that so much of the various branches should be taught as may fit him for any career to which he may afterwards devote himself. Now, experience has proved that there is no art or science to which the study of geometry is not an admirable preparation. This was well known to the ancients; and, although they did not spend so much time as ourselves in the study of mathematics, they never omitted a branch which they, too, regarded as indispensable.

But the opinions of men vary with the times; and one who in our days would venture to recommend the abridgment of the time commonly given, in modern institutions, to natural sciences, and given, not unfrequently with considerable prejudice, to a more solid instruction in literature and moral philosophy, would be censured as the ignorant advocate of an obsolete theory. As material motion has been accelerated by modern inventions, so it is thought possible, in some similar manner, to accelerate intellectual development and the operations of the mind. We have those who undertake to teach everybody every thing, and that in the shortest assignable time; but the competency of the teacher, the progress of the scholar, and the solidity of his acquirements, are matters rather supposed than proved.

But, after all, the teacher is in some respect like a merchant. As the merchant does not consult his own taste, but that of the buyers, so whoever intends to promote the education of youth is compelled to regard the taste of others rather than his own. For, as the merchant aims at gain, so a conscientious promoter of education aims more at the sound training of the heart than at that of the mind. Then, again, the study of mathematics is harmless of itself, and may be pursued without much apprehension by the young; nay, many would be much happier if they allotted to this study time more than lost in the perusal of works of a demoralizing tendency.

The preceding remarks have already furnished the reason of the plan followed in the present elementary work: and, first, since geometry is not to be severed now from the other branches of mathematics, but forms part of the same science with them and succeeds algebra, he who teaches or writes a Geometry for schools supposes the knowledge of algebra, or at least some practice in algebraical language. In the present work, with the exception of the doctrine of ratios and proportions, which is common to all the various branches of mathematics, it may be said that nothing is supposed or borrowed from algebra, except its language; and he who objects to it as a mixing up of algebra with simple geometry would judge as some did of the publications of the Baron of Zach, written with Greek characters, but in the French language, and thought by them to be Greek, when, in fact, it was nothing else but French. But, some would ask, why make use of the algebraic language in geometry? I could ask in my turn, Why do you wish that geometry should succeed algebra? Is it not in order to derive some benefit from algebra? But I will rather propose another question: Is it not you who require to travel over a long journey in a short time? The algebraic language is laconic: it says much in a few words; and that which, if expressed in the old style, would require a book, may be reduced to a few pages by the use of the terminology of algebra, whilst the reasoning remains still as rigorous and as lucid as before. In this manner you secure copiousness of matter and economy at the same time, and the pupil is prevented from losing the practice of algebraic language.

It may be remarked that the use of a different type—a distinction adopted in the Treatise on Algebra—has been discontinued in the present work. This change will, perhaps, not meet with the approbation of all. The reasons which suggested it were, that nearly all of the more difficult parts occur in the last books, and at a time when the minds of the pupils are better prepared to master them. In the first books the few theorems of a more abstruse nature are so explained that a competent teacher may render the comprehension of them an easy task.

The writer of the present elementary work has reason to be grateful to several distinguished persons who were pleased to accept his preceding publications, and by their public and honorable approbation encourage him to finish the work. It was, however, observed that a certain kind of analysis is ill adapted to circumstances; and since the same observation could be renewed on the present occasion, to prevent all misunderstanding, it should be noticed that by the terms analysis and synthesis the writer understands precisely what is understood by logicians. He calls synthesis, or the synthetical method, the proceeding from universal principles and more obvious to particulars and the more recondite truths; or, if it be preferred, from the more elementary and more accessible notions to the more complicated and abstruse; and he calls analysis the process made in the inverted order. Now, if the reader will trouble himself to examine the index, he will see that the order observed in the distribution of the books and of the matters of each book proceeds step by step from the more simple notions to the more complicated. The method, therefore, is thoroughly synthetical, although occasionally, either by way of illustration or corollary, some incidental truth may be treated in a manner apparently or even really analytic. Certainly no one would assert that a stream flows in a direction opposite to its natural course because when it finds lateral ditches in its way it fills them up, and even flows backward, with a portion of its waters. This method greatly contributes to due order and lucidity,—qualities which are occasionally overlooked even in books destined for the instruction of youth, with no small prejudice to the student, who is more puzzled and annoyed by the confusion with which the matter is presented to him than by its inherent difficulties. The writer has sedulously endeavored to avoid this evil; with what success it is the reader's part to judge.

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Elements of Geometry.

INTRODUCTORY ARTICLE.

Object of Geometry. I. Geometry treats of magnitudes.

Dimensions. The dimensions of any body or space, either existing or simply conceived, cannot be more than three. For, suppose a body of any shape placed on a table:—this body, besides the points that are at contact with the table, has other points above them, in succession, from the plane of the table to the top of the body. Now, this extension is one of the dimensions of the body, and is usually called *thickness*. The same body extends itself also in the direction of the length, and in that of the breadth, of the table, and thus we have two more dimensions, which, accordingly, are called *breadth* and *length* of the body. Besides these, no other dimensions can be conceived.

Definitions. Bodies in Geometry are called also *solids*,—that is:—

Solid. A solid is a magnitude having three dimensions.

If we consider only the boundaries of a solid, without any connection with the contiguous internal parts, we have that which in Geometry is called *surface*. Hence, the surface is called, also, the limit of the solid; but, we may more generally say that—

Surface.

A surface is a magnitude having two dimensions. Now, since the boundaries of solids are either plane or curve, so also there are two different kinds of surfaces, called likewise plane and curve surfaces. The plane surface is also simply called a *plane*.

Line.

The boundaries or limits of a surface are the geometrical line; or,

A line is a magnitude having only one dimension; and, since the boundaries of surfaces are either straight or curve, lines also are either *straight* or *curve*.

Point.

The limits of a line are called *points*. The geometrical point, therefore, has no dimensions.

Remarks.

The notions of straight and curve line, plane and curve surface, are clear enough to every one; and it is of no profit to attempt to give an illustration or definition of them.

It is equally easy to see, that if two straight lines coincide in two of their portions, however small, they will coincide in all the other points, even if indefinitely produced, for neither of them ever deviates.

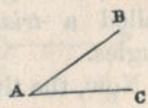
So also a straight line can never be made to coincide with a curve line or globular surface, and a plane surface can never be made to coincide with a curve one: thus a ball, rolled in all directions on a plane, touches the plane always in no more than one point; but if two plane surfaces coincide in any two of their portions, they will evidently coincide in all, even if indefinitely produced.

Now, since magnitudes are the subject of Geometry, and magnitudes admit of one dimension, as lines, either straight or curve,—or two dimensions, as surfaces, either plane or curve,—or three dimensions, as solids, limited by plane or curve boundaries,—the subject of Geometry contains three heads, and three only:—*lines*, *surfaces*, and *solids;* each one of these heads, however, being taken most generally as well to that which concerns the properties of the various magnitudes as to that which regards their mutual relations.

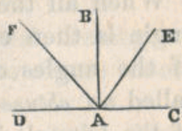
In Elementary Geometry the subject can be embraced only partially; and, besides the straight line, the circular line is the only curve considered by it: it considers plane surfaces and the surfaces or boundaries of those solids which are exclusively taken into consideration.

Preliminaries concerning angles and triangles.

II. Let us add to these general remarks some preliminaries concerning angles and triangles, parallel lines and the circular line. And, first, two straight lines, AB, AC, having only one point, A, common, are said to form an *angle;* the point A is called the *vertex*, and AB, AC, the *sides*, of the angle. The angle is the mutual inclination of the sides, and, consequently, it does not depend on their length. The same letter A is used to designate the angle as well as the vertex; nay, the whole figure is called the angle A, or, more explicitly, the angle BAC.



But when AB stands erect over DC, and does not incline on either side, in this case we cannot rigorously say that the angle BAC and the angle BAD are the mutual inclination of the sides, unless we call inclination the relative position of the two straight lines.



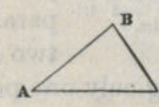
It is evident that any straight line AE, between AB and AC, must be inclined toward AC; and any straight line AF, between AB and AD, must be inclined toward AD.

It is evident, also, that the angles BAD, BAC, are equal to each other. For if we say that BAD is not equal to BAC, we say that the relative position of BA with regard to AD differs from that of the same BA with regard to AC, which is against the supposition.

Right angles. Perpendicular. These angles are called *right angles*, and the straight line AB, forming the two equal angles with CD, is called *normal* or *perpendicular*.

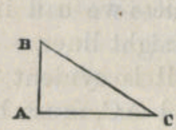
Acute and obtuse angles. Any angle, EAC, less than the right angle, is called an *acute angle;* and any angle, FAC, greater than the right angle, is called an *obtuse angle.*

Triangle. III. The extremities of the sides of any angle may be joined together with another straight line. The figure BAC, arising from this construction, is called a *triangle*, for it contains three angles.



Now, the three sides of a triangle may be either equal or unequal to one another. When the three sides are equal, the triangle is called *equilateral;* when only two sides are equal, the triangle is called *isosceles;* when the three sides are unequal, the triangle is called *scalene.*

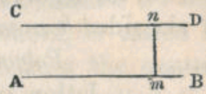
When all the angles of the triangle are acute, the triangle is then called an *acute-angled* triangle; when one of the angles of the triangle is obtuse, the triangle is called an *obtuse-angled* triangle; when one of the angles of the triangle is a right angle, the triangle then is called a *right-angled* triangle, and the side opposite to the right angle is called the *hypothenuse.* Supposing, for instance, A to be a right angle, BC is the hypothenuse.



The horizontal side of the triangle is usually called the *base:* thus, AC is the base of ABC.

Parallel lines.

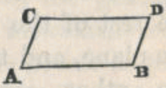
IV. When two straight lines, CD and AB, are on the same plane and keep constantly the same mutual distance, they are called *parallel lines.*



Suppose *mn* to be a movable perpendicular to AB, touching with the upper extremity *n* the other straight line CD; if the same perpendicular, brought at different points along BA, touches invariably the straight line CD with the same extremity *n*, these straight lines are said to preserve the same distance from each other; and, since neither of them will ever deviate from their straight direction, they will always remain at equal distance from each other, even indefinitely produced, and will never meet to form an angle.

Quadrilateral figures and polygons.

V. Now, two parallel lines, CD and AB, may be limited by two other parallel lines, CA and DB, in which case we



Parallelogram.

have a figure of four sides and angles, which we call a *parallelogram.*

Rhombus.

If the four sides of the parallelogram should be all equal, the figure would then be called also a *rhombus.*

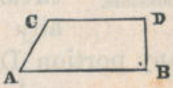
Rectangle.

And if, the sides not being equal, the angles should be all right angles, the figure then would be called a *rectangle.*

Square.

But if all the angles are right, and the sides all equal, the figure would then be called a *square.*

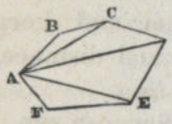
When the two sides CD and AB only are parallel, and the other two inclined to each other, the figure is called a *trapezoid.*



Generally, all figures of four sides are called *quadri* *laterals*, all figures of five sides *pentagons*, and all figures of six sides *hexagons*.

Polygon.

Polygon is the general appellation including figures of any number of sides.

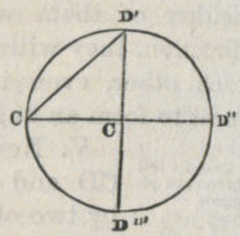


The straight lines AC, AD, AE, drawn from any angle A of the polygon to the opposite angles, are called *diagonals*.

The plane surface included by the sides of the polygon is called *area*, and the sides taken together form the *perimeter* of the polygon.

Circular line.

VI. Suppose the straight line DC to be movable about one of its extremities C,—that is, while C keeps invariably its position on the same point of the plane, the rest of the line turns around on the plane, and traces meanwhile with the other extremity D the line DD′D″D‴. This is the *circular line*, evidently different from the straight line.



Now, since the length of CD remains unchanged, all the points of DD′ . . . . are equally distant from C, which is called the *centre*, and consequently straight lines drawn from the centre to the various points of the circular line are all equal to one another; wherefore some define this line a curve line having all its points equally distant from a central point.

Definitions.

Circle and circumference.

The surface or area limited by the circular line is called the *circle*, and the line itself the *circumference* or *periphery*, although occasionally the circumference also is called circle. Any portion DD′ of the circumference is called an *arc*,

Arc and chord.

and a straight line DD′ drawn from one to another extremity of the arc is called a *chord.* The plane surface or area D*m*D′D, limited by the area and the corresponding chord, is called a *segment.* The line CD drawn from the centre to any point D of the periphery is called the *radius.* The area D*m*D′C, limited by two radii and the portion D*m*D′ of the circumference terminated by the same radii, is called a *sector.* A straight line DD″ which, passing through the centre, touches with its extremities the circumference, is called the *diameter.* The diameter, therefore, is twice the radius. The diameter also bisects the circle and the circumference into two equal parts, called *semicircles* and *semicircumferences.* For, suppose the plane DD′D″D to be turned about the diameter DCD″ so as to make this surface coincide with the other DD‴D″D, the portion DD′D″ of the circumference must then necessarily coincide with DD‴D″, otherwise some of the points of the upper or lower periphery would not be equally distant from the centre. The diameter, therefore, bisects equally circle and circumference.

Segment.

Radius.

Sector.

Diameter.

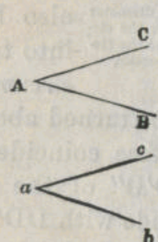
The diameter bisects the circle and the circumference.

BOOK I.

ANGLES AND TRIANGLES.

The first elementary theorems concerning angles and triangles and the measure of the angles afforded by the circular line form the subject of the present book. But, first, let us remark that, when two angles A and *a* are equal to each other, the one may be placed on the other so as perfectly to coincide with it; for, if we place *ab* on AB so as to make the point *a* coincide with A, since *ac* is inclined on *ab* in the same manner in which AC is inclined on AB, the side *ac* also of the angle *a* will coincide with the side AC of the angle A.

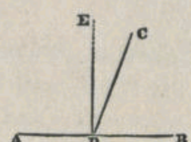
Remark concerning equal angles.



THEOREM I.

When a straight line meets another straight line, the sum of the two angles is equal to two right angles.

Let the straight line CD meet the other straight line AB at D, the line CD meets AB either perpendicularly or not: in the first case the two angles formed are right angles; in the second case let ED be the perpendicular, meeting AB at D; in this manner, (representing the two right angles by the expression $2r$,) we will have,

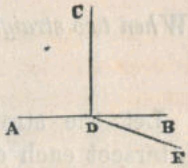


$$ADE + EDB = 2r.$$

Now, the angle ADC is equivalent to ADE + EDC, and the angle CDB is equivalent to EDB − EDC; hence,

$$ADC + CDB = ADE + EDC + ADB - EDC = ADE + ADB = 2r.$$

In whatever manner, therefore, CD meets AB, the sum of the two adjacent angles is equal to two right angles.



Scholium. If we suppose AD and DB to be two separate lines, and CD meeting them in the point of their junction to form two adjacent angles equal or equivalent to two right angles, the two lines must be on the same straight line. Otherwise, suppose that DF is the continuation of the straight line AD; then we have ADC + CDF = $2r$; but by supposition ADC + CDB = $2r$; hence, ADC + CDF = ADC + CDB, and consequently CDB = CDF, which is absurd. We must say, therefore, that AD and DB are on the same straight line.

Remarks and axioms. In demonstrating our last assertion, we have made use of some *axioms* or self-evident principles, which the student may profitably remark here and once forever. First, from the equations

$$ADC + CDF = 2r, \quad ADC + CDB = 2r,$$

we have inferred the other equation,

$$ADC + CDF = ADC + CDB,$$

resting on the axiom that *things that are equal to the same thing are equal to each other*.

Again, from the last equation we have inferred the following:—

$$CDF = CDB,$$

resting on the axiom that *when equals are taken from equals the remainders are equal.*

We have finally inferred that AD and DB must necessarily be on the same straight line, from the absurdity which otherwise would follow, that the whole angle CDF is equal to its portion CDB, resting on the axiom, *every whole is greater than any of its parts.*

THEOREM II

When two straight lines intersect each other, the opposite angles are equal.

Let the straight lines AB and CD intersect each other at the point E; the angles AEC and DEB are called *opposite* or *vertical*, and also the angles AED, CEB.

From the preceding theorem we have

$$AEC + AED = 2r, \; DEB + AED = 2r;$$

hence, $$AEC + AED = DEB + AED,$$

and, consequently,

$$AEC = DEB.$$

In like manner, considering the adjacent angles AED and DEB, and then DEB and BEC, we find

$$AED = CEB.$$

Corollary. That is, the opposite or vertical angles are equal. Hence, if the straight line DC is perpendicular to AB, the four angles are all equal.

Scholium. Whatever be the angles which AB makes with CD, since AED and DEB are equivalent to two right angles, and also AEC and CEB, the sum of the four angles is always equivalent to four right angles.

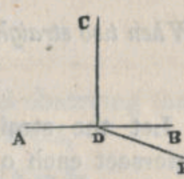
Nay, let any number of lines, CA, CB, CD, meet

Now, the angle ADC is equivalent to ADE + EDC, and the angle CDB is equivalent to EDB − EDC; hence,

$$ADC + CDB = ADE + EDC + ADB - EDC = ADE + ADB = 2r.$$

In whatever manner, therefore, CD meets AB, the sum of the two adjacent angles is equal to two right angles.

Scholium. If we suppose AD and DB to be two separate lines, and CD meeting them in the point of their junction to form two adjacent angles equal or equivalent to two right angles, the two lines must be on the same straight line. Otherwise, suppose that DF is the continuation of the straight line AD; then we have ADC + CDF = 2*r*; but by supposition ADC + CDB = 2*r*; hence, ADC + CDF = ADC + CDB, and consequently CDB = CDF, which is absurd. We must say, therefore, that AD and DB are on the same straight line.



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From the preceding theorem we have

$$AEC + AED = 2r, \; DEB + AED = 2r;$$

hence, $$AEC + AED = DEB + AED,$$

and, consequently,

$$AEC = DEB.$$

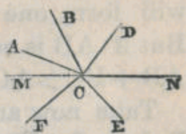
In like manner, considering the adjacent angles AED and DEB, and then DEB and BEC, we find

$$AED = CEB.$$

Corollary. That is, the opposite or vertical angles are equal. Hence, if the straight line DC is perpendicular to AB, the four angles are all equal.

Scholium. Whatever be the angles which AB makes with CD, since AED and DEB are equivalent to two right angles, and also AEC and CEB, the sum of the four angles is always equivalent to four right angles.

Nay, let any number of lines, CA, CB, CD, meet together at C, the sum of the angles about C will be equal to four right angles. For, draw MCN; we will have, first, $BCM + BCN = 2r$; that is, since $BCM = BCA + ACM$, and $BCN = BCD + DCN$,



$$BCA + ACM + BCD + DCN = 2r.$$

In like manner,

$$ECN + ECF + FCM = 2r.$$

Hence, adding together the two sums, and observing that $ACM + MCF = ACF$, and $DCN + NCE = DCE$,

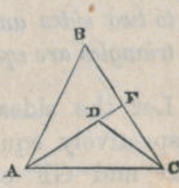
$$ACB + BCD + DCE + ECF + FCA = 4r.$$

That is, the sum of the angles formed by any number of straight lines meeting at one common point is always equal to four right angles.

THEOREM III.

The sum of two straight lines, drawn from any point within the triangle to the extremities of the base, is less than the sum of the other sides.

We need not demonstrate that one of the sides of any triangle is always less than the sum of the other two sides; for if we imagine, for instance, the side AB to be depressed toward AC, keeping the extremity A always immovable and the extremity B invariably united with that of the side BC, we cannot conceive this motion without the sliding of the side BC through the extremity C of the base (to which we suppose it to adhere) and constant increasing of the angle B; and if AB is less than AC, when



AB will coincide with AC, the two sides AB and BC will form one straight line evidently longer than AC. But if AB is equal or greater than AC, much more then $AB + BC > AC$.

Take now any point, D, within the triangle, and from D draw DA, DC to the extremities of the base; produce also AD to F. From the triangle ABF we have

$$AF < AB + BF;$$

And, since *when equals are added to unequals the sums are unequal*, we will have also

$$AF + FC < AB + BF + FC;$$

or, $$AF + FC < AB + BC.$$

Now, $DC < DF + FC$, and, consequently,

$$AD + DC < AD + DF + FC$$

or, $$AD + DC < AF + FC.$$

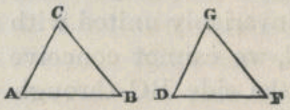
But $AF + FC$ is already less than $AB + BC$; much more then

$$AD + DC < AB + BC.$$

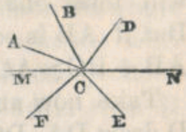
THEOREM IV.

If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are equal.

Let the sides CA and CB of the triangle ABC be respectively equal to the sides GD and GF of the triangle DFG, and the included angle C of the first equal to the included angle G of the second; the two triangles are equal. For, imagine the first ABC



together at C, the sum of the angles about C will be equal to four right angles. For, draw MCN; we will have, first, $BCM + BCN = 2r$; that is, since $BCM = BCA + ACM$, and $BCN = BCD + DCN$,



$$BCA + ACM + BCD + DCN = 2r.$$

In like manner,

$$ECN + ECF + FCM = 2r.$$

Hence, adding together the two sums, and observing that $ACM + MCF = ACF$, and $DCN + NCE = DCE$,

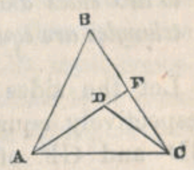
$$ACB + BCD + DCE + ECF + FCA = 4r.$$

That is, the sum of the angles formed by any number of straight lines meeting at one common point is always equal to four right angles.

THEOREM III.

The sum of two straight lines, drawn from any point within the triangle to the extremities of the base, is less than the sum of the other sides.

We need not demonstrate that one of the sides of any triangle is always less than the sum of the other two sides; for if we imagine, for instance, the side AB to be depressed toward AC, keeping the extremity A always immovable and the extremity B invariably united with that of the side BC, we cannot conceive this motion without the sliding of the side BC through the extremity C of the base (to which we suppose it to adhere) and constant increasing of the angle B; and if AB is less than AC, when



AB will coincide with AC, the two sides AB and BC will form one straight line evidently longer than AC. But if AB is equal or greater than AC, much more then $AB + BC > AC$.

Take now any point, D, within the triangle, and from D draw DA, DC to the extremities of the base; produce also AD to F. From the triangle ABF we have

$$AF < AB + BF;$$

And, since *when equals are added to unequals the sums are unequal*, we will have also

$$AF + FC < AB + BF + FC;$$

or,

$$AF + FC < AB + BC.$$

Now, $DC < DF + FC$, and, consequently,

$$AD + DC < AD + DF + FC$$

or,

$$AD + DC < AF + FC.$$

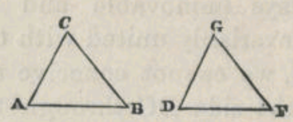
But $AF + FC$ is already less than $AB + BC$; much more then

$$AD + DC < AB + BC.$$

THEOREM IV.

If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are equal.

Let the sides CA and CB of the triangle ABC be respectively equal to the sides GD and GF of the triangle DFG, and the included angle C of the first equal to the included angle G of the second; the two triangles are equal. For, imagine the first ABC

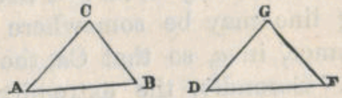


to be placed on the second DEF, so that CA may coincide with GD, A with D, and C with G; since the angle C is equal to G, the side CB must also coincide with GF, and, CB and GF being equal in length, the point B will coincide with F. But if A and B coincide with D and F, the side AB also coincides with DF, and the two triangles are equal.

THEOREM V.

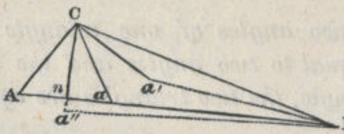
If two angles of one triangle and the included side are equal to two angles and the included side of another triangle, the two triangles are equal.

Let the angles A and B of the triangle ABC be respectively equal to the angles D and F of the triangle DFG, and the included side AB of the first triangle equal to the included side DF of the second. The two triangles are equal. For, placing AB over DF, so that A may coincide with D and B with F, the side AC, being inclined toward AB in the same manner as GD is inclined toward DF, will coincide with GD; and for the same reason CB will coincide with GF; and, therefore, the point C, which is at once on AC and on CB, must necessarily coincide with the point G, which alone is at once on the sides GD and GF, and the two triangles are identical.



THEOREM VI.

If two sides of one triangle are equal to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, the third side of the first triangle is also greater than the third side of the second, and vice versâ.



Let the angle C of the triangle ABC be cut into two angles by any straight line, and let the cutting line be equal in length to the side CA. Three cases may occur:—First, the extremity of the cutting line may be somewhere along the base, AB; for instance, in *a*, so that C*a*, the cutting line, be equal to CA. Secondly, the extremity may fall out of the triangle; for example, in *a''*, C*a''* being again equal to CA. Finally, the extremity may fall within the triangle; for example, in *a'*, when the cutting line is C*a'*, equal in length to the preceding

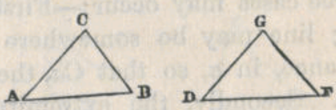
Joining now *a''* and *a'* with B, we have the triangles C*a''*B, C*a'*B, which, together with the triangle C*a*B, have one side, CB, common, which is also one of the sides of the triangle ABC; the sides C*a''*, C*a'*, and C*a*, are equal to CA, but the included angles *a''*CB, *a'*CB, and *a*CB, are less than the included angle ACB. Now, we say that the third side of every one of those three triangles is less than the third side, AB, of the triangle ACB.

With regard to the case of the triangle C*a*B, it is evident that *a*B is less than AB; but also *a'*B is less than to be placed on the second DEF, so that CA may coincide with GD, A with D, and C with G; since the angle C is equal to G, the side CB must also coincide with GF, and, CB and GF being equal in length, the point B will coincide with F. But if A and B coincide with D and F, the side AB also coincides with DF, and the two triangles are equal.

THEOREM V.

If two angles of one triangle and the included side are equal to two angles and the included side of another triangle, the two triangles are equal.

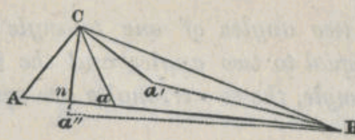
Let the angles A and B of the triangle ABC be respectively equal to the angles D and F of the triangle DFG, and the included side AB of the first triangle equal to the included side DF of the second.



The two triangles are equal. For, placing AB over DF, so that A may coincide with D and B with F, the side AC, being inclined toward AB in the same manner as GD is inclined toward DF, will coincide with GD; and for the same reason CB will coincide with GF; and, therefore, the point C, which is at once on AC and on CB, must necessarily coincide with the point G, which alone is at once on the sides GD and GF, and the two triangles are identical.

THEOREM VI.

If two sides of one triangle are equal to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, the third side of the first triangle is also greater than the third side of the second, and vice versâ.



Let the angle C of the triangle ABC be cut into two angles by any straight line, and let the cutting line be equal in length to the side CA. Three cases may occur:—First, the extremity of the cutting line may be somewhere along the base, AB; for instance, in a, so that Ca, the cutting line, be equal to CA. Secondly, the extremity may fall out of the triangle; for example, in a'', Ca'' being again equal to CA. Finally, the extremity may fall within the triangle; for example, in a', when the cutting line is Ca', equal in length to the preceding

Joining now a'' and a' with B, we have the triangles Ca''B, Ca'B, which, together with the triangle CaB, have one side, CB, common, which is also one of the sides of the triangle ABC; the sides Ca'', Ca', and Ca, are equal to CA, but the included angles a''CB, a'CB, and aCB, are less than the included angle ACB. Now, we say that the third side of every one of those three triangles is less than the third side, AB, of the triangle ACB.

With regard to the case of the triangle CaB, it is evident that aB is less than AB; but also a'B is less than

AB, for we have seen (TH. 3) that $Ca'+a'B < CA+AB$; and, since $Ca' = CA$, it is also

$$a'B < AB.$$

With regard to $a''B$, we have

$$a''B < a''n + nB;$$

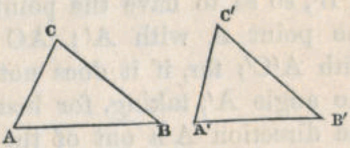
and from the triangle CAn we have

$$CA < An + Cn;$$

hence, $\quad CA + a''B < An + Cn + a''n + nB;$
that is, $\quad CA + a''B < AB + Ca'';$
and, since $CA = Ca''$, we have, also,

$$a''B < AB;$$

that is, the third side, $a''B$, of the triangle $Ca''B$ less than the third side, AB, of the triangle CAB.

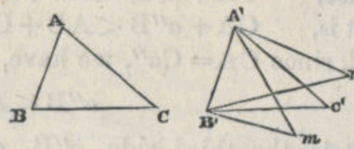


Vice versâ, if the sides CA, CB, of the triangle CAB, are equal to the sides C′A′, C′B′, of the triangle C′A′B′, but the third side, AB, of the first is greater than the third side, A′B′, of the second, the angle C must be greater than C′; for, if it is not greater, it is either equal to or less than C′; but C cannot be equal to C′, for, when two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are identical, and, consequently, we should have AB = A′B′, contrary to the supposition. Nor can the angle C be less than C′; for in this case, according to the preceding demonstration, AB should be less than A′B′; the angle C, therefore, cannot be but greater than C′.

THEOREM VII.

When two triangles have the three sides of the one equal to the three sides of the other, the triangles are equal in all respects.

Let the side AB of the triangle ABC be equal to the side A′B′ of the triangle A′B′C′, and let the sides AC, BC of the former be respectively equal to the sides A′C′, B′C′ of the latter; we say that the two triangles are identical.



For, let us place AB on A′B′, so as to have the point B coinciding with B′ and the point A with A′; AC must necessarily coincide with A′C′; for, if it does not coincide, it will either cut the angle A′, taking, for instance, the direction A′*m* or the direction A′*n* out of the triangle. But, in the first case, the third side B′*m* would be less than B′C′, and, in the second, the third side B′*n* would be greater than B′C′; but the third side is equal to B′C′. When, therefore, AB coincides with A′B′,—that is, B with B′ and A with A′,—C also must coincide with C′, and the whole triangle ABC with A′B′C′.

Observe that, whenever the triangles are identical, the angles opposite to equal sides are equal; *vice versâ*, the sides opposite to equal angles are equal. The use of this remark is frequent.

AB, for we have seen (TH. 3) that $Ca'' + a'B < CA + AB$; and, since $Ca' = CA$, it is also

$$a'B < AB.$$

With regard to $a''B$, we have

$$a''B < a''n + nB;$$

and from the triangle CAn we have

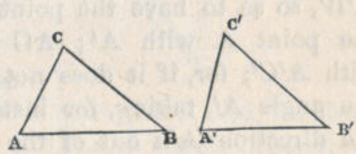
$$CA < An + Cn;$$

hence, $CA + a''B < An + Cn + a''n + nB$;
that is, $CA + a''B < AB + Ca''$;
and, since $CA = Ca''$, we have, also,

$$a''B < AB;$$

that is, the third side, $a''B$, of the triangle $Ca''B$ less than the third side, AB, of the triangle CAB.

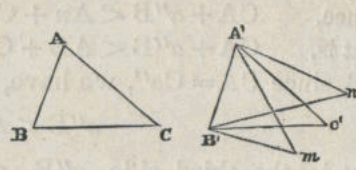
Vice versâ, if the sides CA, CB, of the triangle CAB, are equal to the sides C′A′, C′B′, of the triangle C′A′B′, but the third side, AB, of the first is greater than the third side, A′B′, of the second, the angle C must be greater than C′; for, if it is not greater, it is either equal to or less than C′; but C cannot be equal to C′, for, when two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are identical, and, consequently, we should have AB = A′B′, contrary to the supposition. Nor can the angle C be less than C′; for in this case, according to the preceding demonstration, AB should be less than A′B′; the angle C, therefore, cannot be but greater than C′.



THEOREM VII.

When two triangles have the three sides of the one equal to the three sides of the other, the triangles are equal in all respects.

Let the side AB of the triangle ABC be equal to the side A′B′ of the triangle A′B′C′, and let the sides AC, BC of the former be respectively equal to the sides A′C′, B′C′ of the latter; we say that the two triangles are identical. For, let us place AB on A′B′, so as to have the point B coinciding with B′ and the point A with A′; AC must necessarily coincide with A′C′; for, if it does not coincide, it will either cut the angle A′, taking, for instance, the direction A′*m* or the direction A′*n* out of the triangle. But, in the first case, the third side B′*m* would be less than B′C′, and, in the second, the third side B′*n* would be greater than B′C′; but the third side is equal to B′C′. When, therefore, AB coincides with A′B′,—that is, B with B′ and A with A′,—C also must coincide with C′, and the whole triangle ABC with A′B′C′.

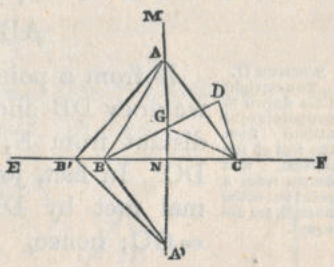


Observe that, whenever the triangles are identical, the angles opposite to equal sides are equal; *vice versâ*, the sides opposite to equal angles are equal. The use of this remark is frequent.

THEOREM VIII.

Two straight lines drawn from any point of a perpendicular to two points of the other line, equidistant from the foot of the same perpendicular, are equal to each other; and vice versâ.

Let the straight line MN be perpendicular to EF, and take on EF two points, B and C, equidistant from the point N of intersection of the two lines: if from any point, A, of the perpendicular we draw to B and C two straight lines, they will be equal to each other. For we have two triangles, ANB, ANC, having the side AN common and BN equal to NC, and, besides, the included angles BNA, CNA also equal; therefore (TH. 4) the triangles are equal, and AB is equal to AC.



Vice versâ, if AB is equal to AC, and BN is equal to CN, AN then must be perpendicular to BC. For the three sides of one triangle being respectively equal to the three sides of the other, their opposite angles are also equal, and, consequently, BNA=ANC; that is, AN is perpendicular to BC.

SCHOLIUM I. The oblique lines drawn from the same point of the perpendicular increase with their distances from the normal.

But if from A we draw AB′ to a point B′, at a greater distance from the foot N of the normal than B is, AB′ is greater than AB. To see it, produce AN to A′, so as to have NA=NA′, and join A′ with B and with B′, we have the triangle AB′A′, and, from a point B within the triangle, two straight lines BA, BA′, drawn to the extremities of the side AA′; therefore, (TH. 3,)

$$AB + BA' < AB' + B'A'.$$

Now, EF is perpendicular to MA′, and A and A′ are two points on MA′ equidistant from the foot N of the perpendicular; therefore,

$$AB = BA', \; AB' = B'A';$$

and, consequently, the first member of the preceding inequality is equal to 2AB, and the second to 2AB′; hence, 2AB < 2AB′, and, therefore,

$$AB < AB'.$$

SCHOLIUM II. Two straight lines drawn to two points equidistant from the foot of the normal, and drawn from a point out of the normal, are unequal.

If from a point D, out of the perpendicular, we draw DB and DC to the two points equidistant from N, then DB will be greater than DC. In fact, joining G, the point of the normal met by DB with C, we will have GB = GC; hence,

$$DB = DG + GC.$$

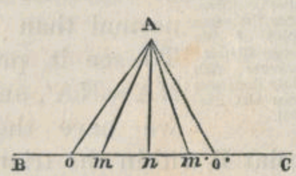
Now, $DC < DG + GC$;

hence, $DC < DB$.

THEOREM IX.

The normal is the shortest line which may be drawn from any point to another line, and the normal is unique.

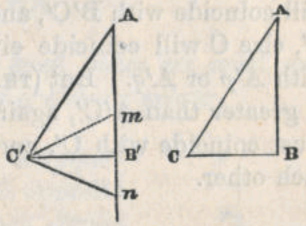
We have seen above that the oblique lines increase with the distance from the normal. Hence, representing by A*n* the normal to BC, and by A*o* and A*m* any two oblique lines, we have A*m* < A*o*; and, if we conceive another oblique line



between Am and An, this will be less than Am, and so on, until we arrive at An; for, beyond An, any straight line drawn from A to BC will be drawn to some point, for example, m', o', &c. equally distant from n as m, o, &c. on the side of B; and, since every one of the oblique lines Am, Ao, &c. is greater than An, so their equals Am', Ao', &c. are likewise all greater than An. And therefore the normal is the shortest line which may be drawn from a given point to another straight line, and it is the only one.

THEOREM X.

Two right-angled triangles having equal hypothenuses and another angle equal, are equal.

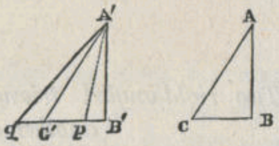


If the hypothenuse AC of the right-angled triangle ABC is equal to the hypothenuse A′C′ of the right-angled triangle A′C′B′, and the angle A of the first triangle is equal to the angle A′ of the second, the two triangles are also equal in the rest. In fact, placing the triangle ACB on A′C′B′, so as to have C coinciding with C′ and A with A′, since the angle A is equal to A′ the side AB also will coincide with the side A′B′, and, besides, the point B will coincide with B′, and, consequently, CB with C′B′. Otherwise, CB will take either the direction C′n or C′m, and then we will have C′m or C′n, together with C′B′, perpendicular to A′B′. But, according to the preceding theorem, there can be only one perpendicular from any point to any straight line. Hence CB coincides with C′B′, and the two triangles are identical.

THEOREM XI.

Two right-angled triangles having equal hypothenuses and another side equal are equal.

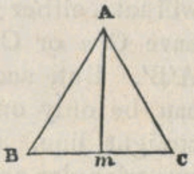
If the right-angled triangles ABC, A′B′C′, have the hypothenuse AC equal to the hypothenuse A′C′ and the side AB equal to the side A′B′, they are equal also with regard to the rest. In fact, the triangle ABC may be placed on A′B′C′, so as to have the point A coinciding with A′ and the point B with B′; and, since B and B′ are right angles, the side BC also will coincide with B′C′, and, besides, C must coincide with C′, else C will coincide either with p or with q and AC with A′p or A′q. But (TH. 8, SCH. I.) A′p is less and A′q is greater than A′C′, against the supposition; therefore C must coincide with C′, and the two triangles are equal to each other.



THEOREM XII.

If from the angle formed by the equal sides of an isosceles triangle we draw a perpendicular to the opposite side, the side and the angle will be divided by it into two equal parts.

From the vertex A of the triangle ABC, having the sides AB, AC equal, draw the perpendicular Am to the opposite side; we will have two right-angled triangles having the hypothenuse AB of one equal to the hypothenuse AC of the other, and



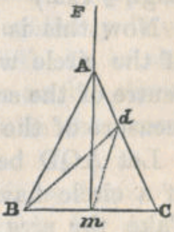
the side Am common to both; hence (TH. 11,) they are identical, and Bm = Cm and BAm = CAm. That is, the perpendicular Am bisects equally the side BC and the angle BAC.

SCHOLIUM. The angles of the isosceles triangle opposite to the equal sides are equal. From the equal triangles AmB, AmC, we have also the angles ABC, ACB equal to each other. Hence, in the isosceles triangle the angles opposite to the equal sides are also equal. In like manner we see that if the triangle would be equilateral the same would have all the angles equal,—that is, the equilateral triangle is also equiangular.

The equilateral triangle is equiangular.

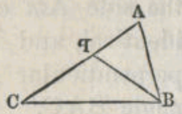
THEOREM XIII.

The sides of a triangle opposite to equal angles are equal, and the sides opposite to greater angles are greater.



When the angles ABC, ACB are equal to each other, the sides AB, AC, opposite to them, are also equal. In fact, divide BC into two equal parts, and from the point m of division draw mF perpendicular to BC; this perpendicular will pass through A. Else let md be the direction of the perpendicular; then, joining d with B, we would have (TH. 8) dB and dC equal, and the angles dBC and dCB also equal; but ABC by supposition is equal to ACB; therefore we would have ABC = dBC, the whole equal to its parts, which involves an absurdity. Hence the bisecting normal can only be mA, and, consequently, AB = AC.

But if the angle ABC is greater than ACB, then the side AC also, opposite to the first angle, is greater than AB, opposite to the second. For, take from the greater angle a portion qBC equal to qCB, then $qC = qB$, and $Aq + qC$ or $AC = Aq + qB$. But $Aq + qB > AB$; hence,



$$AC > AB.$$

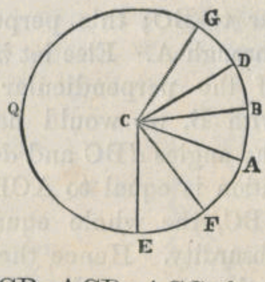
MEASURE OF ANGLES.

How the arcs of circles can be measures of angles.

If two quantities, m and m', are such that when m becomes $2m$ or $3m$, &c. or $\frac{m}{2}$, $\frac{m}{3}$, &c., m' also becomes $2m'$, $3m'$, &c. or $\frac{m'}{2}$, $\frac{m'}{3}$, &c., one of them may be taken as the measure of the other, and this, whatever be the change they undergo. (See Treat. on Alg., § 116.)

Now this is the case with regard to angles and arcs of the circle when the angles have their vertices in the centre of the same circle. Hence one may be used as the measure of the other.

Let AQD be the circumference of a circle having its centre in C. Take the arcs AB, AD, AG, ... in such a manner that we may have



$$AD = 2AB, \; AG = 3AB, \; \&c.$$

Draw to the extremities A, B, D, ... of the arcs the radii CA, CB, CD, &c.; we will have the angles ACB, ACD, ACG, &c.,

increasing in the same manner as the corresponding arcs,—that is,

$$ACD = 2ACB, \ ACG = 3ACB, \ \&c.$$

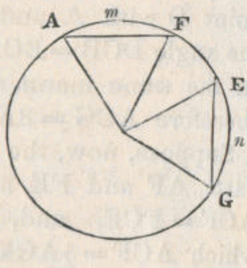
In fact, the sector DCB may be conceived to be turned about the radius CB, so as to make it fall on the sector BCA. Since the arc DB is equal to BA, and both have all their points equidistant from the centre, they must perfectly coincide with each other, and consequently the point D with A and the radius CD with CA; therefore the angle DCB = BCA, and, consequently, ACD = 2ACB. In the same manner, we find that GCD = DCB = BCA; therefore ACG = 3ACB, &c.

Suppose, now, the arc AE to be divided into two equal parts, AF and FE, and draw the radii CE, CF; we have ACF = FCE, and, consequently, ACE = 2ACF, from which ACF = ½ACE; the angle namely corresponding to AF = ½EA is ACF = ½ACE. Divide also the arc AG into three equal parts, AB, BD, DG; the angles ACB, BCD, DCG, are all equal, and, therefore, the angle ACB, corresponding to AB = ⅓AG, is ⅓ACG. In like manner, one-fourth, one-fifth, &c. of any arc will evidently have a corresponding angle one-fourth and one-fifth, &c. of the angle corresponding to the whole arc. The angles, therefore, at the centre increase and diminish as the corresponding arcs.

Vice versâ, the arcs increase and diminish as the corresponding angles; for, if we take DCA, GCA, &c. equal to 2BCA, 3BCA, &c., we will have DCB, GCD, &c. equal to BCA; and, since the sides CA, CB, CD, &c. are all equal, DCB turned about the side CB will have CD coinciding with CA and D with A, and therefore the arc DB with BA. Hence DB = BA. In like manner, GD = DB = BA; and so on. Therefore DA = 2AB, GA = 3AB, &c. But if the angle ACE should be divided into two equal parts by CF, we would find in the same manner that AF=FE, and, consequently, AF=½AE. In like manner, also, from ACG divided into three equal parts by CD, CB, we would find AB=⅓AG, &c. The arc, therefore, and the corresponding angle at the centre, are such quantities that, if one of them increases or diminishes, the other also increases and diminishes in the same manner. The arc, therefore, is the measure of the angle, and *vice versâ*.

Remarks.

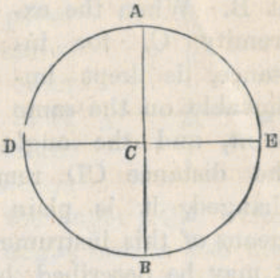
Equal arcs are subtended by equal chords.



We may, from the same principle, remark here, also, that equal arcs are subtended by equal chords; for, when the arc A*m*F is equal to the arc E*n*G, if the one be placed on the other they will perfectly coincide, so as to have the extremities A and F coinciding with E and G. But A and F are the extremities of the chord AF, and E and G the extremities of the chord EG; the two chords, therefore, have the same length,—that is, are equal to each other. *Vice versâ*, when the chords FA, EG are equal, the corresponding arcs also are equal; for, placing AF on EG, so as to have A coinciding with E and F with G, the arc A*m*F must evidently coincide with E*n*G.

Division of the periphery, and value of the angles.

The periphery of the circle is conceived to be divided into 360 equal parts, called *degrees*, and each degree into 60 equal parts, called *minutes*, and each minute again into 60 equal parts, called *seconds*. The manner in which the degrees are expressed is by placing above the number the sign °; the manner in which minutes and seconds are expressed is by placing above the number the signs ′ and ″—the former for the minutes, the latter for the seconds. Thus, for instance, a portion of the circumference embracing 35 degrees 12 minutes and 15 seconds will be represented by 35° 12′ 15″.



Now, if we suppose two diameters, AB, DE, at right angles, the same diameters will cut the circumference into four equal parts, for each angle is measured by an equal arc; but, by dividing the circumference into four equal parts, each part embraces 90°. Hence the measure or value of the right angle is an arc of 90°; and, consequently, the value of two right angles is an arc of 180°, and that of three right angles an arc of 270°. Hence, the right angle is also simply expressed by 90°, two right angles by 180°, and three right angles by 270°.

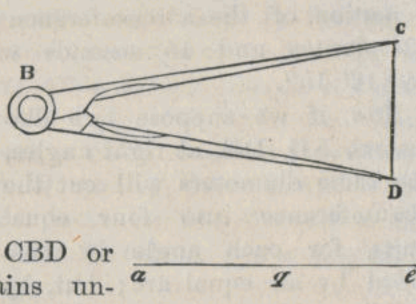
Complement and supplement.

When an arc is less than 90°, what is wanting to complete the 90° is called the *complement* of that arc or of the corresponding angle. Thus, for example, the complement of 60° is 30°. The complement might be taken also negatively when the arc or angle is greater than 90°. Thus, the complement of 110° would be —20°. But it is usually taken in the first manner only. The *supplement* of an arc or angle less than 90° is what is to be added to it to have a semicircle, or 180°; so the supplement of 40° is 140°, and the supplement of 100° is 80°. The supplement also is usually taken positively,—that is, concerning arcs less than 180° only.

PROBLEMS.

Remarks concerning the dividers and the ruler.

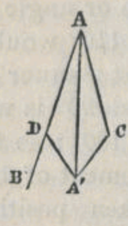
The dividers and the ruler are the two indispensable instruments for drawing plane figures. The dividers consist of two legs movable around a joint at B. When the extremity C, for instance, is kept immovable on the same point, and the angle CBD or the distance CD remains unchanged, it is plain that by



means of this instrument a circumference or a portion of it may be described, being, namely, traced out by the movable extremity D while the other extremity C occupies the centre. The same instrument may be used to measure distances or to cut off from a straight line a portion of it equal to another given line. For instance, open the dividers in such a manner as to have the extremities C and D coinciding with the extremities of the straight line CD; then apply the extremity D of the dividers, thus opened, to the extremity *a* of the straight line *ae*, the other extremity C of the dividers will fall on *q*, for instance, and *aq* is evidently equal to CD.

PROBLEM I. Bisect a given angle equally.

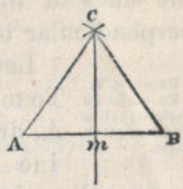
CAB is a given angle to be divided into two equal parts. Take with the dividers AD = AC; then, opening the dividers at pleasure, describe an arc of a circle having the centre in D, and with the same radius describe another arc intersecting the first and having its centre in C: the point A′ of intersection is evidently equally distant from D and from C. Join now A′ with D, with A, and with C; the straight line AA′ divides DAC into two equal parts. In fact, the triangles AA′D, AA′C, have the sides AD = AC, DA′ = CA′, and AA′ common; their angles, therefore, (TH. 7,)



are also equal, and DAA′ opposite to DA′ is equal to CAA′ opposite to CA′.

Problem II. To bisect a given straight line.

AB is a given line to be divided into two equal parts. Describe with the same radius two arcs of circle intersecting each other and having their centres in A and in B, the extremities of the given straight line; then join the point of intersection C with A and with B; bisect then the angle ACB. The straight line C*m*, which bisects the angle, bisects also the opposite side or the given AB. In fact, the triangles CA*m*, CB*m*, besides the common side C*m*, have the side CA equal to the side CB, and the included angles AC*m*, BC*m* also equal; therefore they are identical, and A*m* = *m*B.

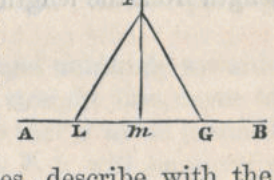


Problem III. From a point given out of a straight line to draw a perpendicular to the same line.

O is a point out of the straight line CB; that is, out of its direction. Draw from this point a perpendicular to CB. Making O the centre of a circle, describe an arc cutting CB in *m* and *n*; then join O with *m* and with *n*, and bisect the angle *m*O*n*; we will have as above the triangles O*mp*, O*np* equal, and consequently the angle O*pm* equal to O*pn*,—that is, O*p* perpendicular to CB.

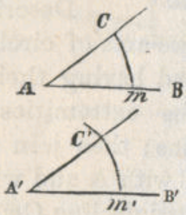
Problem IV. To erect a perpendicular at any point of a given line.

Take on the straight line AB any point *m*. To erect a perpendicular to AB from *m*, take on both sides of *m* two parts of the line, *m*L and *m*G, equal to each other; then, making L and G centres of circles, describe with the same radius two arcs intersecting each other in F; join F with *m*: the straight line F*m* is the required perpen-



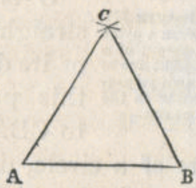
dicular. Because, if we complete the triangles FLm, FGm, the two triangles have all the sides of one equal to the sides of the other, and FmG = FmL; hence, Fm is perpendicular to AB.

PROBLEM V. To make an angle equal to another given angle.



Let BAC be a given angle. To make another angle equal to A, draw the indefinite straight ine A′B′, and then with the same radius describe two arcs of circle, one having the centre in A and the other in A′; then take with the dividers the distance of the two points m and C,—that is, the length of the chord subtending Cm,—and apply this length to the arc m'C′. Now, arcs subtended by equal chords are equal; hence, mC = m'C′, and consequently the angles CAB, C′A′B′, measured by the same arcs, are also equal.

PROBLEM VI. To describe an equilateral or an isosceles triangle on a given straight line.



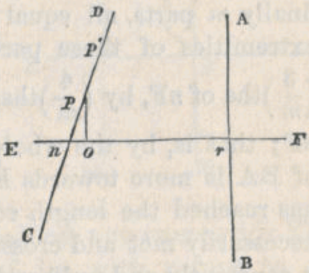
AB is a given straight line, on which an equilateral triangle is to be constructed. With a radius equal to the given line, describe two arcs of circle intersecting each other at C, the first having the centre in B, and the second in A; then complete the triangle CAB; and since both CA and CB are equal to AB, the triangle is equilateral. In like manner an isosceles triangle may be described on AB by taking the radius of a different length from the length of AB.

BOOK II.

PARALLEL AND PROPORTIONAL LINES.

THEOREM I.

When one of two lines is perpendicular to a third line, and the other is oblique, the two lines must necessarily meet each other.



LET the straight line AB be perpendicular to EF, and let another line DC be in any way oblique to EF, the two lines will somewhere meet each other. In fact, supposing the upper part nD of CD to be inclined towards F, it will constantly and uniformly approach rA and go beyond it; for, since rA is perpendicular to EF, it is neither inclined towards E nor towards F; that is, none of its points deviate on either side; hence, nD cannot go constantly and uniformly towards F without approaching in an equal manner rA and passing beyond it.

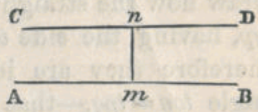
We say that nD goes constantly and uniformly towards F; for it is of the nature of the straight line never to deviate from the same direction, so that if nD is inclined with any part np of itself towards F, it will be equally inclined with any other part of the same np indefinitely produced towards D. Now, to say that np is inclined towards F, means that p is more towards F than n. Let o be a point taken on nF, more towards F than n, and exactly as much more as p is. But no is an aliquot part of nF, for instance, one-fifth or one-tenth; or, more generally, let no be equal to $\left(\frac{1}{m}\right)$th of nF; the point p, therefore, of nD, is more towards F than n, by $\left(\frac{1}{m}\right)$th of nF.

Now, if along nD we take $pp'=np$, on account of the constant and uniform proceeding of the straight line in the same direction, the point p' is more towards F than n and twice as much as p is; that is, the point p' is more towards F than n by $\left(\frac{2}{m}\right)$ths of nF. It is plain that if we take in like manner, on nD, three parts and then four, and finally m parts, all equal to np, we will have the upper extremities of these parts more towards F than n by $\left(\frac{3}{m}\right)$ths of nF, by $\left(\frac{4}{m}\right)$ths of nF, and, finally, by $\left(\frac{m}{m}\right)$ths of nF; that is, by the whole nF. Now, none of the points of BA is more towards F than r is; therefore, before nD has reached the length equal to m times np, it must have necessarily met and crossed rA, after having approached it constantly and uniformly.

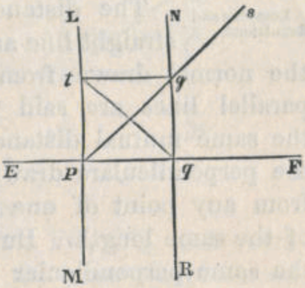
THEOREM II.

The straight line which is vertical to one of two parallels is vertical to the other also; and when two straight lines are perpendicular to a third line they are parallel to each other.

We have said (INTR. ART. 4) that when mn, perpendicular to AB, being brought along at different points of AB, touches with the same extremity n invariably CD, the two



lines AB, CD are parallel to each other and will never meet to form an angle. We add now that *nm*, which is perpendicular to AB, must be perpendicular to CD also, for otherwise C*n*D would be an oblique line to *mn*, and would somewhere meet AB normal to the same *mn*. But if CD somewhere meets AB, the lines AB and CD are no longer parallel; against the supposition, therefore, the perpendicular to one of two parallels must be perpendicular to the other also.



But if two straight lines, LM and NR, are perpendicular to another straight line, EF, they are parallel to each other. Take, in fact, any point *g* in NR, and draw from *p*, *pgs*. Since LM and NR are both perpendicular to EF, their relative position with regard to EF is the same for both, and also with regard to another line making any angle with EF. Hence, the relative position of L*p* with regard to *pg* is the same as that of N*g* with regard to *gs*,—that is, L*pg* = N*gs*. But N*gs* = *pgq*; hence,

$$Lpg = pgq.$$

Draw now from *g* the perpendicular *gt* to LM: we have two right-angled triangles having the hypothenuse common, and, besides, the angle *pgq* of the one equal to the angle *tpg* of the other. Hence, the triangles are identical, and

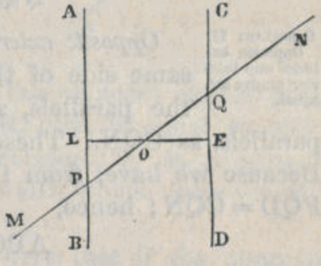
$$tp = gq,\ gt = pq.$$

Draw now the straight line *tq*; we have two triangles *tqg*, *tqp*, having the side *tq* common and $tg=pq$ and $tp=gq$; therefore they are identical, (B. I. TH. 8,) and the like angle $tgq=tpq$,—that is, of the same length of *pq*, and, like *pq*, perpendicular at once to LM and to NR. The same could be demonstrated with regard to any other line drawn perpendicular to LM from any point of NR. Hence, *pq* brought along different points of NR, always perpendicular to the same NR, would invariably touch, with the extremity *p*, the other line LM; therefore the two lines are parallel to each other.

Remarks and definitions. The distance between any point out of a straight line and the line itself is measured by the normal drawn from that point to the line. Now, parallel lines are said to be such as keep everywhere the same mutual distance; which is the same as to say the perpendiculars drawn from the different points, and from any point of one of them to the other, must be of the same length. But, in this case, we have seen that the same perpendicular is common to both. We may, therefore, apply to parallel lines the following definition:—*Parallel lines are those which have a common vertical.* In fact, according to the preceding demonstration, the part of this vertical contained between them is everywhere of the same length.

THEOREM III.

If a straight line meets two parallel lines, the alternate angles made by it are equal to each other.



Let AB, CD be two parallel lines, and MN any other line intersecting them at P and Q. The angles APQ, PQD, or BPQ, PQC, are called *alternate angles*. Those, namely, are the alternate angles which lie on different sides of the secant line within the parallels. Now, the alternate angles are equal to each other.

Divide PQ into two equal parts, PO, OQ, and from O draw OL perpendicular to AB; the same OL produced to E is perpendicular to CD also. Hence, we have the right-angled triangles OEQ, OLP, having the hypothenuse OP of one equal to the hypothenuse OQ of the other, and the angle LOP of the first equal to the angle QOE of the second; therefore, the triangles are equal, and LPO = OQE; that is,

$$APQ = PQD.$$

Now, (B. I. TH. 1,)

$$APQ + QPB = 2r,$$

$$PQD + PQC = 2r;$$

hence, $APQ + QPB = PQD + PQC.$

But $APQ = PQD;$

therefore, $QPB = PQC.$

COROLLARY I. Alternate exterior angles are equal.

The angle CQN and its opposite MPB are called *alternate exterior* angles, and also APM, DQN. Now, CQN = PQD, MPB = APQ. But PQD and APQ are equal to each other; therefore,

$$CQN = MPB.$$

We prove in like manner that

$$NQD = APM.$$

COROLLARY II. Opposite exterior and interior angles are equal.

Opposite exterior and interior angles lie on the same side of the secant line,—the one within the parallels, as APQ, the other out of the parallels, as CQN. These angles are equal to each other. Because we have, from the theorem, APQ = PQD; but PQD = CQN; hence,

$$APQ = CQN.$$

COROLLARY III. The sum of the interior angles on the same side is equal to two right angles.

The angles APQ and PQC, or BPQ and PQD, are the *interior angles on the same side* of the secant. Now, the sum of these interior angles is equal to two right angles. In fact, CQN + CQP = $2r$; but from the last corollory CQN = APQ; hence,

$$APQ + PQC = 2r.$$

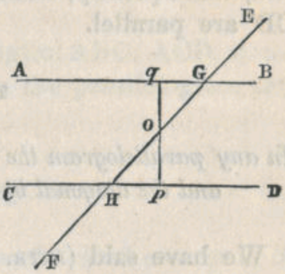
In like manner we would obtain

$$BPQ + PQD = 2r.$$

THEOREM IV.

If two straight lines meet a third line, making the alternate angles equal, the straight lines are parallel.

Let AB and CD be two straight lines met by another, EF, making, with them, the alternate angles AGH, GHD



equal: the two lines are parallel. In fact, divide GH into two equal parts in o, and from o draw oq perpendicular to AB, and produce it on the other side till it meets CD in p. Thus, we have two triangles oqG, opH, equal to each other. For the side oG and the adjacent angles qoG, qGo of the one are equal to the side oH and the adjacent angles poH, oHp of the other. Hence, opH, also, is equal to oqG; but oqG is a right angle; therefore, qp is a perpendicular common to AB and to CD, which, consequently, are parallel to each other.

COROLLARIES. The straight lines are parallel:— 1st. When the alternate exterior angles are equal.

We may observe here that if the alternate exterior angles EGB, CHF are equal, AB and CD are parallel lines; for, when the alternate exterior angles are equal, the alternate interior angles also are equal.

2d. When the sum of the internal angles is equal to two right angles.

But if $BGH + GHD = 2r$, since $BGH + HGA = 2r$, we have

$$BGH + GHD = BGH + HGA;$$

hence, $$GHD = HGA;$$

That is, the alternate interior angles are equal, and, consequently, AB and CD parallels.

3d. When opposite exterior and interior angles are equal.

When the opposite exterior and interior angles BGH, DHF are equal, then, from

$$BGH = DHF,$$

we have $$BGH + GHD = DHF + GHD;$$

but $$DHF + GHD = 2r;$$

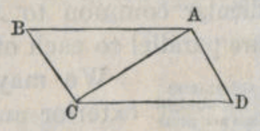
hence, $$BGH + GHD = 2r;$$

and, consequently, from the preceding corollary, AB and CD are parallel.

THEOREM V.

In any parallelogram the opposite sides and angles are equal, and the diagonal bisects equally the parallelogram.

We have said (INTR. ART. 5) that a parallelogram is a quadrilateral figure having equal opposite and parallel sides; but, from the fact that the opposite sides are parallel, it follows that the same sides must be equal and the opposite angles also equal.



Let ABCD be any parallelogram: drawing the diagonal AC, we will have the triangles ACD, ACB equal; for, besides the common side AC, the adjacent angle DAC of the one is equal to the adjacent angle ACB of the other, because alternate angles between the parallels AD and CB. And the adjacent angle ACD of the first is equal to the adjacent angle CAB of the second, because they are alternate angles between the parallels AB and CD. Hence, the two triangles are equal, (B. I. TH. 5,) and, therefore, since the sides opposite to equal angles are equal,

$$AB = DC,\ AD = BC.$$

Moreover, the angles opposite to the common side AC are equal; that is,

$$B = D.$$

Again, since the angles DAC and CAB are respectively equal to the angles BCA and ACD, we have also DAC + CAB = BCA + ACD; that is,

$$DAB = BCD.$$

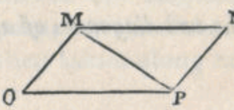
In any parallelogram, therefore, the opposite sides and angles are equal.

From the equality of the triangles ABC, ACD, it follows also that the diagonal bisects the parallelogram into two equal parts.

THEOREM VI.

When the opposite sides of a given quadrilateral are equal, the quadrilateral is a parallelogram.

Let the opposite sides of the quadrilateral NO be equal; that is, MO to NP and MN to OP: the same opposite sides are parallel. For, draw the diagonal MP; we have two triangles having

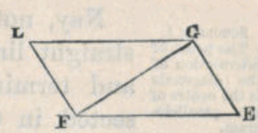


all the sides of one equal to the sides of the other,—that is, MP common, MN = OP and MO = NP; hence, (B. I. TH. 7,) the angles opposite to equal sides are respectively equal in both triangles; that is, O = N and MPO = PMN, OMP = MPN. But alternate angles are equal between parallel lines; therefore MN is parallel to OP and MO is parallel to NP; that is, the quadrilateral NO is a parallelogram.

THEOREM VII.

When two parallel lines are equal, and their corresponding extremities are joined by two other lines, the resulting quadrilateral is a parallelogram.

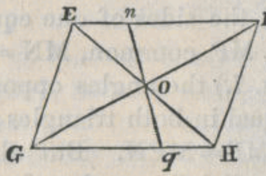
Let LG, FE be two parallel and equal lines. Joining L with F, and G with E, and G with F, we have the triangles FGL, GFE equal to



each other. Because GF is common, GL = EF, and the included angle LGF of the first triangle is equal to the included angle GFE of the second, for they are alternate angles between the parallels LG, FE; hence, (B. I. TH. 4,) the two triangles are equal, and, consequently, the angle LFG also is equal to FGE; hence, LF, GE are parallel, and the quadrilateral LE is a parallelogram.

THEOREM VIII.

The two diagonals of any parallelogram cut each other into two equal parts.



Let EGHF be any parallelogram, and EH, GF its diagonals: the point O of intersection divides both of them equally. In fact, observe, first, that the triangles FOE, GOH are equal to each other, because EF = GH, and the adjacent angles OEF, OFE of the first triangle are respectively equal to the adjacent angles OHG, OGH of the second, being alternate angles between parallels. Hence, the sides opposite to equal angles are also equal; that is,

$$EO = OH, FO = OG;$$

that is, the diagonals in the parallelogram bisect each other.

SCHOLIUM I. The point of intersection of the diagonals is the centre of the parallelogram.

Nay, not only the diagonals, but any other straight line *nq* passing through the point O and terminated by the opposite sides, is bisected in O. For, taking, for instance, the triangles OnF, OqG, we have the side OF of the one equal to the side OG of the other, and the angles nFO, nOF, adjacent to OF, equal to the angles qGO, qOG, adjacent to OG, because nFO and OGq are alternate angles, and nOF, qOG opposite angles; hence, the two triangles are equal, and

$$On = Oq.$$

Any straight line, therefore, passing through the point of intersection of the diagonals, and terminated by the opposite sides of the parallelogram, is bisected in that point. Hence, the point of intersection of the diagonals is called the *centre* of the parallelogram, for its distance from the opposite sides is the same when taken along any straight line.

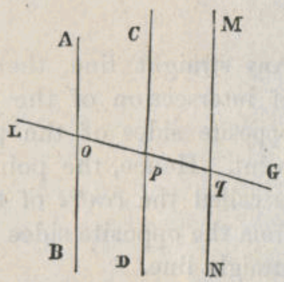
SCHOLIUM II. If the parallelogram is a square or a rectangle, the diagonals are equal.

If the parallelogram has its angles right,—that is, if it is a rectangle or a square,—the diagonals are, in this case, equal to each other. For,

drawing the diagonals AD, BC, we will have the triangles ADC, BCD: the first right-angled in C, the second in D; having, besides, the common side CD, and the side AC of the first equal to the side BD of the second. Hence, the two remaining sides, which are the diagonals, are also equal.

THEOREM IX.

When two straight lines are separately parallel to a third line they are parallel to each other.

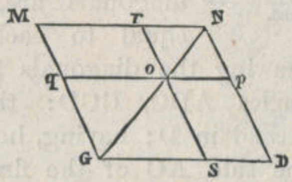


Let the straight line CD be parallel to MN, and also the straight line AB parallel to the same MN: we say that AB and CD are parallel to each other. Draw, in fact, LG cutting the three lines in *opq*: we have C*p*G = M*q*G, and A*o*G = M*q*G; therefore,

$$\text{C}p\text{G}' = \text{A}o\text{G};$$

and, consequently, (TH. IV. COR. 3,) the two straight lines AB, CD are parallel to each other.

COROLLARY. When from any point of the diagonal of a parallelogram we draw parallels to the sides, we have four parallelograms, and two of them equivalent.



Draw from any point *o* of the diagonal NG, *rs* parallel to ND, and *qp* parallel to MN. Since MG, also, is parallel to ND, and GD to MN, the two, *rs* and *qp*, will be respectively parallel to the same MG and GD. Hence, the parallelogram MD is divided by *qp* and *rs* into four parallelograms—M*o*, *rp*, *o*D, *qs*; two of which, namely, M*o*, *o*D, are equivalent to each other. Because the diagonal NG divides the parallelogram MD into two equal triangles, and likewise the two parallelograms *rp*, *qs*; hence,

$$\text{GMN} = \text{GND};$$

and $$\text{G}oq = \text{G}os, \quad \text{N}or = \text{N}op.$$

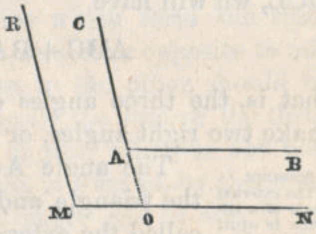
But $\quad GMN = Goq + Nor + oM,$
and $\quad GND = Gos + Nop + oD;$
hence, $\quad Goq + Nor + oM = Gos + Nop + oD;$

and, taking the equal terms from both members,

$$oM = oD.$$

THEOREM X.

When two straight lines, forming an angle, are parallel to two other lines, these make an angle equal to that formed by the first.



The straight lines CA and AB, which form the angle A, are respectively parallel to the straight lines RM, MN. Hence, the angles CAB, RMN are equal to each other. For, produce CA to o, we will have at once,

$$CoN = CAB, \ CoN = RMN;$$

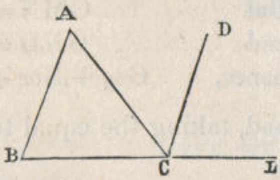
hence, $\quad CAB = RMN.$

THEOREM XI.

The sum of the three angles of any triangle is equal to two right angles.

By means of the parallels we may find that the sum of the three angles of any triangle is equal to two right angles.

Let, in fact, ABC be any triangle; produce the base BC to L, and from C draw CD parallel to AB: we have $DCL + DCB = 2r$; or,



$$DCL + DCA + ACB = 2r.$$

Now, DCL and ACD are respectively equal to ABC and CAB; for DCL and ABC are opposite exterior and interior angles, made by the parallels AB, DC, and the secant BL; the angles ACD and CAB are alternate angles, made by the same parallels and the secant AC. Substituting, therefore, in the first number of the preceding equation, BAC and ABC instead of their equal angles ACD and DCL, we will have

$$ABC + BAC + ACB = 2r;$$

that is, the three angles of any triangle taken together make two right angles, or their sum is equal to 180°.

SCHOLIUM. The external angle of a triangle is equal to the two opposite internal.

The angle ACL, formed by the side AC of the triangle and another side BC produced, is called the external angle. But $ACL = ACD + DCL$; hence, also,

$$ACL = CAB + ABC;$$

that is, any external angle of a triangle is equal to the two opposite internal.

COROLLARY I. In any triangle there can only be one right or one obtuse angle.

From the preceding theorem it follows, first, that in any triangle there cannot be more than one either obtuse or right angle; for, if we suppose two obtuse or two right angles, in both cases the sum of the three angles would exceed 180°; hence, when one of the angles of a given triangle is either an obtuse or a right angle, the other two are both acute angles.

COROLLARY II. When two of the angles of any triangle are known, the third angle may be inferred from them.

Now, if two of the angles A, B, C of any triangle are known, or even the sum $A+B$ of two of them, the third angle C may be easily inferred; for

$$A+B+C=180°;$$

hence, $$C=180°-(A+B).$$

COROLLARY III. The values of the equal angles of a right-angled isosceles triangle, and of an equilateral triangle, are always the same.

It is well known that the angles of a triangle opposite to equal sides are equal, and consequently the isosceles triangle has two angles equal to each other, and the equilateral triangle has all its angles equal. Hence, the equal angles of a right-angled isosceles triangle are each measured by an arc of 45°. For, observe first that the equal sides of a right-angled isosceles triangle must be the sides which form the right angle, otherwise the right angle would be opposite to one of them, and the angle opposite to the other should be also a right angle, and consequently the sum of the three angles greater than two right angles, which is not possible. The right angle, therefore, is formed by the equal sides. Call, now, A the right angle and B and C the other two; we will have from $A+B+C=180°$, and from $A=90°$, $B+C=90°$.

But $B=C$; hence, $B+C=2C=2B$;

hence, $$2C=2B=90°;$$

that is, $$C=B=45°.$$

When the triangle is equilateral, and consequently equiangular, we have

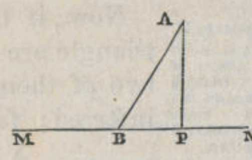
$$A+B+C=3A=3B=3C=180°;$$

and, therefore,

$$A=B=C=60°.$$

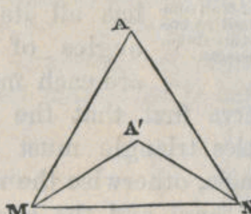
The measure, namely, of any angle of the equilateral triangle is 60°.

COROLLARY IV. When one straight line makes two unequal angles with another straight line, and from any point of the former we draw a perpendicular to the latter, the perpendicular must fall on the side of the acute angle.



Let the angle MBA which AB makes with MN be an obtuse angle, and consequently ABN an acute one. If from any point A of BA we draw AP perpendicular to MN, this must fall on the side of the acute angle; for should it fall on the side of the obtuse angle, the triangle formed by AB, the normal, and a part of MN, would have the sum of its angles greater than two right angles.

COROLLARY V. When two triangles have a common side, and the angle of the one opposite to this side is within the other triangle, the same angle is greater than the other opposite angle.



Let MN be the common side of the two triangles AMN, A′MN; of the two angles A and A′ opposite to it, the latter is greater than the former; for

$$A + AMN + ANM = 180°,$$
$$A' + A'MN + A'NM = 180°;$$

hence, $A + AMN + ANM = A' + A'MN + A'NM.$

Now, $A'MN = AMN - AMA'$, $A'NM = ANM - ANA'$; therefore,

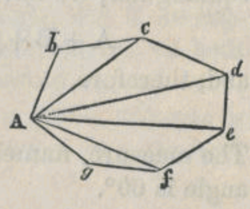
$$A + AMN + ANM = A' + AMN + ANM - (AMA' + ANA'),$$

and, consequently,

$$A = A' - (AMA' + ANA');$$

that is, $A < A'$.

COROLLARY VI. The internal angles of a polygon are equal to two right angles as many times as there are sides in the polygons, minus two.

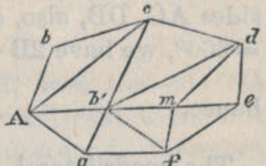


The polygon ABC . . . . may be divided into as many triangles as there are sides in the polygon, minus two. For, drawing from A the diagonals A*c*,

Af, these two diagonals with four sides cb, bA and fg, gA of the polygon form two triangles; that is, two triangles less than the number of sides. But, drawing Ad, Ae to the other angles, we have evidently as many triangles as there are remaining sides of the polygon. The polygon, therefore, can be divided into as many triangles as there are sides in it minus two, so that if the number of sides is n, the number of triangles will be $n-2$. Observe, now, that the internal angles of the polygon embrace all the angles of the triangles and no more than them; therefore, since the sum of the angles of any triangle is equal to 180°, and the number of triangles in a polygon of n sides is $n-2$, the sum of the internal angles of the polygon is

$$(n-2)\,180^\circ;$$

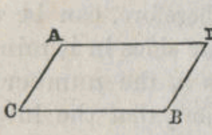
that is, as many times two right angles as there are sides in the polygon minus two.



We have supposed every one of the internal angles of the given polygon to be less than 180°, and consequently the external angles all greater than 180°. But let the triangle Acb, formed by the sides Ab and bc of the same given polygon and the diagonal Ac, be turned about Ac so as to take the position $Ab'c$ on the plane of the polygon. Thus we have the polygon $Ab'cd$ . . in which the internal angle b' is greater than 180°. Now, drawing from b' the diagonals $b'g$, $b'f$, &c., we have the polygon $Ab'c$ . . . divided into the same number of triangles in which Abc . . . . has been divided, and therefore the sum of the internal angles is the same in both. It is plain, likewise, that if we join, for instance d with m, a point on the diagonal $b'e$, and f also with m, so that the internal angle fmd also be greater than 180°, the number of triangles remains the same, and the same consequently the sum of the internal angles. The polygon, therefore, may contain several internal angles greater than 180°, the sum of all, however, remaining equal to $(n-2)\ 180°$.

COROLLARY VII. A quadrilateral in which two of the opposite angles are equal, and the other two also equal to each other, is a parallelogram.

The sum of the angles of any quadrilateral is therefore $2 \cdot 180°$ or 360°. Now, let the angle A be equal to its opposite B and D equal to C; we will have



$$A + B + C + D = 2A + 2C = 360°;$$

hence, $$A + C = 180°.$$

But when the sum of the internal angles made by a straight line, AC, with two other lines, is 180°, the two lines (B. II. TH. 4. COR. 2) are parallel; hence, the sides AD and CB of the quadrilateral are parallel. But the sides AC, DB, also, are parallel; for, from $A + B + C + D = 360°$, we have $2B + 2C = 360°$;

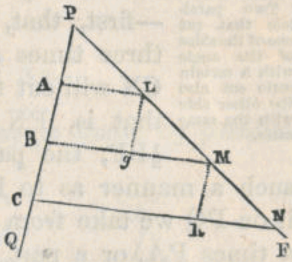
hence, $$B + C = 180°.$$

The quadrilateral, hence, is a parallelogram.

THEOREM XII.

Parallel lines cutting equally one of the sides of an angle cut equally also the other side.

Divide the side PQ of the angle QPF into any number of equal parts PA, AB, BC, &c., and from each point of division draw AL, BM, CN, &c. all parallel to one another; the sections PL, LM, MN, &c. of the side PF made by these parallels are all equal.



Draw, in fact, from L, Lg parallel to PQ; we have two triangles, gLM, APL, equal to each other: for Lg = AB and AB = AP; hence,

$$AP = Lg;$$

Lg and gM, being respectively parallel to PA and AL, form equal angles,—that is,

$$PAL = LgM.$$

Moreover, the opposite exterior and interior angles made by PN and the parallels PA, Lg are also equal, namely,

$$APL = gLM.$$

The triangles, therefore, PAL, LgM, have the side PA equal to the side Lg, and the angles adjacent to the first equal to the angles adjacent to the second; therefore, they are equal also in the rest,

and $$PL = LM.$$

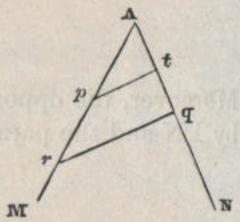
We prove, in like manner, with the triangles PAL, MhN,

that $$PL = MN;$$

or, with the triangles LgM, MhN, that LM = MN, &c. Hence, parallel lines that cut equally one of the sides of an angle cut also equally the other side.

COROLLARY I. Two parallels that cut one of the sides of the angle with a certain ratio cut also the other side with the same ratio.

It is easy to see, from the preceding theorem,—first, that, taking on PQ, for instance, PC three times as great as PA, the parallels AL, CN will cut the side PF with the same ratio,—that is, PN = 3PL, or, taking PA equal to $\frac{1}{2}$BP, the parallels AL, BM will cut PM in such a manner as to have PL = $\frac{1}{2}$PM; and, in general, if on PQ we take from P a part equal to two, three, ... n times PA, or a part equal to one-half, one-third, ... one-nth of PA, and from A and the different points in which these parts terminate we draw parallels, the sections of the other side PF will give the same ratio as those of the first side.



Let now tp and qr be any two parallels cutting the sides of the angle MAN; we say that the ratio $\frac{Aq}{At}$ of the segments of the side AN is equal to the ratio $\frac{Ar}{Ap}$ of the segments of the side AM. In fact, should Aq be twice three times, &c. as long as At, or one-half, one-third, &c. of At, in like manner Ar would be twice, three times, &c. as long as Ap, or one-half, one-third, &c. of Ap. So that when the ratio $\frac{Aq}{At}$ becomes equal to 2, 3, ... or equal to $\frac{1}{2}$, $\frac{1}{3}$, ... the ratio $\frac{Ar}{Ap}$ also becomes 2, 3, ... $\frac{1}{2}$, $\frac{1}{3}$; .... but when two quantities increase and decrease together in this manner, whatever be the value given to one, the same value is to be given to the other also; (Treat. on Alg., § 116;) hence, whatever be the ratio $\frac{Aq}{At}$ of the segments of the side AN, we will always have

$$\frac{Aq}{At}=\frac{Ar}{Ap};$$

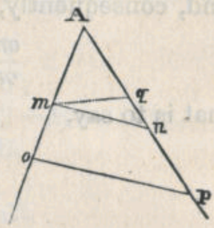
that is,

$$Aq : At :: Ar : Ap.$$

The segments of the sides of an angle made by parallel lines are proportional.

COROLLARY II. *Vice versâ*, the straight lines that cut the sides of an angle proportionally, are parallel lines.

Vice versâ, suppose mn and op to cut the sides of the angle A in such a manner as to give



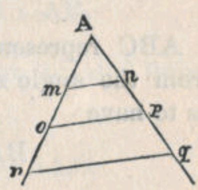
$$\frac{Ap}{An}=\frac{Ao}{Am},$$

or, $Ap : An :: Ao : Am;$

the two straight lines mn and op are parallel. For, if mn is not parallel to op, let the parallel to op be mq: then $\frac{Ao}{Am}=\frac{Ap}{Aq}$; but by supposition $\frac{Ao}{Am}=\frac{Ap}{An}$; hence, also, $\frac{Ap}{An}=\frac{Ap}{Aq}$; that is, $An=Aq$; which is impossible. No other line, therefore, drawn from m, is parallel to op, except mn.

COROLLARY III. The segments of the sides of any angle between parallel lines are proportional.

Let now the sides of the angle A be met by three parallel lines mn, op, rq; we will have



$$Ao : Am :: Ap : An,$$
$$Ar : Ao :: Aq : Ap.$$

But, from the doctrine of proportions, (Treat. on Alg., § 119,) the two preceding give the others:

$$Ao - Am : Ao :: Ap - An : Ap,$$
$$Ar - Ao : Ao :: Aq - Ap : Ap;$$

that is,

$$om : Ao :: pn : Ap,$$
$$ro : Ao :: qp : Ap,$$

from which we infer

$$\frac{om}{pn} = \frac{Ao}{Ap}, \quad \frac{ro}{qp} = \frac{Ao}{Ap};$$

and, consequently,

$$\frac{om}{pn} = \frac{ro}{qp}, \text{ and } \frac{om}{ro} = \frac{pn}{qp};$$

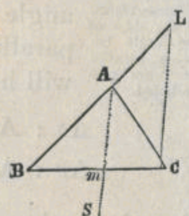
that is to say,

$$om : pn :: ro : qp,$$
$$om : ro :: pn : qp.$$

The portions, namely, of the sides of any angle between parallel lines are proportional.

THEOREM XIII.

When a straight line bisects equally one angle of a triangle, it cuts the opposite side into two segments proportional to the other sides, and vice versâ.



ABC represents any triangle. Draw from the angle A, AS in such a manner as to have

$$BAS = SAC;$$

the same straight line cuts the opposite side BC in two parts, Bm, mC, which form a proportion with the sides AB, AC. In fact, produce BA to L, and let AL = AC. Join L with C: we will have (B. II. TH. 11, SCH.)

$$BAC = ALC + ACL.$$

Now, $ALC = ACL$;
hence, $ALC + ACL = 2ALC$,
and, $BAC = 2BAS$.
Therefore, $2BAS = 2ALC$,
or, $BAS = ALC$.

Hence, (B. II. TH. 4, COR. 3,) AS, LC are parallel lines; and, consequently,

$$BL : BA :: BC : Bm;$$

from which $BL - BA : BA :: BC - Bm : Bm$;
or, $AL : AB :: mC : mB$.
Now, $AL = AC$;
hence, $AC : AB :: mC : mB$.

Vice versâ, when AS cuts BC into two segments proportional to the adjacent sides, the same AS bisects equally the angle A. For, producing BA to L, so as to have $AL = AC$, and joining L with C, we will have

$$Bm : mC :: BA : AL;$$

and, also, $Bm + mC : mC :: BA + AL : AL$.
That is, $BC : mC :: BL : AL$;
or, $BC : BL :: mC : AL :: mB : AB$.

Hence, (B. II. TH. 12, COR. 2,) Am, LC are parallel lines; and, consequently,

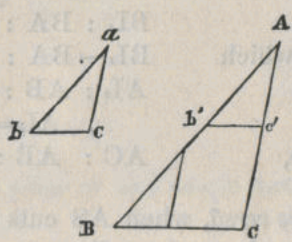
$$BAm = ALC, \text{ and } mAC = ACL = ALC.$$

Hence, $BAm = mAC$.

THEOREM XIV.

When the three angles of one triangle are equal to the three angles of another triangle, the sides of the two triangles are proportional, and vice versâ.

When the angles a, b, c of the triangle bac are respectively equal to the angles A, B, C of the triangle BAC, the two triangles are called *similar*, and the sides opposite to equal angles are proportional. In fact, since $a = A$, placing ab on AB so as to have the point a coinciding with A, the side ac also must coincide with AC, and let b', c' be the points of coincidence of b and c with AB and AC: the triangles then $Ab'c'$, Abc are identical, and b and c being respectively equal to B and C, we will have



$$b' = B,\ c' = C;\ \text{hence, } BC,\ b'c' \text{ are parallel lines;}$$

and, consequently, $AB : AC :: Ab' : Ac'$;
or, $AB : AC :: ab : ac$.

In like manner, if the sides of the angle abc are made to coincide with the sides of the angle ABC, we find

$$BA : BC :: ba : bc$$

From these two proportions we have the equations $AB \cdot ac = AC \cdot ab$, $BA \cdot bc = BC \cdot ba$, and, consequently,

$$AB = \frac{AC}{ac} ab;\ BA = \frac{BC}{bc} ba;$$

from which $$\frac{AC}{ac} = \frac{BC}{bc};$$

and $$\frac{AC}{BC} = \frac{ac}{bc};$$

or, $$AC : BC :: ac : bc.$$

The sides, therefore, of similar triangles opposite to the equal angles are proportional.

Vice versâ, when the sides of two triangles are proportional, the triangles are similar. For, supposing the proportions

$$ab : ac :: AB : AC,$$
$$ab : bc :: AB : BC.$$

Take on AB a segment $Ab' = ab$, and from b' draw $b'c'$ parallel to BC: we will have

$$Ab' : Ac' :: AB : BC.$$

Hence, $$Ab' : Ac' :: ab : ac;$$

or, $$Ab' \cdot ac = Ac' \cdot ab.$$

Now, $Ab' = ab$; hence, also, $Ac' = ac$. But $Ab'c'$ is simi lar to ABC; hence,

$$Ab' : b'c' :: AB : BC;$$

and, since $$AB : BC :: ab : bc,$$
$$Ab' : b'c' :: ab : bc;$$

from which $$Ab' \cdot bc = b'c' \cdot ab.$$

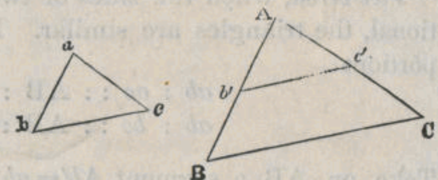
But $$Ab' = ab; \text{ hence, } bc = b'c'.$$

The two triangles, therefore, $Ab'c'$, abc have all the sides of one equal to the sides of the other, and, consequently, are identical. But $Ab'c'$ is similar to ABC; hence, abc, also, is similar to ABC.

In similar triangles, the sides opposite to equal angles are called proportional sides, and generally the proportional sides of triangles and of all polygons are called *homologous* sides.

THEOREM XV.

When two sides of one triangle are proportional to two sides of another triangle, and the angles included by the proportional sides are equal, the triangles are similar.



Let the sides *ab*, *ac* of the triangle *abc* be proportional to the sides AB, AC of the triangle ABC, and the included angles *a* and A be equal to each other: the triangles are similar. Supposing AB greater than *ab*, take on AB, A*b*′ = *ab*, and from *b*′ draw *b*′*c*′ parallel to BC: we will have at once

$$\text{AB} : \text{AC} :: ab : ac,$$
$$\text{AB} : \text{AC} :: \text{A}b' : \text{A}c'$$

Hence, $\text{A}b : ac :: \text{A}b' : \text{A}c'$;

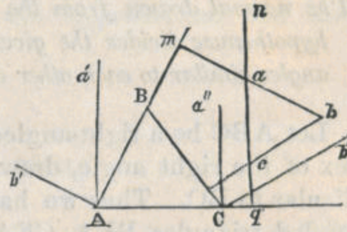
from which $ab \cdot \text{A}c' = ac \cdot \text{A}b'$.

But $ab = \text{A}b'$; hence, also, $ac = \text{A}c'$, and the two triangles *abc*, A*b*′*c*′, having two sides and the included angle equal, are equal triangles. But A*b*′*c*′ is a triangle similar to ABC; hence, *abc* also is similar to ABC, and, with AB : AC : : *ab* : *ac*, we will have also

$$\text{AB} : \text{BC} :: ab : bc,$$
$$\text{AC} : \text{BC} :: ac : bc.$$

THEOREM XVI.

Two triangles which have their sides mutually perpendicular are similar.



Let the sides ab, ac, bc of the triangle abc be respectively perpendicular to the sides AB, AC, BC of the triangle ABC: then will they be similar. For, draw from A, Aa' perpendicular to AC, and Ab' perpendicular to AB, we will have the angles $b'Aa'$, man equal to each other: for mb and $b'A$ are both perpendicular to the same AB, and, consequently, parallel; and nq and $a'A$ are both perpendicular to the same AC, and, consequently, likewise parallel to each other; hence,

$$b'Aa' = man.$$

Now, $b'Aa' = BAC$; for each one of these two angles is equal to $90° - a'AB$, and $man = bac$; therefore, the preceding equation is equivalent to

$$BAC = bac.$$

Again, draw from C, Ca'' perpendicular to AC, and Cb'' perpendicular to BC; we will have

$$a''Cb'' = acb.$$

But $\qquad a''Cb'' = ACB$

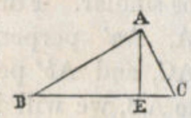
hence $\qquad ACB = acb.$

The angles A and C, therefore, of the triangle ABC are equal to the angles a and c of the triangle abc; hence, also, the third angle of the first is equal to the third angle of the second, and the triangles are similar.

THEOREM XVII.

The normal drawn from the vertex of the right angle to the hypothenuse divides the given triangle in two right-angled triangles similar to each other and to the given triangle.

Let ABC be a right-angled triangle. From A, the vertex of the right angle, draw AE perpendicular to BC. Thus we have two right-angled triangles BEA, CEA, both similar to the given triangle BAC, and, consequently, similar to each other. For



the triangles BAC, BAE, besides the right angle, have the angle B common; hence, the third angle of the first must be equal to the third of the second, namely,

$$ACB = BAE,$$

and the two triangles are similar. In like manner, the triangles BAC, CAE, besides the right angle, have the angle C common;

hence, also, $$ABC = CAE,$$

and the triangles BAE, CAE are similar to each other and to the given.

COROLLARY I. The normal drawn from the right angle to the hypothenuse is mean geometrical proportional between the segments.

Now, from the triangles AEB, AEC we have

$$BE : EA :: EA : EC;$$

that is, the normal AE, drawn from the right angle to the hypothenuse, is a mean geometrical proportional between the segments EB and EC.

COROLLARY II. Either side about the right angle is mean geometrical proportional between the hypothenuse and the adjacent segment.

From the triangles AEB, ACB we have

$$BE : BA :: BA : BC;$$

and from the triangles AEC, ABC we have

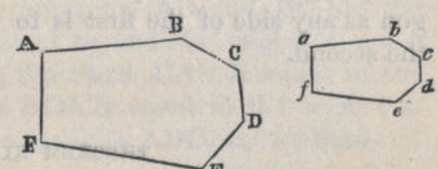
$$CE : AC :: AC : BC;$$

hence, either side about the right angle is a mean geometrical proportional between the hypothenuse and the adjacent segment.

THEOREM XVIII.

The perimeters of similar polygons are to each other as their homologous sides.

Those polygons are called similar which have the angles of one equal to the angles of the other and the sides about equal angles or homologous sides proportional. Let, for example, ABCDEF and *abcdef* be two such polygons, having namely the angles A, B, C . . . of the one respectively equal to the angles *a*, *b*, *c* . . . of the other, and the sides AB, AF about A proportional to the sides *ab*, *af* about *a*, and also the sides AB, BC about B proportional to the sides *ab*, *bc* about *b*, &c. Now, we say that the perimeters of the two polygons are to each other as any of the sides of one polygon is to the homologous side of the other.



In fact, from the proportional sides we have

$$AF : af :: AB : ab :: BC : bc :: CD : cd, \&c.$$

That is to say,

$$\frac{AF}{af}=\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}=\&c.$$

If, therefore, we call ρ the ratio $\frac{AF}{af}$, we have

$$\frac{AF}{af}=\rho, \frac{AB}{ab}=\rho, \frac{BC}{bc}=\rho, \ldots \ldots$$

and $\quad AF=\rho af, AB=\rho ab, BC=\rho bc$, &c.;
from which

$$AF+AB+BC+\ldots=\rho[af+ab+bc+\ldots]$$

and, consequently,

$$\frac{AF+AB+BC+\ldots}{af+ab+bc+\ldots}=\rho=\frac{AF}{af}=\frac{AB}{ab}=\frac{BC}{bc}=\&c.;$$

that is,

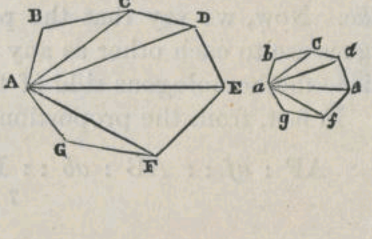
$$AF+AB+BC+\ldots : af+ab+bc+\ldots :: AF : af,$$
$$:: AB : ab,$$
$$:: BC : bc, \&c.$$

But $AF+AB+\ldots$, $af+ab+\ldots$ are the perimeters of the two polygons. Therefore, the perimeter of one polygon is to the perimeter of another similar polygon as any side of the first is to the homologous side of the second.

THEOREM XIX.

Similar polygons are divisible into an equal number of similar triangles, and vice versâ.

Supposing again ABC . . . *abc* . . . to be two similar polygons, having the angles A, B, C . . . . of the one equal to the angles *a*, *b*, *c* . . . . of the



other; draw the diagonals AC, AD . . . , *ac*, *ad* . . . : the resulting triangles ABC, ACD . . . are respectively similar to the triangles *abc*, *acd*. . . . And first, ABC is certainly similar to *abc*; for B = *b*, and the sides about B are proportional to the sides about *b*; hence, (B. II. TH. 15,) ABC and *abc* are similar; and, consequently,

$$BCA = bca,$$

and $\quad BC : bc :: AC : ac.$
But $\quad BCD = bcd;$
hence, $\quad ACD = acd.$
Moreover, $\quad BC : bc :: CD : cd;$
hence, $\quad AC : ac :: CD : cd;$

therefore, the two triangles ACD, *acd*, also, are similar to each other. In like manner we demonstrate the similarity of the triangles ADE, *ade*, &c.

Vice versâ, if two polygons can be divided into an equal number of similar triangles equally disposed, the polygons also are similar. Because, when the triangle ABC is similar to *abc* and equally disposed with regard to the remaining parts of the polygons, AB and BC are the homologous sides of *ab* and *bc*, and the angles B and *b* are equal to each other, the angle BAC is equal to the angle *bac*, and the angle BCA is equal to the angle *bca*. In like manner, from the triangles ADC, *adc* we have

$$ACD = acd;$$

and since BCA = *bca*, we have also

$$BCD = bcd.$$

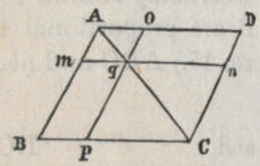
Again, from the same similar triangles ACB, *acb* we have

$\quad AC : ac :: CD : cd.$
But $\quad AC : ac :: CB : cb;$
hence, $\quad CD : cd :: CB : cb.$

In like manner, we find the angles CDE, DEF . . . .

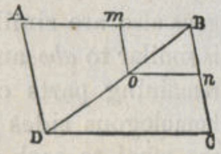
of the first polygon equal to the angles *cde*, *def* . . . . of the second, and the sides about them proportional; hence the similarity of the two polygons.

COROLLARY I. Two out of the four parallelograms into which a given parallelogram is divided by the parallels to the sides drawn from any point of the diagonal are similar to each other and to the given.



From any point q of the diagonal AC of a given parallelogram draw mn, op parallel to the sides; the parallelogram will be divided into four parallelograms, two of which are bisected by the diagonal AC. Now, these two are similar to each other and similar to BD. For the triangles Aoq, Amq into which the parallelogram mo is divided are similar to the triangles ADC, ABC into which the given BD is divided; hence the two parallelograms are similar. For the same reason pn is similar to BD and to mo.

COROLLARY II. When two similar parallelograms have a common angle, with coinciding homologous sides, they have also one of the diagonals coinciding.



Let, now, mn, AC be two similar parallelograms, having the sides Bm, Bn of the one homologous to the sides AB, BC of the other and the angle B common: the diagonals Bo and BD of the two parallelograms must then coincide with each other. For Bo divides mn into two triangles, omB, onB respectively similar to the triangles DAB, DCB in which DB divides the parallelogram AC; hence, mBo is equal to ABD. But AB, mB coincide; hence, also, Bo and BD.

REMARKS

On regular and symmetrical polygons.

When all the sides and all the angles of a polygon are equal, the polygon is then called a *regular polygon*.

Now, the number of sides is equal to the number of angles. Hence, if we suppose the polygon to contain n sides, it will contain n angles. But the sum of the n internal angles of a polygon is (B. II. TH. 10, COR. 6) $180° \cdot (n-2;)$ therefore, the measure of each angle of a regular polygon of n sides is $\frac{(180°)\cdot(n-2,)}{n}$, or $\left(1-\frac{2}{n}\right)\cdot 180°$, whatever may be the length of the sides.

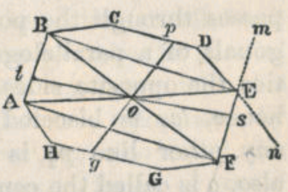
Two regular polygons having the same number of sides are evidently similar to each other.

When every side of the polygon has its opposite side equal and parallel, the polygon is called *symmetrical*. Hence, the parallelogram is to be reckoned among symmetrical polygons.

We will subjoin here some few theorems concerning symmetrical polygons.

THEOREM XX.

Opposite angles in symmetrical polygons are equal.



Let the sides AB, BC, CD, DE of the symmetrical polygon ACEG be respectively equal and parallel to the opposite sides EF, FG, GH, HA: the angles, also, A, B, C, D must be respectively equal to their opposite E, F, G, H. In fact, producing the sides DE and FE to n and m, we have the angle mEn = DEF. But Fm and Dn are parallel to AB and AH; hence, mEn = BAH; and, therefore,

$$BAH = DEF.$$

We prove, in like manner, that

$$B = F, C = G, D = H.$$

THEOREM XXI.

The diagonals joining opposite angles in a symmetrical polygon are mutually cut into two equal parts in the centre; as is also any straight line passing through the centre and terminating at the perimeter.

The diagonal drawn from B to its opposite angle F is cut into two equal parts by the diagonal drawn from the angle A to the opposite E, and *vice versâ*. For, joining B with E and A with F, since AB and EF are equal and parallel, the quadrilateral AFEB is a parallelogram; hence, its diagonals bisect each other in o. But, for the same reason, the diagonal from C to G bisects BF; that is, it passes through o, and then it is bisected itself, and so is the diagonal drawn from D to H. The point o is called the centre.

We have seen (B. II. TH. 8) that any straight line which passes through the point of intersection of the two diagonals of a parallelogram and reaches with its extremities the opposite sides, is likewise bisected in that point; hence, tos is bisected in o, and, for the same reason, any other line pq is bisected in o. For this reason, also, o is called the centre.

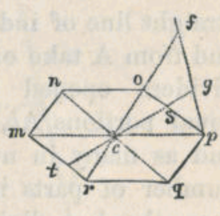
COROLLARY. Any diagonal joining the opposite angles of a symmetrical polygon bisects the perimeter and also the area of the polygon.

Since AB, BC . . . . are respectively equal to EF, FG, ... the perimeter ABCDE on one side of the diagonal is equal to the perimeter on the other side; the diagonal AE then divides into two equal parts the perimeter. The same is to be said of any other diagonal.

The triangle A*o*B is equal to the triangle E*o*F; and, joining C with G and D with H, we have a succession of triangles above the diagonal AE equal to the corresponding triangles below it; the area, therefore, ACEA is equal to AGEA. Any diagonal, therefore, joining the opposite angles of a symmetrical polygon divides into two equal parts the area of the polygon.

THEOREM XXII.

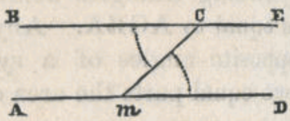
Any polygon having a centre is symmetrical.



Let *mopr* be a polygon having a centre in *c*; that is, such a point, in which the straight lines passing through it and reaching the perimeter are cut into two equal parts: the polygon is necessarily symmetrical. Join, in fact, *m* with *c*, and produce *mc* to *p*, we will have *mc* = *cp*; now, from *p* draw a parallel to *mr* equal in length to the same *mr*: this parallel will coincide with *po*. Otherwise, suppose *pf* to be the parallel to *mr*, and from *r* draw *rcf*; we will have two triangles *mcr*, *fcp* having the opposite angles at *c* equal, and, by supposition, the alternate angles *cmr*, *cpf* also equal, the side *mc* moreover equal to the side *cp*. Hence, *cf* would be equal to *cr*, and consequently also to *co*, which is impossible unless *f* coincides with *o*; therefore, *po* is equal and parallel to *mr*. In like manner, *no* is equal and parallel to *rq*, and *nm* equal and parallel to *qp*, and the polygon is symmetrical.

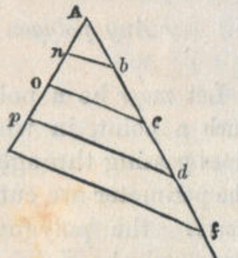
PROBLEMS.

PROBLEM I. From a given point, draw a straight line parallel to another.



Let AD be a straight line, and C a point from which a parallel is to be drawn to AD. Join C with any point *m* of AD, and then draw CB, (B. I. PROB. 5,) making with C*m* an angle equal to C*m*D: BE and AD are then parallel to each other.

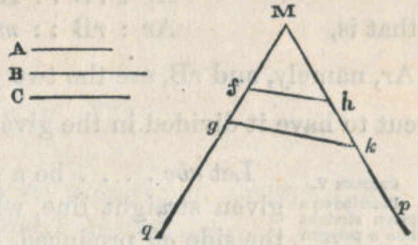
PROBLEM II. Divide a given line into equal parts.



AB is a given line to be divided into a certain number of equal parts. From A draw any other straight line of indefinite length; and from A take on it—with the dividers opened at pleasure—equal portions A*b*, *bc*, *cd*, ... and as many in number as the number of parts in which AB is to be divided. Join then the last division *f* of A*m* with B, and from *d*, *c*, *b* draw parallels to *f*B: these parallels divide AB (B. II. TH. 12) into the required number of equal parts.

PROBLEM III. To find the fourth proportional to three given straight lines.

A, B, C are three given straight lines, which, together with another x to be found, ought to be in proportion as follows:—

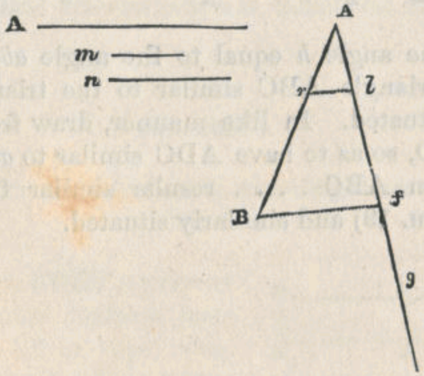


$$A : B :: C : x.$$

To find the fourth proportional x, take two indefinite straight lines, Mq, Mp, making any angle M; and on Mq take Mf = A, Mg = B; on Mp take Mh = C. Join f with h, and from g draw gk parallel to fh: the segment Mk is (B. II. TH. 12, COR. 1) the fourth proportional x.

PROBLEM IV. Divide a given line in a given ratio.

Let A be a straight line to be divided into two such parts that their ratio be equal to the ratio of the given straight lines m and n.



Take for this purpose AB = A, and from A draw the indefinite straight line Ag, making any angle with AB. On Ag take Al = m and lf = n. Join then f with B, and from l draw lr parallel to Bf: we will have (TH. 12, COR. 1)

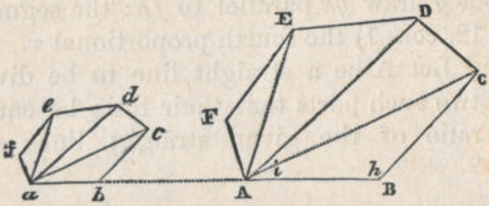
$$Ar : rB :: Al : lf;$$

that is, $$Ar : rB :: m : n.$$

Ar, namely, and rB, are the two parts in which A is to be cut to have it divided in the given ratio $\frac{m}{n}$.

Problem V. Describe on a given straight line a polygon similar to another given polygon and similarly situated.

Let abc . . . . be a given polygon and AB a given straight line, which we may take along the side ab produced, or parallel to it. In the given polygon draw the diagonals ac, ad, ae; draw then from A, AC, making with AB the angle i equal to the angle cab which the diagonal ac makes with the side ab; draw also BC, making



with AB the angle h equal to the angle abc. Thus, we have the triangle ABC similar to the triangle abc and similarly situated. In like manner, draw from A and C AD and CD, so as to have ADC similar to adc, &c.: thus the polygon ABC . . . . results similar to the given abc . . . (TH. 19) and similarly situated.

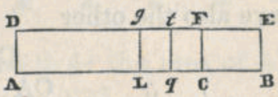
BOOK III.

COMPARISON OF PLANE SURFACES LIMITED BY STRAIGHT LINES.

THE determination of the area of any surface is made by referring it to some standard adopted as unity of measure; thus, for instance, we say the surface of a field is so many square yards, taking the square yard as unity of measure; the surface of a country is so many square miles, taking the square mile as unity of measure, &c.; in the same manner as the length of any straight line is determined by referring it to some other length taken as unity of measure,—an inch, a foot, &c. The determination, therefore, of any surface contains an implicit comparison; hence, this determination is called the *comparison of surfaces.*

THEOREM I.

Two rectangular areas having the same height are to each other as their bases.



Let ACFD, CBEF represent two rectangular surfaces having the side CF or *height* common, and the side AC or *base* of the one different from the base CB of the other: the two areas CD, CE are to each other as their respective bases.

In fact, take on the greater base CA a segment CL = CB, and draw Lg perpendicular to AC; we will have the rectangle LF equal to AB. For, turning FL about FC, the point g will coincide with E and L with B, and consequently all the sides of one with all the sides of the other. In like manner, if we take along CA and from L another segment equal to LC = CB, and finish the rectangle, we will have another area equal to the preceding; and with three segments we will obtain three equal areas, &c.; that is, if the base CA is twice, three times, four times, &c. the base CA, the area of the rectangle CD is twice, three times, four times, &c. the area of the rectangle CE. But if we take C$q = \frac{1}{2}$CL $= \frac{1}{2}$CB, and draw qt perpendicular to AB, we will have the rectangles qF, qg equal to each other, and consequently $q\text{F} = \frac{1}{2}\text{C}g = \frac{1}{2}\text{CE}$. Also, if we divide CL into three equal parts and draw perpendiculars to CL so as to complete the rectangles, we will have three equal rectangles, and consequently every one of them equal to $\frac{1}{3}\text{C}g = \frac{1}{3}\text{CE}$, &c.; that is, if CA becomes one-half, one-third, &c. of CB, the corresponding rectangle CD becomes, likewise, one-half, one-third, &c. of CE. Therefore, whatever be the length of CA compared with CB, or, what is the same, whatever be ρ in the equation

$$\text{CA} = \rho \cdot \text{CB};$$

with this equation (see Treat. on Alg., § 116) we will have also the other

$$\text{CD} = \rho \cdot \text{CE};$$

hence, in all cases, $\frac{\text{CA}}{\text{CD}} = \frac{\text{CB}}{\text{CE}}$, or $\frac{\text{CA}}{\text{CB}} = \frac{\text{CD}}{\text{CE}}$;

that is, $\text{CA} : \text{CB} :: \text{CD} : \text{CE}.$

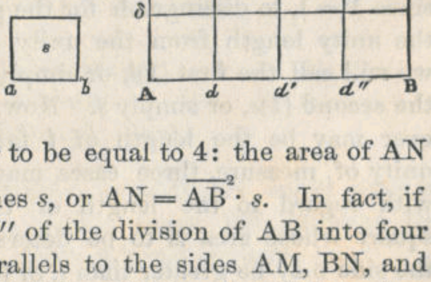
The areas, namely, of any two rectangles having the same height are to each other as their bases.

Remarks. We may here observe that whatever be the unity of measure of the bases CB, CA, their ratio CA, CB is invariably the same; for from $CA = \rho CB$ we will always have $\frac{CA}{CB} = \rho$.

Observe, also, that the number of times the unity of measure is contained in a certain linear length, or the quotient of the linear length divided by the unity of measure, is that which we call *numerical value* of that length. Hence, the preceding theorem may be expressed also as follows:—

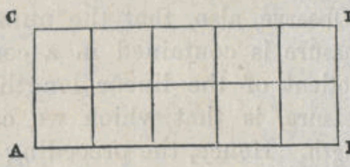
The areas of two rectangular surfaces are to each other as the numerical values of their bases.

The area, also, of a surface may be numerically expressed, taking the area of a square as unity of measure. But the area of the square varies with the length of its side; hence, the unity square supposes a linear unity corresponding to it.



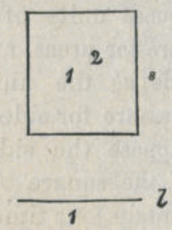
To facilitate the understanding of the numerical value of areas, let s represent the square unity of measure for areas, and its side ab the unity of measure for sides, and suppose the side AB of the square AN to contain four times ab or to be equal to 4: the area of AN will contain $4^2 = 16$ times s, or $AN = \overline{AB}^2 \cdot s$. In fact, if from each point d, d', d'' of the division of AB into four equal parts we draw parallels to the sides AM, BN, and from each point $\delta, \delta', \delta''$ of the division of AM into four equal parts we draw parallels to the sides MN, AB, we have the area AN divided into sixteen squares equal to *s*. If, *vice versâ*, we suppose AB to be the linear unity of measure, and AN or S the unity of measure for areas, the side of *s* then, numerically expressed, is $\frac{1}{4}$; now $(\frac{1}{4})^2=\frac{1}{16}$; in fact, *s* is the sixteenth part of S.

With regard to rectangles, suppose the base AB to contain five times the linear unity of measure *ab*, and the height AC to contain the same unity twice; the area of the rectangle will be $2 \cdot 5$ times *s*.



Drawing, in fact, from each point of the division of the base into five equal parts, parallel lines to the sides AC, BD, and from the point of division of the height a parallel to the base, we have the area CB divided into ten squares, all equal to *s*. Hence, the product of the numerical value of the base by the height of the rectangles, multiplied by *s*, gives the area of the rectangle. But let us examine the question in a more general point of view.

Let *l* or 1 represent the linear length of the side of the square 1^2 or *s*, taken as the unity of measure of surfaces. And, since $1^2=1$, to distinguish for the present the unity length from the unity surface we will call the first (1)*l*, or simply *l*, and the second (1)*s*, or simply *s*. Now, whatever may be the length of *l* taken as unity of measure, three cases may occur with regard to the length of the side of any given square whose area is to be determined. The length of the side may be greater than *l*, or less than *l*, or equal to *l*; and, representing it generally by νl, or ν, in the first case we will have $\nu > 1$, in the second $\nu < 1$, and in the last

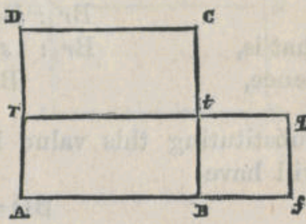


case $\nu = 1$; but in all these cases the area of the square will be expressed by $\nu^2 \cdot s$, or simply ν^2, as will appear from the following theorem:

THEOREM II.

The area of the square is expressed by the product of $s = 1$, *multiplied by the square of the numerical value of its side.*

Let, first, the side AB of the given square be greater than $l(=1.)$ Produce AB to f, so as to have $Bf = 1$; take also $Bt = 1$, and finish the square Bq and the rectangle Br; we will have (TH. 1)



$$Bq : Br :: Bf : BA;$$

that is, $s : Br :: 1 : \nu$,

and, therefore, $Br = \nu s$.

Comparing now rB with DB, we have

$$rB : DB :: rA : DA;$$

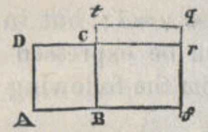
that is, $rB : DB :: 1 : \nu$.

Hence, $DB = Br \cdot \nu$;

but $Br = \nu s$;

hence, $DB = \nu^2 \cdot s = \nu^2$.

If ν should be a whole number, we could express this equation by saying,—the square DB contains the unity square s as many times as there are units in ν^2.



Suppose, now, the side AB of the given square to be less than $l\,(=1.)$ Produce AB to f, so as to have $\mathrm{B}f=1$, and finish the square $\mathrm{B}q$; produce, also, DC to r; we will have

$$\mathrm{BD} : \mathrm{B}r :: \mathrm{AB} : \mathrm{B}f;$$

that is, $$\mathrm{BD} : \mathrm{B}r :: \nu : 1,$$

hence, $$\mathrm{BD} = \mathrm{B}r \cdot \nu.$$

Compare, now, $\mathrm{B}r$ with $\mathrm{B}q$; we will have

$$\mathrm{B}r : \mathrm{B}q :: \mathrm{BC} : \mathrm{B}t;$$

that is, $$\mathrm{B}r : s :: \nu : 1,$$

hence, $$\mathrm{B}r = \nu \cdot s.$$

Substituting this value in the preceding equation, we will have

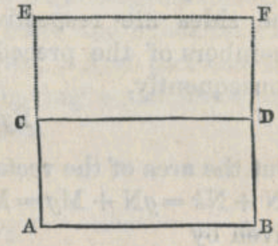
$$\mathrm{BD} = \nu^2 \cdot s = \nu^2.$$

If, finally, the side of the given square is equal to $l\,(=1)$, it is plain that also in this case $\nu^2 \cdot s$, or ν^2, expresses the area of the square.

Therefore, in all cases, whatever may be the linear unity l and the corresponding unity of surfaces s, the area of any square is expressed by the product of s multiplied by the square of the numerical value of its side. Now, the numerical value of the side is commonly expressed by the side itself; and s, on account of being equal to 1, is not expressed; therefore, the area of any square having AB for one of its sides is simply represented by $\overline{\mathrm{AB}}^2$.

THEOREM III.

The area of a rectangle is given by the product of the numerical value of the base into that of the height multiplied by $s = 1$.



Let AD represent any rectangle. Produce AC and BD to E and F, so as to have AE = BF = AB, and finish the square AF. We will have

$$AD : AF :: AC : AE;$$

or, $$AD : AF :: AC : AB;$$

hence, $$AD = AF\frac{AC}{AB}.$$

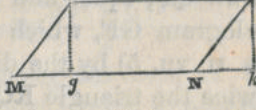
But $$AF = \overline{AB}^2 \cdot s;$$

therefore, $$AD = AB \cdot AC \cdot s,$$

or, simply, $$AD = AB \cdot AC.$$

THEOREM IV.

The area also of any parallelogram is given by the vroduct of the base into the height.



Let MNOP be any parallelogram, having MN for its base, and gP (the common perpendicular to the opposite sides) for its height. The area of this parallelogram is given by the product Pg · MN multiplied, as it is understood, by $s = 1$.

Observe, in fact, that by drawing Pg and Ok perpendicular to MN we have the rectangle $PgkO$, whose area is equal to that of the parallelogram PMNO; for

$$PgkO = PgNO + ONk,$$
$$PMNO = PgNO + PMg.$$

Now, the triangles ONk, PMg are equal to each other; for $MP = ON$, $Pg = Ok$, and the included angle P of the one is equal to the included angle O of the other, because the sides are respectively parallel; hence, the second members of the preceding equations are identical, and, consequently,

$$PgkO = PMNO.$$

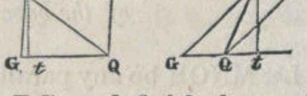
But the area of the rectangle is given by $Pg \cdot gk$; and $gk = gN + Nk = gN + Mg = MN$; hence, the area of PMNO is given by

$$Pg \cdot MN,$$

the product of the base into the height.

THEOREM V.

The area of any triangle is given by half the product of the base into the height.



Let RGQ be any two triangles. Draw from R, Rt perpendicular to the base or to the side produced. From Q draw QF, equal and parallel to RG, and finish the parallelogram GF, which is divided into two equal triangles (B. II. TH. 5) by the diagonal RQ. Hence, GF is equal to twice the triangle RQG; or,

$$RQG = \frac{GF}{2}.$$

Now, $$GF = GQ \cdot Rt;$$

hence, $$RQG = \frac{GQ \cdot Rt}{2}.$$

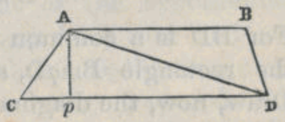
The area, namely, of any triangle is one-half the area of the parallelogram having the same base and the same height.

COROLLARY. Parallelograms and triangles having equal bases and equal heights have also equal areas.

From this and the preceding theorems we easily infer that any two parallelograms having the same base or equal bases and equal heights must have also equal areas. And likewise the areas of two triangles having equal bases and equal heights are equal to each other.

THEOREM VI.

The area of a trapezoid is given by the product of the vertical to the parallel sides into half the sum of the same sides.



Let AB, CD be the parallel sides of any trapezoid. Draw from A, Ap perpendicular to both, and draw also the diagonal AD: we will have the area of the trapezoid divided into two triangles, having the common height Ap, and CD for the base of one, and AB for the base of the other. Now, the area of ADB is given by $\frac{AB \cdot Ap}{2}$, and the area of ADC is given by $\frac{CD \cdot Ap}{2}$;

hence, $$CB = \frac{AB \cdot Ap}{2} + \frac{CD \cdot Ap}{2} = Ap\frac{AB + CD}{2}.$$

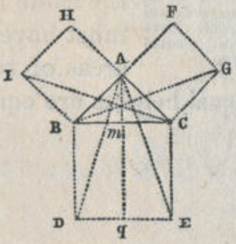
SCHOLIUM. Concerning the areas of polygon.

It is plain that by taking the sum of the areas of the triangles into which a polygon may be divided, we will obtain the area of the polygon itself.

THEOREM VII.

The area of the square described on the hypothenuse is equal to the sum of the areas of the squares described on the other sides of the right-angled triangle.

First demonstration. Let ABC be a triangle right-angled at A. Describe the square BE on the hypothenuse and the squares AG, AI on the other sides: we will have



$$BE = AG + AI.$$

Draw, in fact, from A, Amq perpendicular to BC and DE, and draw also the diagonal AD; we have (B. III. TH. 3 and 5)

$$BmqD = 2\ ABD.$$

For BD is a common base to the triangle ABD and to the rectangle BmqD, and Dq is their common height. Draw, now, the diagonal CI: we have in like manner

$$BIHA = 2\ IBC.$$

For BI is a base common to BH and to IBC, and IH is their common height. Now, the triangle ABD is equal to the triangle IBC, because the side AB of the one is equal to the side BI of the other, both being sides of the same square, and the side BD of the first is equal to the side BC of the other, for the same reason. But the included angle ABD of the first is also equal to the included angle IBC of the second; because ABD is equal to ABC plus a right angle, and IBC is likewise equal to ABC plus a right angle; hence, (B. I. TH. 4,)

$$ABD = IBC.$$

And consequently, from the preceding equations,

$$BmqD = BIHA.$$

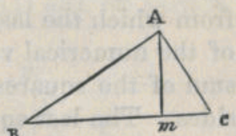
In like manner, drawing the diagonals AE, BG, we find

$$CmqE = CGFA;$$

hence, $\quad BmqD + CmqE = BIHA + CGFA;$

that is, $\quad BCED = BIHA + CGFA$

Second demonstration. We may arrive at the same conclusion by another process. We have seen (B. II. TH. 17) that the normal Am drawn from the right angle to the hypothenuse divides ABC into two triangles similar to the given one, and, consequently,



$$Bm : AB :: AB : BC,$$
$$Cm : AC :: AC : BC.$$

Now, call ν the numerical value of the hypothenuse BC, measured with the linear length $l = 1$, and let ν', ν'' be the numerical values of AB and AC, and δ the numerical value of mC, all the lengths being measured with the same l: the two preceding proportions will then be equivalent to

$$(\nu - \delta) : \nu' :: \nu' :: \nu$$
$$\delta : \nu'' :: \nu'' : \nu;$$

from which $\quad (\nu - \delta)\ \nu = \nu'^2,$

$$\delta\nu = \nu''^2,$$

and, consequently,

$$(\nu - \delta)\,\nu + \delta\nu = \nu'^2 + \nu''^2;$$

or, $\quad \nu^2 = \nu'^2 + \nu''^2$

Now, calling, as before, s the square constructed on the side $l=1$, and multiplying by s both members of the last equation, we have

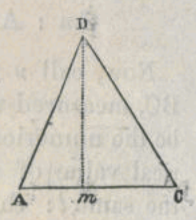
$$\nu^2 s = \nu'^2 s + \nu''^2 s.$$

But $\nu^2 \cdot s$ is the area of the square constructed on the hypothenuse BC, or, according to the ordinary expression, $\nu^2 s = \overline{BC}^2$; and likewise $\nu'^2 s = \overline{AB}^2$, $\nu''^2 s = \overline{AC}^2$;

hence, $$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$$

SCHOLIUM I. Different meaning of the two equations. The last equation, therefore, expresses that the area of the square on the hypothenuse is equal to the sum of the areas of the squares on the other sides. But the preceding equation, $\nu^2 = \nu'^2 + \nu''^2$, from which the last is inferred, expresses that the square of the numerical value of the hypothenuse is equal to the sum of the squares of the numerical values of the other sides. The last equation, however is commonly taken to signify both of them.

SCHOLIUM II. Connection between the areas of the squares constructed on the sides of any triangle. Let DAC be any acute-angled triangle. To find the relation between the square on one of its sides—for instance, AD—and the squares on the other two sides, draw from one of the adjacent angles, for instance, D, the normal Dm to the opposite side, which will meet it somewhere between A and C. Thus, we have the right-angled triangles mDA, mDC, from which



$$\overline{AD}^2 = \overline{Am}^2 + \overline{Dm}^2,$$
$$\overline{DC}^2 = \overline{Dm}^2 + \overline{mC}^2;$$

and, substituting in the first equation the value of $\overline{Dm}^2$ deduced from the second,

$$\overline{AD}^2 = \overline{Am}^2 + \overline{DC}^2 - \overline{mC}^2.$$

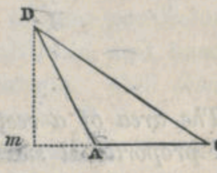
Now, $\mathrm{A}m = \mathrm{AC} - m\mathrm{C}$;

hence, $\overline{\mathrm{A}m}^2 = \overline{\mathrm{AC}}^2 - 2 \cdot \mathrm{AC} \cdot m\mathrm{C} + \overline{m\mathrm{C}}^2$

and, consequently,

$$\overline{\mathrm{AD}}^2 = \overline{\mathrm{AC}}^2 + \overline{\mathrm{DC}}^2 - 2\mathrm{AC} \cdot m\mathrm{C}.$$

But $\mathrm{AC} \cdot m\mathrm{C}$ is the area of a rectangle having AC for base and mC for altitude. Hence, *the area of the square constructed on any side* AD *of the acute-angled triangle* ADC *is equal to the difference between the sum of the squares constructed on the other two sides and the double rectangle having for base one of these two sides and for altitude the segment of the same side between the angle opposite to* AD *and the perpendicular drawn to it from its opposite angle.*



The same can be proved of the square constructed on either side about the obtuse angle of an obtuse-angled triangle. Let, for instance, DA be one of these sides; produce the base CA, and draw from D, Dm perpendicular to Cm. Thus, we have two right-angled triangles DAm, DCm, from which we have

$$\overline{\mathrm{DA}}^2 = \overline{\mathrm{A}m}^2 + \overline{\mathrm{D}m}^2,$$
$$\overline{\mathrm{DC}}^2 = \overline{\mathrm{D}m}^2 + \overline{m\mathrm{C}}^2;$$

and, substituting in the first of these equations the value of $\overline{\mathrm{D}m}^2$, taken from the second,

$$\overline{\mathrm{DA}}^2 = \overline{\mathrm{A}m}^2 + \overline{\mathrm{DC}}^2 - \overline{m\mathrm{C}}^2.$$

Now, $\overline{\mathrm{A}m}^2 = (\mathrm{C}m - \mathrm{AC})^2 = \overline{\mathrm{C}m}^2 + \overline{\mathrm{AC}}^2 - 2\mathrm{C}m \cdot \mathrm{AC}$;

hence, $\overline{\mathrm{DA}}^2 = \overline{\mathrm{AC}}^2 + \overline{\mathrm{DC}}^2 - 2\mathrm{C}m \cdot \mathrm{AC}.$

Let us, finally, see how the area of the square constructed on the side opposite to the obtuse angle is given by the sum of two squares and a double rectangle. Observe that from the preceding equations

$$\overline{DC}^2 = \overline{Dm}^2 + \overline{mC}^2,\ \overline{DA}^2 = \overline{Am}^2 + \overline{Dm}^2,$$

we have $\overline{DC}^2 = \overline{mC}^2 + \overline{DA}^2 - \overline{Am}^2.$

Again, from $Cm = CA + Am$

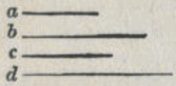
we have $\overline{Cm}^2 = \overline{CA}^2 + \overline{Am}^2 + 2CA \cdot Am;$

hence, $\overline{DC}^2 = \overline{CA}^2 + \overline{DA}^2 + 2CA \cdot Am.$

THEOREM VIII.

The area of a rectangle constructed on the extremes of four proportional sides is equal to the area of the rectangle constructed on the mean sides.

Let the straight lines a, b, c, d be proportional, so that we have



$$a : b :: c : d.$$

Since from this proportion we have

$$a \cdot d = b \cdot c,$$

we infer that the area of the rectangle having one of the extreme terms for its base and the other extreme for its altitude is equal to the area of the rectangle having one of the mean terms for its base and the other mean term for its altitude.

COROLLARY. The square on the mean proportional is equivalent to the rectangle on the extremes.

Hence, if we suppose $b = c$,—that is, b a mean proportional between a and d,—

then, since $a \cdot d = b^2$,

the area of the square of the mean proportional term is equal to the area of the rectangle constructed on the extremes.

THEOREM IX.

Parallelograms and triangles having the same base are to each other as their altitudes: or, having the same altitudes, are to each other as their bases.

Call A the altitude and B the base of one parallelogram or one triangle, and A′, B′ the altitude and base of another parallelogram or another triangle. Call also the first parallelogram—that is, its area, P and the second P′, or the first triangle T and the second T′: we will have

$$P = A \cdot B, \quad P' = A' \cdot B';$$

or,
$$T = \frac{A \cdot B}{2}, \quad T' = \frac{A' \cdot B'}{2}.$$

Hence,
$$\frac{P}{P'} = \frac{A \cdot B}{A' \cdot B'}, \quad \frac{T}{T'} = \frac{A \cdot B}{A' \cdot B'};$$

that is, $P : P' :: A \cdot B : A' \cdot B'$; $T : T' :: A \cdot B : A' \cdot B'$.

Suppose, now, $B = B'$;

then $P : P' :: A : A'$; or, $T : T' :: A : A'$.

Suppose B not equal to B′, but A equal to A′;

then $P : P' :: B : B'$; or, $T : T' :: B : B'$.

SCHOLIUM. When the bases are reciprocally as their altitudes, the areas of the parallelograms or of the triangles are equal.

But if the bases B and B′ are inversely or reciprocally as the altitudes A and A′; that is,

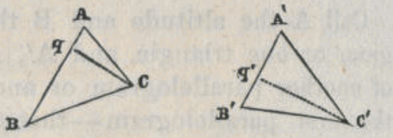
if $$A : A' :: B' : B;$$

then, since $A \cdot B = A' \cdot B'$,

the parallelograms or triangles have equal areas.

THEOREM X.

The areas of two triangles having one equal angle are to each other as the products of the sides about the equal angles.



Let ABC, A′B′C′ be two triangles having the angle A of the one equal to the angle A′ of the other. Draw from C, Cq perpendicular to the opposite side AB, and from C′ $C'q'$ perpendicular to the opposite side A′B′: the triangles AqC, $A'q'C'$, are then similar to each other.

Hence, $$Cq : C'q :: AC : A'C';$$

or, $$\frac{Cq}{C'q'} = \frac{AC}{A'C'}.$$

Now, $$ABC \text{ or } T = \frac{AB \cdot Cq}{2},$$

and $$A'B'C' \text{ or } T' = \frac{A'B' \cdot C'q'}{2};$$

hence, $$\frac{T}{T'} = \frac{AB \cdot Cq}{A'B' \cdot C'q'} = \frac{AB}{A'B'} \cdot \frac{Cq}{C'q'} = \frac{AB}{A'B'} \cdot \frac{AC}{A'C'};$$

or, $$T : T' :: AB \cdot AC :: A'B' \cdot A'C'.$$

Corollary I. *If the areas of the two triangles are equal, the sides about the equal angles are reciprocal.*

If we suppose the areas T and T′ to be equal to each other, then we have also

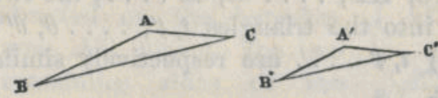
$$AB \cdot AC = A'B' \cdot A'C';$$

from which $$\frac{AB}{A'B'} = \frac{A'C'}{AC};$$

that is, $$AB : A'B' :: A'C' : AC.$$

Corollary II. *The areas of similar triangles are as the squares of the homologous sides.*

Let the triangles BAC, B′A′C′ be similar to each other, the angles A, B, C being respectively equal to the angles A′, B′, C′.



Now, since $A = A'$, calling, as above, T and T′ the areas of the two triangles, we have, from the theorem,

$$\frac{T}{T'} = \frac{AB}{A'B'} \cdot \frac{AC}{A'C'}.$$

But, from the similarity of the triangles, we have also

$$AB : A'B' :: AC : A'C'; \text{ or, } \frac{AB}{A'B'} = \frac{AC}{A'C'};$$

hence, $$\frac{T}{T'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{AC}^2}{\overline{A'C'}^2};$$

and, since $$AC : A'C' :: BC : B'C',$$

therefore, also, $$\frac{T}{T'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

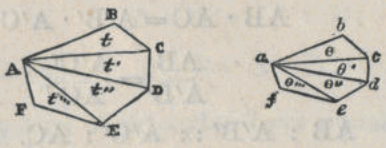
That is,

$$T : T' :: \overline{AB}^2 : \overline{A'B'}^2 :: \overline{AC}^2 : \overline{A'C'}^2 :: \overline{BC}^2 : \overline{B'C'}^2.$$

The areas, namely, of two similar triangles are as the squares of the homologous sides.

COROLLARY III. The areas of similar polygons are as the squares of their homologous sides.

Hence, also, the areas of similar polygons are as the squares of the homologous sides.



Let, in fact, ABC . . . . , *abc* . . . . be two similar polygons, whose sides AB, BC, CD . . . . are respectively homologous to the sides *ab*, *bc*, *cd* . . . . Drawing the diagonals AC, AD, ... *ac*, *ad* . . . . , the two polygons are divided into the triangles $t, t', \ldots . \theta, \theta' \ldots .$, and (B. II. TH. 19) t, t' . . . . are respectively similar to θ, θ' . . . . ; hence,

$$t : \theta :: \overline{AB}^2 : \overline{ab}^2, \text{ or } \frac{t}{\theta} = \frac{\overline{AB}^2}{\overline{ab}^2};$$

$$t' : \theta' :: \overline{CD}^2 : \overline{cd}^2, \text{ or } \frac{t'}{\theta'} = \frac{\overline{CD}^2}{\overline{cd}^2};$$

$$t'' : \theta'' :: \overline{DE}^2 : \overline{de}^2, \text{ or } \frac{t''}{\theta''} = \frac{\overline{DE}^2}{\overline{de}^2};$$

$$t''' : \theta''' :: \overline{FE}^2 : \overline{fe}^2, \text{ or } \frac{t'''}{\theta'''} = \frac{\overline{FE}^2}{\overline{fe}^2}.$$

Now, (B. II. TH. 18,) first, from the similarity of the polygons, we have $\frac{AB}{ab} = \frac{CD}{cd} = \frac{DE}{de} \ldots .;$

hence, also, $\frac{t}{\theta} = \frac{t'}{\theta'} = \frac{t''}{\theta''} = \ldots . = \frac{\overline{AB}^2}{\overline{ab}^2} = \frac{\overline{CD}^2}{\overline{cd}^2} = \ldots .$

Secondly, from the equality of the ratios,

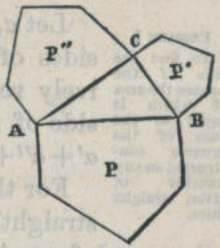
$$\frac{t + t' + t'' + \ldots .}{\theta + \theta' + \theta'' + \ldots .} = \frac{t}{\theta} = \frac{t'}{\theta'} = \ldots . = \frac{\overline{AB}^2}{\overline{ab}^2} = \frac{\overline{CD}^2}{\overline{cd}^2} = \ldots .$$

But $t + t' + t'' + \ldots$ is the area of the polygon ABC ...., and $\theta + \theta' + \theta'' + \ldots$ is the area of the polygon abc .... Hence, calling P and P′ the areas of the two polygons, we have $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{ab}^2} = \frac{\overline{CD}^2}{\overline{cd}^2} = \ldots$

that is,
$$P : P' :: \overline{AB}^2 : \overline{ab}^2;$$
$$P : P' :: \overline{CD}^2 : \overline{cd}^2, \text{ \&c.}$$

COROLLARY IV. If the sides of a right-angled triangle are homologous sides of similar polygons, the area of the polygon on the hypothenuse is equal to the sum of the areas of the polygons on the other sides.

Let three similar polygons, P, P,′ P″, be constructed on the hypothenuse AB, and on the two remaining sides of the right-angled triangle ABC, so as to have the side AB of P homologous to CB of P′ and to CA of P″. From the preceding corollary we have



$$\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{CB}^2},\ \frac{P}{P''} = \frac{\overline{AB}^2}{\overline{CA}^2};$$

and, consequently,

$$\frac{\overline{AB}^2}{P} = \frac{\overline{CB}^2}{P'} = \frac{\overline{CA}^2}{P''}.$$

And, calling R this common ratio, we have, also, with

$$\frac{\overline{AB}^2}{P} = R,$$

$$\overline{CB}^2 = P'R,\ \overline{CA}^2 = P''R,$$

and
$$\overline{CB}^2 + \overline{CA}^2 = R(P' + P'');$$

hence,
$$\frac{\overline{CB}^2 + \overline{CA}^2}{P' + P''} = R = \frac{\overline{AB}^2}{P};$$

or, $$\frac{\overline{CB}^2+\overline{CA}^2}{\overline{AB}^2}=\frac{P'+P''}{P}.$$

But $\overline{CB}^2+\overline{CA}^2=\overline{AB}^2$; hence, $\dfrac{P'+P''}{P}=1$; and, consequently,
$$P=P'+P''.$$

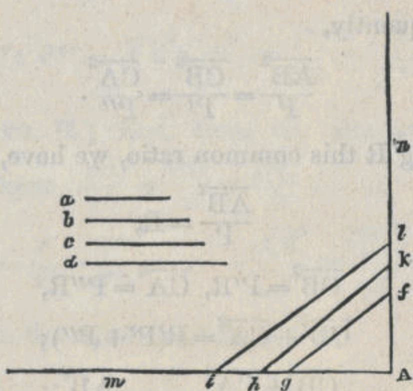
PROBLEMS.

Problem I. To find the side of the square the area of which is equal to the sum of the squares constructed on any number of given straight lines.

Let a, b, c, d be four given straight lines or sides of squares whose areas may be respectively represented by a, a', a'', a'''. Find the side of the square having for its area $A=a+a'+a''+a'''$.

For this purpose, let us draw two indefinite straight lines Am, An at right angles to each other, and from A take Af, Ag, equal to the sides a and b; join f with g; we will have

$$\overline{fg}^2=\overline{fA}^2+\overline{Ag}^2=a+a'.$$



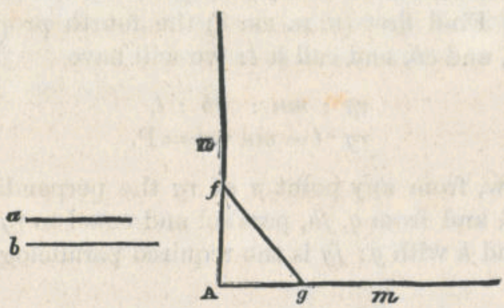
Take, now, on Am, A$h = fg$, and on An, A$k = c$, and join k with h: we have

$$\overline{kh}^2 = \overline{\mathrm{A}h}^2 + \overline{\mathrm{A}k}^2 = a + a' + a''.$$

Take, finally, on Am, A$i = kh$, and on An, A$l = d$, and join l with i: we will have

$$\overline{il}^2 = \overline{\mathrm{A}i}^2 + \overline{\mathrm{A}l}^2 = a + a' + a'' + a''':$$

il, therefore, is the side required.



PROBLEM II. To find the side of the square the area of which is equal to the difference between the unequal areas of two squares constructed on two given lines.

Let a, b be two straight lines or sides of two squares a and a'; find the side of another square whose area may be equal to $a' - a$. Taking again the indefinite lines Am, An at right angles, and on An taking A$f = a$, with the radius b and centre f describe an arc of a circle so as to cut Am in g, and join f with g:

we will have $$\overline{fg}^2 = a';$$

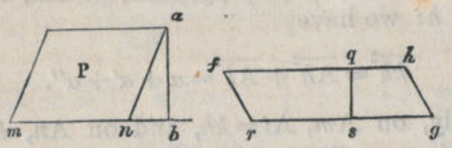
but $$\overline{fg}^2 = \overline{f\mathrm{A}}^2 + \overline{\mathrm{A}g}^2 = a + \overline{\mathrm{A}g}^2;$$

hence, $$a' = a + \overline{\mathrm{A}g}^2,$$

and, consequently,

$$\overline{\mathrm{A}g}^2 = a' - a.$$

Ag, therefore, is the side of the square the area of which is equal to the difference of the given squares.



PROBLEM III. Construct on a given side a parallelogram whose area is equal to that of another parallelogram.

Let P be a given parallelogram having mn for base and ab for altitude, and let rg be a given straight line on which a parallelogram is to be constructed having the area equal to P.

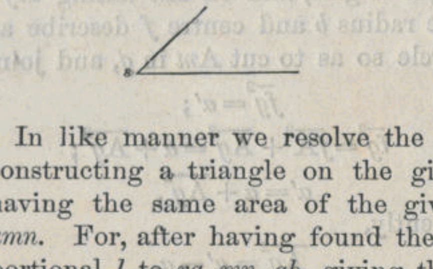
Find first (B. II. PR. 3) the fourth proportional to rg, mn, and ab, and call it l: we will have

$$rg : mn : ab : l,$$

and

$$rg \cdot l = mn \cdot ab = \mathrm{P}.$$

Draw, now, from any point s of rg the perpendicular sq equal to l, and from q, fh, parallel and equal to rg; join f with r, and h with g: fg is the required parallelogram.



PROBLEM IV. To construct a triangle having the same area of another triangle and one angle equal to a given angle.

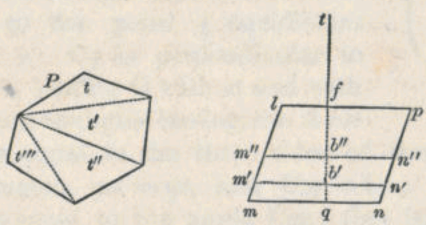
In like manner we resolve the problem of constructing a triangle on the given side rg having the same area of the given triangle amn. For, after having found the fourth proportional l to rg, mn, ab, giving this l for the altitude of the triangle to be constructed on rg, the two triangles will have the same area.

But if one of the angles of the new triangle is to be equal to the given angle *s*, then, after having found *l*, take on any perpendicular *tq* to *rg* a part $tf = l$, and from *f* draw an indefinite parallel to *rg*: then draw from *r*, *rv*, making with *rg* the angle $vrg = s$; join *v* with *g*, and we will have the required triangle.

PROBLEM V.

To construct on a given side and with a given angle a parallelogram whose area is equal to that of a given triangle, or a triangle whose area is equal to that of a given parallelogram.

This problem is resolved in the same manner as the two preceding, with this difference,—that the altitude of the parallelogram is to be taken equal to one-half of the fourth proportional *l* in the first case, and equal to twice *l* in the second case.



PROBLEM VI.

To find the altitude of a parallelogram to be constructed on a given base and whose area is to be equal to the area of a given polygon.

Let P be a given polygon which may be divided into the triangles *t*, *t'* *t''* . . . . Let, also, *mn* be the base on which a parallelogram is to be constructed having the same area as P. What will be the altitude of the parallelogram?

Find, first, the fourth proportional to *mn* the base and the altitude of the triangle *t*, and, drawing the indefinite *qt* perpendicular to *mn*, take on it *qb'* equal to one-half of the found fourth proportional. Take, again, *b'b''* equal to one-half of the fourth proportional to *mn* the base and the altitude of the triangle *t'*, and so on. Let, now, *qf* be the sum of all the halved fourth proportionals: it will also be the altitude of the parallelogram having the same area as the polygon.

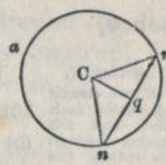
For, draw from f, lp parallel to mn, and from m and n, ml, np parallel to each other, and from b', from b'', &c., $m'n'$, $m''n''$, &c. parallel to mn: we will have the parallelograms $m'n$, $m''n'$, &c. having the same areas as the triangles t, t', &c. Hence, mp is a parallelogram whose area is equal to that of P, and whose altitude is qf.

BOOK IV.

THE CIRCLE.

THEOREM I.

A straight line drawn from the centre and bisecting a chord is perpendicular to it, and vice versâ.



Let mn be any chord in the circle anm, and let Cq be a straight line drawn from the centre to the point q equidistant from m and n: Cq is perpendicular to mn. Because, joining C with m and with n, we have two triangles having the three sides of one equal to the three sides of the other, Cq, namely, common, $qm = qn$, and Cm = Cn; hence, the angle Cqm is equal to the angle Cqn; that is, Cq is perpendicular to mn.

Vice versâ, if Cq is drawn perpendicularly to mn, the chord mn is bisected in q. For the right-angled triangles Cqn, Cqm, besides the right angles at q and the common side Cq, have the hypothenuse Cm of the one equal to the hypothenuse Cn of the other; hence, the two triangles are equal, and $qm = qn$.

THEOREM II.

A straight line drawn from the centre and bisecting the chord, when produced, bisects the arc also, and vice versâ.

The straight line Cq, drawn from the centre to the point q of mn, equidistant from the extremities m and n, and produced, bisects the corresponding arc mrn. In fact, since the angles mCq, qCn are equal to each other, the arcs also (B. I., Meas. of Angles) are equal; that is, $mr = nr$.

Vice versâ, if we draw the radius Cr to the middle point r of the arc mrn, the same radius will bisect the corresponding chord. Because the triangles mCq, nCq have the sides mC, Cq and the included angle of the one equal to the sides nC, Cq and the included angle of the other; hence,

$$mq = qn.$$

COROLLARY I. Of two chords intersecting each other and not passing through the centre, one must divide the other unequally.

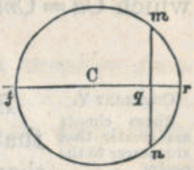
Let two chords nm, pq intersect mutually in o: om and on must be unequal, or else op and oq. For, if we suppose $mo = no$, then co is perpendicular to mn; and if po and oq also are equal, the same Co would be perpendicular to pq also, and the angle Coq would be equal to Com, which is impossible.

COROLLARY II. The straight line which bisects any chord, and forms right angles with it, passes through the centre.

Let now the straight line fo be drawn perpendicularly to mn, and let it pass through the point o equidistant from the extremities of the chord, the same fo must pass through C, the centre of the circle. Else, drawing from the centre of the circle a straight line to o, this line would be perpendicular to mn, and we would have two perpendiculars to mn meeting in o, which is impossible.

COROLLARY III. The straight line, also, which bisects the chord and the corresponding arc, or which bisects the arc and is perpendicular to the chord, passes through the centre.

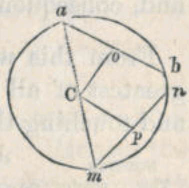
If the straight line *fr* bisects the arc *mn* in *r* and the corresponding chord in *q*, it must pass through the centre. For, if it does not pass through it, draw from the centre a straight line to *q*: this line produced will pass through *r*; hence, *rf* and the line drawn from the centre to *q* coincide from *q* to *r*, and, consequently, in the supposition that *rf* avoids the centre, we would have two straight lines coinciding from *r* to *q*, and then deviating from each other; which is impossible.



If *rf* bisects the arc, and is perpendicular to the chord, it must likewise pass through the centre; for from *r* only one perpendicular can be drawn to *mn*. But if from the centre we draw a perpendicular to *mn*, this perpendicular passes through *r*; hence, *rf* passes through the centre.

COROLLARY IV. The chords that are equidistant from the centre are equal, and *vice versâ*.

When the perpendiculars C*o*, C*p*, drawn to the chords *ab*, *mn*, are equal, the chords are said to be equidistant from the centre. Now, when chords are equidistant from the centre they are equal to one another. In fact, join C with *a* and with *m*: we have two right-angled triangles, C*oa*, C*pm*, from which



$$\overline{Ca}^2 = \overline{Co}^2 + \overline{oa}^2,\ \overline{Cm}^2 = \overline{Cp}^2 + \overline{pm}^2.$$

Now, $Ca = Cm$, and, by supposition, $Co = Cp$;

therefore, $\overline{ao}^2 = \overline{pm}^2$,

and, consequently, $ao = pm$;

and, $ab = mn$.

If, *vice versâ*, $ab = mn$, we will have them equidistant from the centre, because, drawing Co and Cp perpendicular to them, and then Ca and Cm, we have from the right-angled triangles the same preceding equations, in which $Ca = Cm$ and $ao = pm$; hence,

$$Co = Cp.$$

COROLLARY V. Those chords are greater that are nearer to the centre.

But if Co is less than Cp, that is, if ab is nearer to C than mn,—then $ab > mn$; because from the same right-angled triangles Cao, Cmp we have

$$\overline{Co}^2 + \overline{ao}^2 = \overline{Cp}^2 + \overline{pm}^2,$$

and $$\overline{ao}^2 = \overline{pm}^2 + \left(\overline{Cp}^2 - \overline{Co}^2\right).$$

Now, $Cp > Co$; hence, $\overline{Cp}^2 - \overline{Co}^2$ is a positive difference;

hence, $\overline{ao}^2 > \overline{pm}^2$, or $ao > pm$;

and, consequently, $ab > mn$.

From this we infer, besides, that the diameter is the greatest of all the straight lines drawn within the circle and touching the periphery with their extremities.

SCHOLIUM. The greater chord subtends the greater arc, and *vice versâ*.

Join now C with b and with n: the two triangles aCb, nCm have the sides Ca, Cb of the one equal to the sides Cm, Cn of the other, but the third side ab of the first greater than the third side mn of the other. Hence, (B. I. TH. 6,) $aCb > nCm$, and, consequently, the arc $ab > mn$.

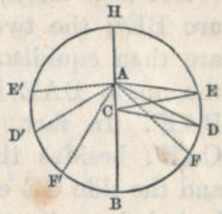
Vice versâ, if the arc $ab > mn$, or $aCb > nCm$, from the same triangles we have $ab > mn$.

Remark.

It is well understood that we take the arcs less than the semi-periphery; for any chord subtends two arcs, one greater and one less than the semi-periphery.

THEOREM III.

The greatest of all straight lines drawn to the periphery from some point out of the centre is that which passes through the centre. The others constantly diminish the more they recede from the centre.



Let A be any point out of the centre, and AE, AD, ... AB be lines drawn to different points of the periphery, but the last passing through the centre. Join C with E and with D: from the triangles ACE, ACD we have AD > AE, because AC and CE of the one are equal to AC and CD of the other. But the angle ACE is less than ACD; hence, (B. I. TH. 6,)

$$AD > AE.$$

We prove, in like manner, that AF > AD, and so, likewise, AD′ > AE′, AF′ > AD′, and so on. Hence, the more the straight lines drawn from A to the periphery approach AB the more they increase in length. Hence, AB is the greatest of all; and, since the more they approach to AH (a continuation of BA) on either side the more they diminish, AH is then the least of them all.

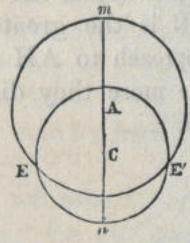
THEOREM IV.

Those straight lines drawn to the periphery from a point out of the centre and equidistant from the greatest are equal to one another.

Let now the arc BE be equal to the arc BE′; the two lines AE and AE′ are then equidistant from AB, because the angle BAE is equal to the angle BAE′. In fact, the triangles CAE, CAE′, besides the common side AC, and the side CE equal to CE′, have the included angles ACE, ACE′ also equal, being measured by the arcs HE, HE′, equal to each other. Hence, also, the angle BAE is equal to BAE′. But, from the same triangles, we have AE = AE′; hence, the two lines equidistant from AB are equal to each other.

It is plain that only two such lines can be equal; for any other line, AD for instance, or AG, approaches either AB or AH. In the first case it is greater, in the second less, than AE.

COROLLARY. Two circles having different centres can intersect each other in two points only.



It follows, hence, that two circles, E*m*E′, E*n*E′, of which the first has the centre in A, and the second in C, cannot intersect each other in more than two points. For, if we suppose the circle E*n*E′ to be met by E*m*E′ in more than two points, then, drawing from A or from C straight lines to these points of intersection, we would have more than two lines equal drawn from A to the periphery EnE′ or from C to the periphery EmE′.

THEOREM V.

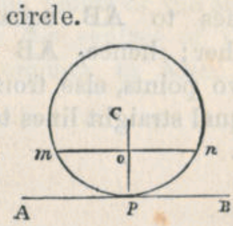
The tangent to the circle is perpendicular to the radius drawn to the point of contact, and vice versâ.

We call a tangent to the circle a straight line AB, which, however produced on both sides, remains always out of the periphery, but touches it in a point *m*. Now, if from the centre of the circle we draw the radius C*m* to the point *m*

of contact, it will be perpendicular to the tangent; for any other straight line drawn from C to AB must be greater than C*m*. But (B. I. TH. 9) the shortest line drawn from C to AB is perpendicular to it; hence, C*m* is perpendicular to AB; and, *vice versâ*, if AB meets the extremity of the radius and forms right angles with it, it is a tangent to the circle. For, drawing from C any other straight line to AB, it will be greater than C*m*, (B. I. TH. 8, SCH. 1,) and, consequently, out of the circle.

COROLLARY I. When the arc subtended by a chord is bisected and a tangent to the circle touches the point of section, it is parallel to the chord.

Let the arc *mpn* subtended by *mn* be cut into two equal parts in *p*, and let AB be tangent to the circle in the same point; *mn* and AB are parallel.



For C*p* is perpendicular to *mn* and perpendicular to AB.

COROLLARY II.

When the arc is bisected and a straight line passing through the point of section is parallel to the chord, it is also a tangent to the circle.

But if the arc *mpn* is bisected in *p*, and from *p*, AB is drawn parallel to *mn*, AB is a tangent to the circle. For, since C*p* is perpendicular to *mn* and AB parallel to *mn*, the radius C*p* is perpendicular also to AB, and AB is a tangent to the circle in *p*.

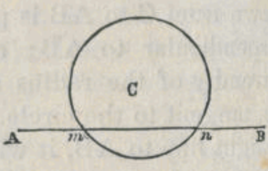
COROLLARY III.

When a tangent is parallel to a chord, the point of contact is the middle point of the arc subtended by that chord.

Let now the arc *mpn* be cut somewhere in *p*, and let AB touch the circle in the same point: if AB is parallel to *mn*, then *mp* is equal to *pn*. For, drawing C*p*, we have C*p*A, and, consequently, C*om*, right angles; but when the radius is perpendicular to the chord, it bisects both the chord and the arc; hence, *p* is the middle point of the arc *mpn*.

THEOREM VI.

The secant to the circle cannot meet it in more than two points.



The secant differs from the tangent, for it enters within the circle. Now, from the point C, or centre of the circle, we cannot draw more than two straight lines to AB equal to each other; hence, AB cannot meet the periphery but in two points, else from C we could draw more than two equal straight lines to AB.

THEOREM VII.

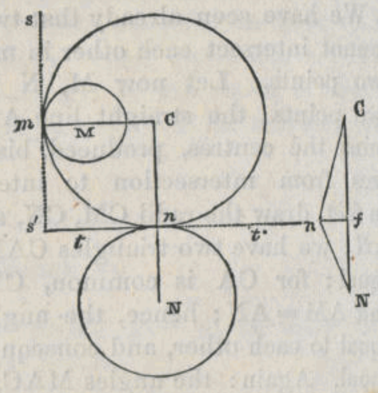
When two circles meet in two points, the straight line which joins the centres bisects the arcs and the chord between the intersections.

We have seen already that two circles cannot intersect each other in more than two points. Let now M, N be these two points: the straight line AC which joins the centres, produced, bisects the arcs from intersection to intersection. In fact, draw the radii CM, CN, and AM, AN: we have two triangles CAM, CAN equal; for CA is common, CM = CN, and AM = AN; hence, the angles, also, MC*s*, NC*s* are equal to each other, and consequently the arcs M*s*, N*s* are equal. Again: the angles MAC, CAN are equal to each other; hence M*r* = *r*N. Now, LQ bisects both peripheries; hence, from M*s* = *s*N, and M*r* = *r*N, we infer ML = LN, MQ = QN.

Now, the straight line drawn from the centre to the middle point of any arc bisects also the chord subtending that arc. But *r* and *s* are the middle points of the arcs M*r*N, M*s*N, and *rs* passes through the centres of the circles; hence, the chord MN, common to both, is bisected by *rs*.

THEOREM VIII.

When two circles touch each other in one point only, the straight line which passes through the centres passes also through the point of contact.



Two circles may touch each other in one single point in two ways. The one is with the circles, both external,—that is, with the point of contact between them, as *n*, or with the point of contact on the same side, as *m*. In both cases, the straight line passing through the centres, passes also through the point of contact. Because, drawing, in the first case, from the centres the radii C*n*, N*n* to the point of contact, if these two radii are not in the same straight line, let C*f*N (the side of the annexed triangle) be the straight line joining the two centres, and C*n*, N*n* the radii drawn to the point of contact. But C*f*N, by not passing through the point of contact, must cross some space out of the circles, and be consequently greater than the sum of the two radii. Now, CN in the triangle is, on the contrary, less than the sum of the radii C*n* + *n*N. It is, therefore, impossible that CN passes out of the point of contact. Hence, the normal *tnt'* drawn to CN is the common tangent of both circles. In the second case, draw *ss'* tangent to the external circle in *m*, the point of contact; the same ss' must necessarily be tangent also to the internal. Draw, then, from the centres C and M the radii to m: these two radii must be on the same straight line; otherwise, we could draw two perpendiculars to ss' from the same point m.

THEOREM IX.

The angle having its vertex at the centre is twice the angle at the periphery when both terminate at the extremities of the same arc.

The arc on which the angles rest is either less, or greater, or equal to half the periphery, or 180°. In the first supposition three cases may take place. And, first, let the side PM of the *inscribed* angle MPN pass through the centre C of the circle. Join C with N: thus we have another angle whose sides pass through the same extremities M and N of the arc MN, but having the vertex at the centre. Now, from the isosceles triangle PCN we have

First Case.

$$CPN = CNP;$$

and, consequently,

$$CPN + CNP = 2\,CPN.$$

But the external angle to

$$MCN = CPN + CNP;$$

hence,

$$MCN = 2\,CPN.$$

Second Case. But let the centre be within the angle MPN. Draw then from P, PQ passing through the centre. We will have

$$MCQ = 2\,MPQ,\ NCQ = 2\,NPQ.$$

Hence, $MCQ + NCQ = 2(MPQ + NPQ)$;

that is, $MCN = 2\,MPN.$

Third Case. Let, finally, the centre C be out of the angle MPN, and draw again PQ through the centre, and join C with M and N: we have

$$QCM = 2\,QPM,\ QCN = 2\,QPN.$$

Hence,

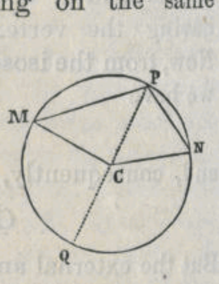
$$QCN - QCM = 2(QPN - QPM;)$$

that is, $MCN = 2\,MPN.$

In any case, therefore, when the arc MN is less than 180°, the angle at the centre is the double of the inscribed angle.

But, also, when the arc MN is greater than 180°, the angle at the centre, measured by this arc, is twice as great as the inscribed angle resting on the same arc.

Fourth Case. Let, in fact, NQM be an arc greater than 180°. The angle at the centre, measured by this arc, embraces the angles MCQ and QCN.

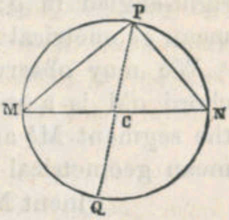


But $MCQ = 2\,MPQ,\ QCN = 2\,QPN$;

hence, $MCQ + QCN = 2(MPQ + QPN)$;

that is, $MCQ + QCN = 2\,MPN.$

Fifth Case. In the supposition that the arc MN is equal to the semi-periphery, then the angle at the centre becomes a straight line and diameter of the circle, equivalent to an angle of 180°; hence, the inscribed angle MPN is a right angle. Drawing, in fact, again PCQ, we will find, as in the preceding cases,



$$\text{MCQ} + \text{QCN} = 2\,\text{MPN}.$$

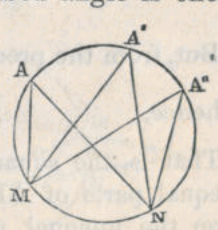
But $\text{MCQ} + \text{QCN} = 180°$;

hence, $\text{MPN} = 90°$.

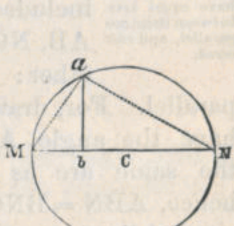
The inscribed angle, namely, whose sides pass through the extremities of the diameter, is a right angle.

Several corollaries may be now easily inferred.

COROLLARY I. Measure of inscribed angles. The measure of an inscribed angle is one-half of the corresponding included arc. Hence, the inscribed angles A, A′, A″ . . . ., including the same arc MN, are all equal to one another. And, as the angles including the semi-periphery are right angles, so those including an arc greater than the semi-periphery are obtuse; and those including an arc less than the semi-periphery are acute angles.



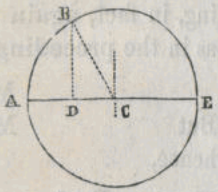
COROLLARY II. The perpendicular drawn to the diameter from any point of the periphery is a mean geometrical proportional between the segments. The perpendicular ab, drawn from any point a of the periphery to the diameter MN, is a mean geometrical proportional between the segments Mb, Nb. For, joining a with M and with N, we have the triangle MNa



right-angled in a; hence, (B. II. TH. 17, COR. 1,) ab is a mean geometrical proportional between Mb and bN.

We may observe, also, that (B. II. TH. 17, COR. 2) the chord aM is a mean geometrical proportional between the segment Mb and the diameter; or the chord aN is a mean geometrical proportional between the adjacent segment Nb and the diameter.

COROLLARY III. *Concerning the squares and rectangles constructed on the equal and unequal sections of the same straight line.*



Divide AE into two equal parts in C, and describe the circle ABE, having the centre in C; divide, also, AE unequally in D, and draw the perpendicular DB. Join, also, B with C. Now, since AC = BC, and $\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2$, we have

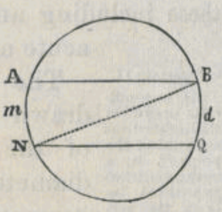
$$\overline{AC}^2 = \overline{BD}^2 + \overline{DC}^2$$

But, from the preceding corollary, $\overline{BD}^2 = AD \cdot DE$;

hence, $$\overline{AC}^2 = AD \cdot DE + \overline{DC}^2.$$

That is, the square constructed on AC, one of the two equal parts of AE, is equal to the rectangle constructed on the unequal parts AD, DE, plus the square of the intermediate segment DC.

COROLLARY IV. *Chords which have equal arcs between them are parallel, and vice versâ.*



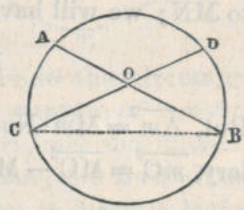
Let the arcs AmN, BdQ included by the chords AB, NQ be equal to each other: the chords are parallel. For, draw BN, and we will have the angle ABN measured by the same arc as the angle BNQ; hence, ABN = BNQ; hence, also, AB and AQ are two parallel lines.

Vice versâ, if AB and NQ are parallel lines, the included arcs are equal. For we have ABN = BNQ; but equal angles are measured by equal arcs, and the measure of the angle at B is half the arc A*m*N; the measure of the angle at N is half the arc B*d*Q; hence,

$$AmN = BdQ.$$

COROLLARY V. Half the sum of the arcs included between two chords intersecting each other is the measure of the angle formed by the chords.

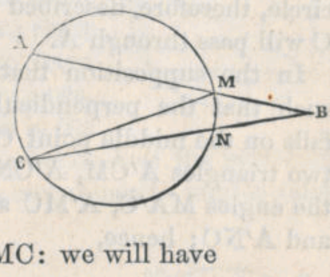
Let the chords AB, CD intersect each other in O: half the sum of the arcs AC, DB will be the measure of the angle AOC or BOD.



For, drawing CB, we have AOC = OBC + OCB. Now, the measure of OBC is $\frac{1}{2}$AC, and the measure of OCB is $\frac{1}{2}$DB, therefore, $\frac{1}{2}$(AC + DB) is the measure of DOB. In like manner, $\frac{1}{2}$(AD + CB) is the measure of the angle COB.

COROLLARY VI. The angle formed by two chords intersecting each other out of the circle is half the difference of the arcs included by it.

But if the chords AM, CN meet each other in a point B out of the circle, the angle ABC formed by them is then measured by half the difference of the arcs AC, MN included by it. Draw, in fact, MC: we will have



$$AMC = MCN + MBN;$$

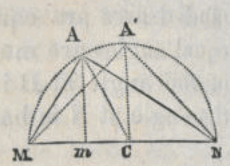
and, consequently,

$$MBN = AMC - MCN.$$

Now, half the arc AC is the measure of the angle AMC, and half the arc MN is the measure of the angle MCN; hence, the measure of the angle ABC is

$$\tfrac{1}{2}(AC - MN.)$$

COROLLARY VII. The circle having for its diameter the hypothenuse of a right-angled triangle will pass through the vertex of the right angle.



Let MN be the hypothenuse of the right-angled triangle AMN; bisect it in C, and join C with A; draw, also, from A the perpendicular Am to MN; we will have

$$\overline{CA}^2 = \overline{Am}^2 + \overline{mC}^2$$

But, $\overline{Am}^2 = Mm \cdot Nm$, and, from the third preceding corollary, $\overline{mC}^2 = \overline{MC}^2 - Mm \cdot Nm$; hence,

$$\overline{CA}^2 = Mm \cdot Nm + \overline{MC}^2 - Mm \cdot Nm = \overline{MC}^2,$$

and, consequently,

$$CA = MC;$$

that is, the points M, A, N are equidistant from C. A circle, therefore, described with the radius CM and centre C will pass through A.

In the supposition that the right-angled triangle be such that the perpendicular drawn from the vertex A′ falls on the middle point C of the hypothenuse, then the two triangles A′CM, A′CN are equal to each other, and the angles MA′C, A′MC are respectively equal to NA′C and A′NC; hence,

$$MA'C + NA'C = 2\,MA'C,$$
$$A'MC + A'NC = 2\,A'MC.$$

But $MA'C + NA'C = 90°$, and $A'MC + A'NC = 90°$;

therefore, $$2\,MA'C = 2\,A'MC,$$

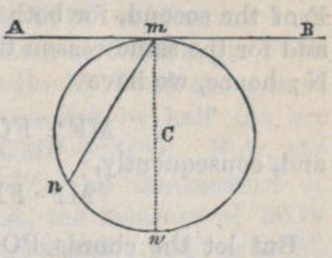
or, $$MA'C = A'MC;$$

and, consequently,

$$MC = CA'.$$

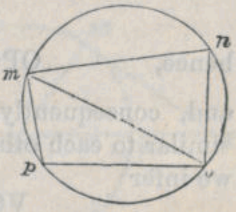
COROLLARY VIII. The angles which a chord drawn from the point of contact makes with the tangent are measured by half the arcs subtended by it.

The angles A*mn*, B*mn*, which the tangent AB makes with the chord *mn*, are measured by half the arcs *mn*, *mn'n*. For, drawing *mn'* perpendicular to AB, it will pass (TH. 5) through the centre C and divide the circumference into two equal parts. Now, the measure of A*mn'* or B*mn'* is one-half of the semi-periphery, and the measure of *nmn'* is $\frac{1}{2}nn'$. But $Amn = Amn' - nmn'$, and $Bmn = Bmn' + n'mn$; hence, the measure of A*mn* is $\frac{1}{2}mnn' - \frac{1}{2}nn' = \frac{1}{2}(mnn' - nn') = \frac{1}{2}mn$, and the measure of B*mn* is $\frac{1}{2}mn'n$.



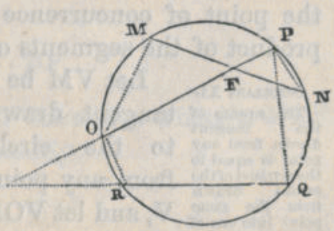
COROLLARY IX. A quadrilateral inscribed in a circle has its opposite angles equivalent to two right angles.

Let *mnop* be a quadrilateral inscribed in the circle: the opposite angles *n* and *p* make together the sum of two right angles. For, drawing *mo*, we have the periphery divided into two parts, *mpo*, *mno*, and one-half of each is the measure of the opposite angle; hence, half the periphery, or 180°, is the measure of the two opposite angles taken together.



COROLLARY X. The product of the segments of one chord is equal to the product of the segments of another.

Let the chords MN, PO meet each other at F within the circle: we will have MF · FN = PF · FO. In fact, the triangles MFO, PFN are similar to each other; because the angle MFO is equal to its opposite PFN, and the angle M of the first triangle is equal to the angle



P of the second, for both are measured by the same arc; and for the same reason the angle O is equal to the angle N; hence, we have

$$MF : FO :: FP : FN.$$

and, consequently,

$$MF \cdot FN = FO \cdot FP.$$

But let the chords PO, QR meet each other in V out of the circle. Join P with Q, and O with R; we will have the angles ORQ and OPQ making together two right angles, and likewise the angles POR, PQR; hence,

$$ORQ + OPQ = 180°,$$
$$POR + PQR = 180°;$$

but, also,

$$ORQ + ORV = 180°,$$
$$POR + ROV = 180°;$$

hence,

$$OPQ = ORV, \; PQR = ROV;$$

and, consequently, the two triangles VOR, VQP are similar to each other. And from their homologous sides we infer

$$VQ : VO :: VP : VR;$$

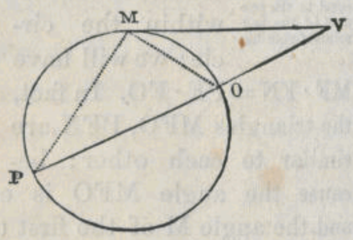
hence,

$$VQ \cdot VR = VP \cdot VO;$$

that is, even when the chords meet each other out of the circle, the product of the segments of the one between the point of concurrence and the circle is equal to the product of the segments of the other.

Corollary XI. The square of the tangent drawn from any point is equal to the product of the secant drawn from the same point into one of its segments.

Let VM be a tangent drawn to the circle from any point V, and let VOP be any secant drawn from the same point. Join M with P and O: we



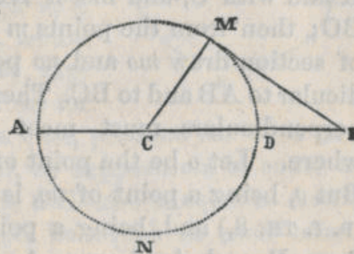
will have two triangles VMO, PMV similar to each other; for the angle V is common, and the angle VMP of the one is equal to the angle MOV of the other; because the angle VMP is measured by half the arc POM, and the angle MOV = OMP + MPO. Now, the measure of OMP is one-half of PO, and the measure of MPO is one-half of MO; hence, the measure of MOV is one-half of POM, and, consequently, PMV = MOV. Now, from the homologous sides of the two similar triangles we have

$$PV : MV :: MV : OV;$$

and, consequently,

$$\overline{MV}^2 = PV \cdot OV.$$

COROLLARY XII. Concerning the square and rectangles of the segments of a straight line and of the line itself. Let the straight line AD be bisected in C, and let DF be added to it, or let AD be produced to F. With the centre C and the radius CA describe the circle AMN, which will pass through D and from F draw FM tangent to the circle: we will have from the preceding corollary



$$\overline{MF}^2 = AF \cdot DF.$$

Join, now, C with M: we will have the triangle MCF right-angled in M, and, consequently,

$$\overline{CF}^2 = \overline{CM}^2 + \overline{MF}^2$$

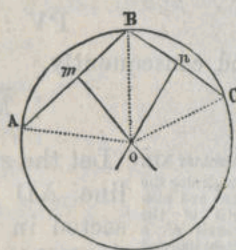
But $$CM = CD;$$

hence, $$\overline{CF}^2 = \overline{CD}^2 + AF \cdot DF,$$

the square, namely, of half the line AD plus the rectangle constructed on the sides AD+DF; and DF is equal to the square constructed on $\frac{1}{2}$AD + DF.

THEOREM X.

Three points that are not in the same straight line are certainly on the periphery of a circle.



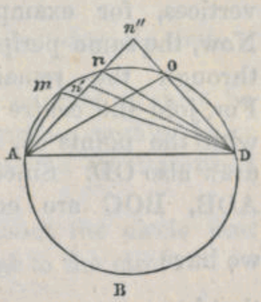
Let A, B, C be three points not situated on the same straight line; the same points must be on the periphery of a circle. For join B with A and with C, and bisect AB and BC; then from the points *m* and *n* of section draw *mo* and *no* perpendicular to AB and to BC. These two perpendiculars must meet somewhere. Let *o* be the point of their common intersection. But *o*, being a point of *no*, is equidistant from B and C, (B. I. TH. 8,) and, being a point of *mo*, is also equidistant from B and A; hence, A*o* = *o*B = *o*C. Therefore, describing the circle with the centre *o* and the radius *o*A, this circle must pass through B and C also.

COROLLARY. Any triangle may be inscribed in a circle.

Any polygon whose angles or vertices are on the periphery of a circle is called an inscribed polygon; and when the polygon has three sides, an inscribed triangle; when four, an inscribed quadrilateral, &c. Now, since the vertices of any triangle are three points not situated on the same straight line, any triangle may be inscribed in a circle.

THEOREM XI.

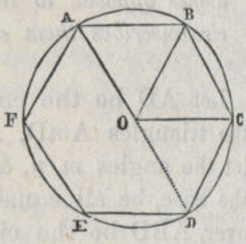
When any number of triangles have the same base, and the angles opposite to the base are all equal, the same circle circumscribes them all.



Let AD be the common base of the triangles AmD, AnD, &c., and let the angles m, n, &c., opposite to the base, be all equal. Let, moreover, ABD be the circle in which the triangle AmD is inscribed: the same circle will pass through n, o, &c. For else n, for example, would be either within the periphery, suppose in n', or out of the circle, for instance in n''. In the first case, produce An' to n and join n with D. Now, An'D, by supposition, is equal to AmD. But the inscribed angle AnD also is equal to AmD; hence, An'D and AnD would be equal to each other, which is impossible, because (B. II. TH. 10, SCH.) An'D $=$ $n'n$D $+$ nDn'. In the second case, from the point of intersection of An'' with the periphery draw the chord nD: we have again the inscribed angle AnD $=$ AmD, and, by supposition, An''D also equal to the same angle, which is impossible. The vertex n, therefore, of the triangle AnD must be on the periphery, and the same can be proved of the other triangles.

THEOREM XII.

Any regular polygon may be inscribed in the circle.



Let ABCD . . . . represent any regular polygon. The periphery of a circle may pass through three vertices, for example, A, B, C. Now, the same periphery must pass through the remaining vertices. For, join the centre O of the circle with the points A, B, and C, and draw also OD. Since the triangles AOB, BOC are equal to each other and isosceles,

we have $$OBA = OBC;$$

that is, $$OBC = \tfrac{1}{2}ABC.$$

Now, the angle $BCD = ABC$, and the angle $BCO = CBO$;

hence, $$BCO = OCD.$$

But the sides of regular polygons are all equal; hence, the triangles OBC, ODC, besides the common side OC, have the side BC equal to CD; hence, the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other;

therefore, $$OD = OB;$$

and, consequently, the point D is equally distant from the centre O as the point B, which belongs to the periphery; hence, D also belongs to the same periphery. We prove, in like manner, that E is equidistant from the centre as the points A, B, C, D, and so on.

Hence, since we may always describe a circumference passing through three vertices of any polygon, and since, when the polygon is regular, the periphery which passes through three successive vertices passes also through all the others, any regular polygon may, consequently, be inscribed in the circle.

THEOREM XIII.

Any regular polygon may be circumscribed about the circle.

Let now a regular polygon of any number of sides be inscribed in the circle ABO . . . . : another regular polygon of the same number of sides may be circumscribed about the circle.

We call polygon circumscribed about the circle that polygon which has all its sides tangent to the circle.

Draw from the centre C, Cm, Cm' perpendicular to the sides AB, BO of the inscribed polygon. Since these sides are chords of the circle, the perpendiculars bisect them, and, produced to n and n', bisect the arcs also.

But $BnA = Bn'O$;

hence, $Bn = Bn'$.

Draw now the radius CB, which will be the common hypothenuse of the right-angled triangles CBm, CBm', and the triangles are equal to each other; because, besides the common hypothenuse, the angle BCn of the one is equal to the angle BCn' of the other, having equal arcs for measure.

Draw from n the tangent nb, and from n' the tangent

$n'b$; these two tangents must meet at a point b of the radius CB produced; for the triangles Cnb, $Cn'b$ have the side Cn of the one equal to the side Cn' of the other, and the angles adjacent to the equal sides likewise equal; hence, the hypothenuse of the one must have the same length as the hypothenuse of the other; but the two hypothenuses are on the same straight line and have one extremity, C, common; the other extremity also, then, must be common.

In like manner, the tangent drawn from the middle point n'' of the arc AF meets ab in a point of CA produced; and the tangent drawn from the middle point n''' of the arc OD meets bo in a point o of CO produced, &c. The tangents drawn from the points n, n', n'', &c. . . . form a polygon; and the radii drawn to the vertices of the inscribed polygon meet, if produced, the vertices of the circumscribed one; and the sides of the circumscribed polygon are evidently the same in number as those of the polygon inscribed.

It is now easy to see how the circumscribed polygon is a regular polygon, having, namely, all its sides and all its angles equal. And, with regard to the angles, the angle b of the circumscribed polygon is equal to the angle B of the inscribed one; for ba and bo are respectively parallel to BA and BO; and, in like manner, all the other angles of the circumscribed polygon are equal to the corresponding angles of the inscribed polygon. But the angles of the inscribed polygon are all equal; hence, also, the angles of the circumscribed polygon are equal. With regard to the sides: from the similar triangles ABC, abC, we have $ab : \mathrm{AB} :: \mathrm{C}b : \mathrm{CB}$;

hence, $$ab = \mathrm{AB} \cdot \frac{\mathrm{C}b}{\mathrm{CB}}.$$

And, in like manner, from the similar triangles BOC, boC, we infer,

$$bo = \mathrm{BO} \cdot \frac{\mathrm{C}b}{\mathrm{CB}}.$$

But $\mathrm{AB} = \mathrm{BO}$;

hence, also, $ab = bo$.

In the same manner we demonstrate that bo is equal to the following side, and so on.

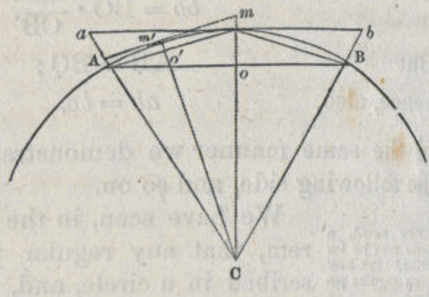
Vice versâ, a circle may be inscribed in any given regular polygon.

We have seen, in the preceding 12th theorem, that any regular polygon may be inscribed in a circle, and, when inscribed, each side becomes a chord. Now, chords of equal length are equidistant from the centre, (TH. 2, COR. 4;) that is, the perpendiculars drawn from the centre to every one of them are all equal. Now, describing a circle with the same centre and with a radius equal to the common distance, the circle will have all the sides of the polygon tangent, and the polygon will be circumscribed about it.

THEOREM XIV.

The circle may be considered as the limit of inscribed and circumscribed regular polygons whose sides increase constantly in number, or as a regular polygon of an infinite number of sides.

Let AB represent the side of a regular polygon inscribed in the circle, and ab the parallel side of the corresponding circumscribed and similar polygon. Drawing the radius Cm at the point of contact, it bisects AB and ab, forming right angles with both sides; the segment, moreover, mo of the radius, being the common perpendicular between the parallels ab, AB, is the measure of their mutual distance, which is the same with regard to all the other sides of the polygons.



Join, now, A with m: the chord Am will be the side of a regular polygon inscribed in the circle and having double the number of the sides of the polygon to which AB belongs. In fact, joining m with B, we will have a chord equal to mA, and bisecting likewise all the remaining arcs subtended by the other sides equal to AB, and joining the middle points with the extremities of the arcs; for each chord equal to AB we will have two equal to Am, and forming angles equal to one another because measured by equal arcs. Draw, now, the radius Cm' perpendicular to Am, and from m' a tangent to the circle. In the same manner as Am represents the side of a polygon containing the double of the sides of that to which AB belongs, so the tangent drawn from m' and included within the angle ACm represents the side of the circumscribed regular polygon having the same number of sides. $o'm'$, moreover, is the distance between the sides of the two polygons inscribed and circumscribed. Now, let us compare this distance, which is equal to m'C $-$ Co', with the distance $mo = m$C $-$ Co. But the normal Co is less than any oblique line drawn from C to AB, and, consequently, it is much more less than Co', which goes beyond AB; hence, since m'C, mC are radii of the same circle,

$$m'C - Co' < mC - Co,$$

or,

$$m'o' < mo.$$

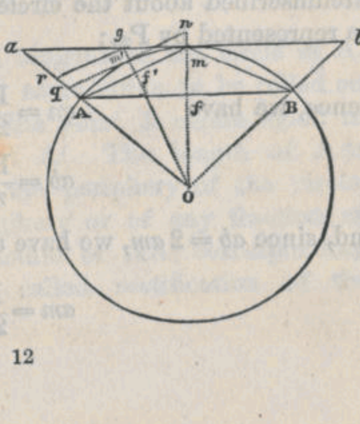
Now, duplicating again the number of sides of both polygons inscribed and circumscribed, and continuing indefinitely this duplication, we will have the two polygons constantly approaching coincidence; but they cannot approach each other without approaching at the same time the periphery of the circle between them, and they could not coincide with each other without coinciding at the same time with the periphery. Hence, the periphery of the circle is the limit towards which regular polygons, inscribed as well as circumscribed, tend, when their sides constantly increase in number; or the circle itself may be considered as a polygon having an infinite number of sides.

SCHOLIUM I. But also without duplicating the number of sides, but only increasing it in any manner, we come to the same conclusion; for the two polygons, inscribed and circumscribed, approach each other by increasing the number of their sides.

SCHOLIUM II. The perimeter of the inscribed polygon increases and that of the circumscribed polygon decreases by duplicating the number of sides.

Since the periphery of the circle is constantly between the perimeters of the two polygons, inscribed and circumscribed, and since by increasing the number of the sides of the polygons they approach constantly each other, we infer that the perimeter of the inscribed polygon increases and that of the circumscribed polygon decreases by increasing the number of sides.

And, in fact, let AB be one of the n sides of the regular inscribed polygon whose perimeter we will represent by p_n: we will have



$$AB = \frac{1}{n} p_n.$$

Divide the arc AmB into two equal parts in m, and join A with m: Am will be one of the $2n$ sides of another regular inscribed polygon; and, calling p_{2n} the perimeter of this polygon, we will have

$$Am = \frac{1}{2n} p_{2n}.$$

Join, now, B with m: we will have $Am + mB = 2\,Am$; But from the last equation

$$2Am = \frac{1}{n} p_{2n};$$

hence, $$Am + mB = \frac{1}{n} p_{2n}.$$

Now, $$Am + mB > AB;$$

hence, $$\frac{1}{n} p_{2n} > \frac{1}{n} p_n;$$

that is, $$p_{2n} > p_n.$$

Draw, now, from O the radius Om', perpendicular to Am, and produce it to g on the side ab of the circumscribed polygon of n sides, and call P_n the perimeter of the same polygon. The tangent qn, limited by the sides oa, on, and passing through the middle point of the arc Am, is one of the $2n$ sides of another regular polygon, which may be circumscribed about the circle, and whose perimeter may be represented by P_{2n};

hence, we have $$qn = \frac{P_{2n}}{2\,n},$$

$$ab = \frac{P_n}{n};$$

and, since $ab = 2\,am$, we have also

$$am = \frac{P_n}{2\,n}.$$

Now, $am > qn$; for, drawing from g, gr parallel to qn, we have, from the increasing proportional sides of the triangles rgO, qm'O,

$$rg > qm'.$$

But $ag > rg$; hence,

$$ag > qm';$$

and from the equal triangles gmO, nm'O we have

$$gm = m'n;$$

hence,
$$ag + gm > qm' + m'n;$$

that is,
$$am > qn,$$

and, consequently,
$$\frac{P_n}{2n} > \frac{P_{rn}}{2n};$$

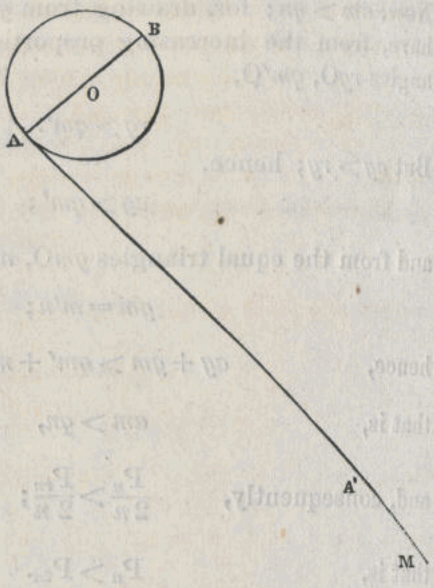
that is,
$$P_n > P_{2n}.$$

COROLLARY I. The periphery of the circle is less than the perimeter of any circumscribed polygon, and greater than the perimeter of any inscribed polygon.

Hence, we infer this important corollary: that the periphery of the circle is constantly between the perimeters of the two polygons; for they approach at the same time each other and the periphery, the one constantly increasing and the other constantly diminishing.

COROLLARY II. Concerning the rectification of the periphery.

Let AM be a tangent to the circle at A, and suppose the same circle to be rolled on the tangent till the point A comes again in contact with the tangent at A′. The length of AA′ is evidently equivalent to the periphery of the circle, and the value of this periphery or of any fraction of it given by AA′, or a fraction of AA′ corresponding to that of the circle, is called rectification of the periphery.



Now, between the radius and the periphery, as we will see, there is not a common measure. Taking, however, the radius of the circle as unity of measure, we may have the value of the periphery given by units and fractions of unity as near to the exact value as we may desire. Let, in fact, a represent one of the n sides of the regular polygon circumscribed about the circle, and let a' be one of the n sides of the regular polygon inscribed in the same circle AB: if we take two straight lines having the one $n \cdot a$ for length, and the other $n \cdot a'$, the two straight lines represent the perimeters of the two polygons in the same manner in which AA′ represents the periphery of the circle; hence, according to the preceding corollary, we have

$$n \cdot a > \mathrm{AA'} > n \cdot a'.$$

Now, if for any number n of sides we may obtain the values of $n \cdot a$ and $n \cdot a'$, given by the radius and fractions of the radius, a numerical value between that of $n \cdot a$ and that of $n \cdot a'$ is the value of AA′ given by the same radius. But, by increasing indefinitely n, $n \cdot a$ and $n \cdot a'$ approach indefinitely each other and AA′; hence, much more, any value between them approaches the same AA′. But let us see how $n \cdot a$ and $n \cdot a'$ may be numerically given by the radius taken as unity of measure.

Perimeters of the polygons given by the radius.

Let mn be the side of a regular hexagon inscribed in the circle BCAp. Draw the radii Om, On at the extremities of the side: we will have $mOn = \frac{360°}{6} = 60°$. But in the triangle mOn, $mO = nO$;

hence also $$nmO = mnO.$$

But $$nmO + mnO + mOn = 180°;$$

that is, $$2\,nmO + 60° = 180;$$

hence, nmO, and, consequently also, $mnO = 60°$;

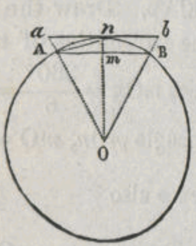
and, therefore, (B. I. TH. 13,)

$$mn = mO = nO.$$

That is, the side of the regular inscribed hexagon is equal to the radius of the circle, and, consequently, making $Om = 1$, the perimeter of the hexagon, numerically given by the radius, is 6.

Now, when the numerical value of the regular inscribed polygon of ν sides is given, we may infer from this the numerical value of the corresponding circumscribed polygon and that of the regular inscribed polygon of 2ν sides, as we will presently see. Hence, from the numerical value of the inscribed hexagon we infer the numerical value of the circumscribed hexagon and of the inscribed regular polygon of twelve sides. Again, from the numerical value of the inscribed polygon of twelve sides we infer the numerical value of the circumscribed polygon of the same number of sides, and that of the inscribed polygon of twenty-four sides, &c.

To see how, from the primeter p_ν of the regular inscribed polygon of ν sides, we may obtain the perimeter P_ν of the corresponding circumscribed polygon; let AB represent one of the ν sides of the inscribed polygon, and ab the parallel side of the corresponding circumscribed polygon: the radius On, drawn to the point n of contact, is perpendicular to both sides, and bisects both of them. Now, from the right-angled triangle AmO



we have $\overline{AO}^2$, or $1 = \overline{Om}^2 + \overline{Am}^2$;

and, consequently, $Om = \sqrt{1 - \overline{Am}^2}$,

and from the similar triangles anO, AmO,

we have $an : Am :: 1 : mO$;

hence, $an = \dfrac{Am}{Om}$;

that is, $an = \dfrac{Am}{\sqrt{1 - \overline{Am}^2}}$;

and, consequently,

$$2\,an, \text{ or } ab = \frac{2Am}{\sqrt{1 - \overline{Am}^2}} = \frac{AB}{\sqrt{1 - \overline{Am}^2}}.$$

Now, AB is one of the ν sides of the perimeter p_ν;

that is, $AB = \dfrac{p_\nu}{\nu}$;

and, in like manner, $ab = \frac{P_\nu}{\nu}$.

Again, since $Am = \frac{AB}{2}$,

we have also $\overline{Am}^2 = \frac{\overline{AB}^2}{4} = \frac{\overline{p_\nu}^2}{4\nu^2}$.

Hence, from the preceding equation,

$$\frac{P_\nu}{\nu} = \frac{p_\nu}{\nu\sqrt{1 - \frac{\overline{p_\nu}^2}{4\nu^2}}};$$

and, consequently,

$$P_\nu = \frac{2\nu \cdot p_\nu}{\sqrt{4\,\overline{\nu}^2 - \overline{p_\nu}^2}} \quad (a)$$

a formula giving the value of the perimeter P_ν by that of the corresponding inscribed p_ν. Let us now pass to see how $p_{2\nu}$, or the perimeter of a regular inscribed polygon of 2ν sides, may be given likewise by p_ν.

Draw An, which is the side of the regular inscribed polygon, having $p_{2\nu}$ for perimeter.

Since $nm = On - Om = 1 - \sqrt{1 - \overline{Am}^2}$,

and $\overline{An}^2 = \overline{Am}^2 + \overline{mn}^2$,

therefore, $\overline{An}^2 = \overline{Am}^2 + 1 - 2\sqrt{1 - \overline{Am}^2} + 1 - \overline{Am}^2 =$

$$2\left(1 - \sqrt{1 - \overline{Am}^2}\right)$$

and $$An = \sqrt{2\left(1 - \sqrt{1 - \overline{Am}^2}\right)}.$$

Now, $$An = \frac{p_{2\nu}}{2\nu},$$

and $$Am = \frac{AB}{2} = \frac{p_\nu}{2\nu};$$

hence, $$\frac{p_{2\nu}}{2\nu} = \sqrt{2\left(1 - \sqrt{1 - \frac{p_\nu^2}{4\nu^2}}\right)},$$

and $$p_{2\nu} = 2\nu\sqrt{2\left(1 - \sqrt{1 - \frac{p_\nu^2}{4\nu^2}}\right)} \quad (a');$$

a formula by which the perimeter of the inscribed polygon of 2ν sides is given by that of the inscribed polygon of ν sides.

Nothing else remains to be found to obtain a series of the numerical values of the inscribed and corresponding circumscribed polygons of six, of twelve, of twenty-four sides, &c. Thus, for example, by making, in (a'), $p_\nu = 6$, or supposing the regular inscribed polygon of ν sides to be a hexagon, we will find $p_{2\nu}$, or $p_{12} =$ 6.2116571 . . . . ; and, substituting this value of p_{12}, instead of p_ν, in the formula (a), we will obtain the numerical value of the perimeter of the circumscribed polygon of twelve sides,—that is, $P_{12} = 6.4307806 \ldots$. Substituting, then, in (a'), the found value of p_{12} instead of p_ν, we will find p_{24}, &c. . . . . Continuing in this manner, we will obtain

$$p_{6144} = 6.2831850 \ldots,$$
$$P_{6144} = 6.2831858 \ldots,$$

for the numerical values of the inscribed and circumscribed polygons of 6144 sides, the radius of the circle being taken as the unity of measure. Now, the numerical value of the periphery of the circle given by the radius is, as we have seen above, between the numerical values of the polygons inscribed and circumscribed, whatever may be the number of their sides; hence, it is also between p_{6144} and P_{6144}. But these two values are equal to each other as far as the sixth decimal figure; the same number, therefore, as far as the sixth decimal figure, represents also the numerical value of the periphery.

The periphery of the circle is usually expressed by 2π; hence,

$$2\pi = 6.283185 \ldots;$$

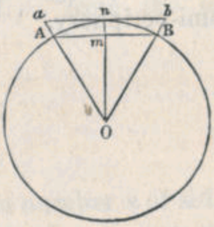
and, consequently, we have, also,

$$\pi = 3.141592 \ldots$$

for the numerical value of the semi-periphery.

THEOREM XV.

The area of the circle is equal to the product of the radius into the semi-periphery.



The area of the circle is evidently between the areas of the two polygons inscribed and circumscribed. Let, now, ν be the number of the sides of the inscribed and of the circumscribed polygon, and let AB and ab represent one of their respective sides. Now, the two polygons are divisible into as many triangles equal to AOB and aOb as there are sides; and the area of

AOB is $\frac{1}{2}AB \cdot Om$, and the area of aOb is $\frac{1}{2}ab \cdot On$. Hence, the area of the inscribed polygon of ν sides is

$$(\tfrac{1}{2}AB \cdot Om)\nu,$$

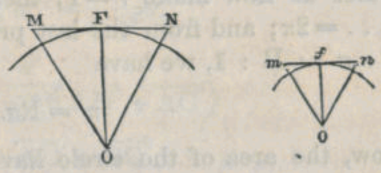
and that of the circumscribed is

$$(\tfrac{1}{2}ab \cdot On)\nu.$$

Observe, now, that $\frac{1}{2}AB \cdot \nu$ and $\frac{1}{2}ab \cdot \nu$ are half the perimeters of the inscribed and circumscribed polygons; hence, the areas of both polygons are given by the product of half the perimeter into the perpendicular drawn to any of their sides from the centre of the circle. Now, by increasing indefinitely the number ν of sides, the perimeters of the two polygons approach each other, and the difference between On and Om approaches zero, and, consequently, the areas of the two polygons are becoming identical; but then only will they be identical when Om will become equal to On, and when the perimeters of the polygons coincide with the periphery of the circle. But the semi-perimeter is then changed into the semi-periphery, and the perpendicular drawn from the centre to any side is changed into the radius. The common area, therefore, of the two polygons, which is the same as that of the circle, is given by the product of the radius into the semi-periphery.

THEOREM XVI.

The area of the circle, having R for radius, is numerically expressed by $R^2 \cdot \pi$.



Let OF, *of* be the radius R and r of two circles, and let MN, *mn*, touching the circle in F and f, be the sides of any two regular polygons circumscribed about them, containing, however, the same number n of sides. In this supposition the triangles MON, *mon* are similar to each other;

hence, $$MN : mn :: MO : mo.$$

And from the similar triangles MFO, *mfo* we have, also,

$$MO : mo :: FO : fo;$$

hence, $$MN : mn :: R : r.$$

Now, MN, *mn* are the nth parts of the perimeters of the polygons circumscribed about the circles; and, calling these perimeters respectively P and p, from the last proportion we will have

$$\frac{1}{n}P : \frac{1}{n}p :: R : r;$$

that is, $$P : p :: R : r.$$

Now, this ratio does not depend on the number n of sides, which may be indefinitely increased; hence, we will have also the same ratio when the two polygons coincide with the peripheries; and, calling $2\pi'$ the periphery whose radius is R, and $2\pi''$ the periphery whose radius is r, we will have

$$2\pi' : 2\pi'' :: R : r;$$

and, also,
$$\pi' : \pi'' :: R : r;$$

that is, *the peripheries or semi-peripheries of two circles are to each other as the radii of the same circles.*

Let us now make $r = 1$; then $2\pi''$ becomes 6.283185 .... $= 2\pi$; and from the last proportion, which becomes $\pi' : \pi :: R : 1$, we have

$$\pi' = R\pi.$$

Now, the area of the circle having R for radius and π' for semi-periphery is given by $R \cdot \pi'$; hence, from the last equation the same area is given, also, by

$$R^2 \cdot \pi.$$

COROLLARY I. The areas of two circles are to each other as the squares of their radii or diameters.

Let now R and R′ be the radii of two different circles, and let a, a' represent their areas: we will have

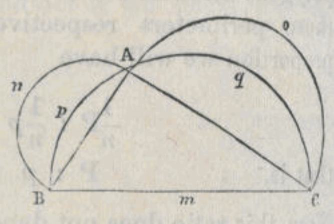
$$a = R^2\pi,\ a' = R'^2\pi;$$

hence,
$$a : a' :: R^2 : R'^2,$$

and, also,
$$a : a' :: (2R)^2 : (2R')^2.$$

COROLLARY II. The sum of the areas of the lunulæ is equal to that of the corresponding right-angled triangle.

Let ABC be a triangle right-angled in A, and with m, the middle point of the hypothenuse, as centre, and mB as radius, describe the semicircle BpqC, which will pass through A; describe, also, on AB and AC, taking them as diameters, the semicircles BnA, AoC. The surfaces ApBn, AqCo are called *lunulæ*



or *lunæ*. Now, the sum of the areas of the lunulæ is equal to the area of the triangle ABC. Call, in fact, a', a'', a''', the areas of the circles having BC, AB, AC for diameters: we will have

$$a' : a'' :: \overline{BC}^2 : \overline{AB}^2,$$

$$a' : a''' :: \overline{BC}^2 : \overline{AC}^2;$$

hence, $$a'' = \frac{\overline{AB}^2}{\overline{BC}^2} a', \quad a''' = \frac{\overline{AC}^2}{\overline{BC}^2} a',$$

and $$a'' + a''' = \frac{a'}{\overline{BC}^2}\left(\overline{AB}^2 + \overline{AC}^2\right)$$

But $$\overline{AB}^2 + \overline{AC}^3 = \overline{BC}^2;$$

hence, $$a'' + a''' = a';$$

and, also, $$\tfrac{1}{2}a'' + \tfrac{1}{2}a''' = \tfrac{1}{2}a';$$

that is, $$ABn + ACo = BCqp.$$

Now, $$ABn = ApBn + ABp,$$

$$ACo = AqCo + ACq,$$

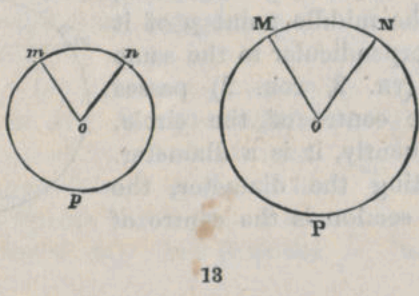
and $$BCqp = ABp + ACq + ABC.$$

Hence, substituting and eliminating the equal terms to be found in both members, we will have

$$ApBn + AqCo = ABC.$$

Corollary III. Similar arcs are to one another as their radii, and similar sectors are to one another as the squares of their radii.

Let r', r'' be the radii of the two circles *mnp*, MNP, and call π', π'' their semi-



peripheries. Let, also, mn, MN be two similar arcs; that is, each containing the same number of degrees and fraction of a degree; that is, if $mn=\frac{p}{q}\pi'$, MN be equal to $\frac{p}{q}\pi''$, and

$$mn : \text{MN} :: \pi' : \pi''.$$

Now, $\pi' : \pi'' :: r' : r''$;

hence, $mn : \text{MN} :: r' : r''$.

It is evident, moreover, that the sector mno takes as much of the area of its own circle as the sector MON takes of the area of its own; so that, calling a' and a'' the areas of the two circles, and $\frac{r}{s}a'$ the area of the sector mno, $\frac{r}{s}a''$ will be the area of the sector MNO, and we will have

$$mno : \text{MNO} :: a' : a''.$$

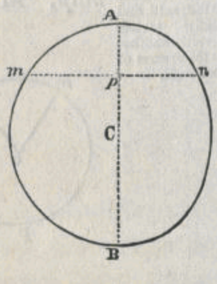
But $a' : a'' :: r'^2 : r''^2$;

hence, $mon : \text{MON} :: r'^2 : r''^2$.

PROBLEMS.

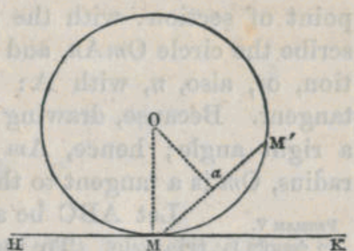
Problem I. *To find the centre of a given circle.*

Let AmBn be a given circle, the centre of which is to be found.



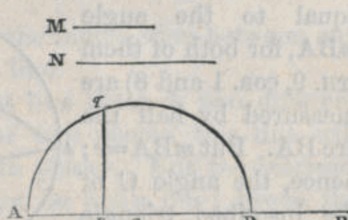
Draw any chord mn, and from the middle point p of it draw AB perpendicular to the same mn; AB (th. 2, cor. 2) passes through the centre of the circle, and, consequently, it is a diameter. Now, bisecting the diameter, the point C of section is the centre of the circle.

Problem II. To describe a circle which shall touch a given line in a given point and pass through another given point.



Let M and M′ be two given points, the first on the straight line HK, and the other out of it. To describe a circle which touches HK in M and passes through M′, join first M with M′ and bisect MM′. From the point *a* of section draw *a*O perpendicular to MM′, and from M draw MO perpendicular to HK. The point O of intersection is equidistant from M and M′; hence, a circle having the centre in O and described with the radius OM will pass through M′; but the same circle touches also (TH. 5) HK in M.

Problem III. To find the mean geometrical proportional between two given straight lines.



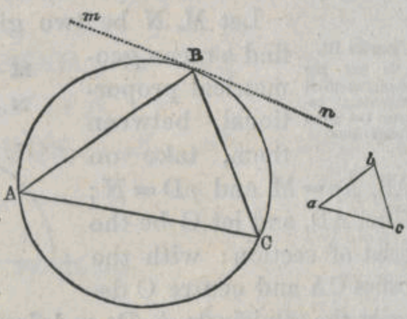
Let M, N be two given straight lines. To find a mean geometrical proportional between them, take on AB, A*p* = M and *p*D = N; bisect AD, and let C be the point of section: with the radius CA and centre C describe the semicircle A*q*D, and draw from *p*, *pq* perpendicular to AB: we will have (TH. 9, COR. 2) *pq* a mean geometrical proportional between the segment A*p* = M, and *p*D = N.

Problem IV. From a given point to draw a tangent to the circle.

Let *mpn* be a given circle and A a given point out of it. To draw a tangent from A to the circle, join first A with O, the centre of the circle, and bisect AO. Let R be the point of section: with the radius RO and centre R describe the circle $OmAn$, and join the point m of intersection, or, also, n, with A: Am or An is the required tangent. Because, drawing Om, we have the angle AmO a right angle; hence, Am being perpendicular to the radius, Om is a tangent to the circle.

PROBLEM V. To describe in a given circle a triangle similar to another triangle.

Let ABC be a given circle and abc a given triangle. To inscribe in the circle a triangle similar to abc, draw from any point B the tangent mn, and also two chords BA, BC, the first making with mn the angle mBA equal to the angle c of the given triangle, and the second the angle nBC equal to the angle a of the same triangle. Join, then, A with C, and the inscribed triangle BAC will be similar to the given triangle. In fact, the angle BCA is equal to the angle mBA, for both of them (TH. 9, COR. 1 and 8) are measured by half the arc BA. But $mBA=c$; hence, the angle C of the inscribed triangle is equal to the angle c of the given triangle.



In like manner the angle A of the inscribed triangle is equal to the angle a of the given triangle, and, consequently, the third angle B of the first triangle is equal to the third angle of the second, and the triangles are similar to each other.

BOOK V.

THE STRAIGHT LINE AND THE PLANE.

THEOREM I.

The intersection of two plane surfaces is a straight line.

It is evident that a straight line cannot coincide with a plane in two different points at any distance from each other without coinciding altogether with all its other points.

It is likewise evident that the intersection between any two surfaces cannot be but a line.

Now, if we draw a straight line through two different points of the intersection of two planes, this line will coincide altogether with both planes. But the intersection of the two planes cannot be but a line; hence, the straight line being at the same time on both planes, it must coincide with their intersection, which, consequently, is likewise a straight line.

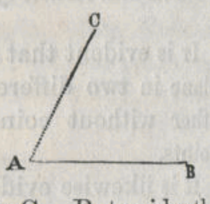
THEOREM II.

An indefinite number of planes may pass through the same straight line.

When two points are given in space, or the straight line which joins them, we may conceive a plane passing through them, which, if revolved about them, is evidently capable of taking an indefinite number of different positions, and, consequently, an indefinite number of planes may pass through the same straight line.

THEOREM III.

Only one plane may pass through three different points not situated on the same straight line.



Let A, B, C be any three points not situated on the same straight line. Join two of them—for example, A and B—with the straight line AB, and let a plane pass through AB. This plane may be turned in such a manner as to pass also through C. But evidently the same plane, being raised above or depressed below that point, must in all cases escape it; hence, two lines AB, AC, forming an angle, determine the position of a plane, because the extremities of these lines are three points not situated on the same straight line; hence, only one plane may pass through them, and the two lines will coincide with it.

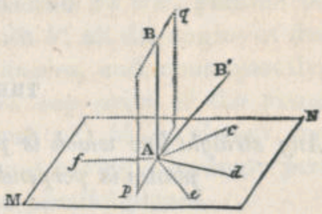
COROLLARY. Two lines forming an angle determine the position of a plane.

THEOREM IV.

Only one perpendicular to the plane may pass through the same point.

A straight line AB is said to be perpendicular to the plane MN when it is perpendicular to every straight line

Ac, Ad, Ae, &c. on the plane and passing through its foot. In one of the following theorems we will see how such a perpendicular may be erected from any point of a given plane. Now, we say that only one perpendicular may pass through the same point.

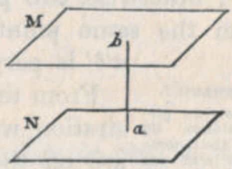


Suppose, in fact, AB′ to be another perpendicular to the plane, having the point A common with the plane and with AB, and let BA*d* be the plane determined by AB, AB′, and A*d* the intersection of this plane with MN. Since A*d* on the plane MN passes through the foot of both perpendiculars, the two angles BA*d*, B′A*d* will be both right angles, and, consequently, BA*d* = B′A*d*; that is, a part equal to the whole, which is impossible; hence, another perpendicular AB′ cannot be erected on MN from A besides AB.

Perpendicular and parallel planes.

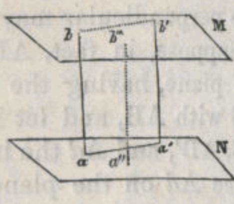
Any plane *pq* which passes through the perpendicular AB is said to be perpendicular to the other.

Two planes M, N, to which the same *ab* is perpendicular, are said to be parallel to each other. The reason of this expression will appear in the following theorem:—



THEOREM V.

Any straight line which is perpendicular to one of the parallel planes is perpendicular to the other also.



Let M, N be two parallel planes, and ab their common perpendicular; let, also, $a'b'$ be perpendicular to N: the same $a'b'$ will be perpendicular to M also. In fact, join a with a'; the plane determined by ab, aa' may be conceived as generated by a straight line coinciding first with ab and then passing successively through all the points of aa' and always parallel to ab. Now, this movable line is, with regard to both planes M and N, in the same relative angular position as ab; hence, it will remain constantly perpendicular to both. But when the movable line will pass through a' it will coincide with $a'b'$; otherwise, two perpendiculars could be drawn to N from the same point a', which is not possible; hence, $a'b'$ is perpendicular to both planes.

COROLLARY I. *The two perpendiculars lie on the same plane and are parallel to each other.*

From the process of the preceding demonstration we see that the two perpendiculars are on the same plane and parallel to each other. Nay, if we imagine any other straight line $a''b''$ perpendicular to N, $a''b''$ will be parallel to ab and to $a'b'$; and ab, $a''b''$ will be both on one common plane, and $a'b'$, $a''b''$ on another common plane.

COROLLARY II. *Two planes having a common perpendicular are equidistant everywhere.*

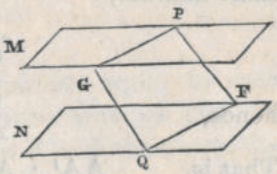
Hence, also, the two planes M, N are equidistant everywhere. For, take the common perpendicular ab as the measure of this distance: then the distance of the point a', taken at pleasure, will be measured by $a'b'$. parallel to ab. But, joining a with a', b with b', all the angles of the quadrilateral ba' will be right angles, and, consequently, $b'a' = ba$: hence, the distance of any point of the plane N from the corresponding point of M is always the same. For this reason, two planes having a common perpendicular are called parallel planes.

COROLLARY III. Parallel planes can never meet each other.

Hence, also, two parallel planes can never meet each other, even if they would be produced beyond all limits.

COROLLARY IV. The intersections of two parallel planes made by another plane are parallel lines.

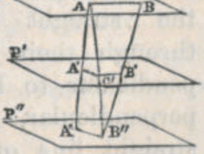
It follows, moreover, that if two parallel planes M, N are met by another plane PQ, the intersections PG and QF are two parallel lines; because PG, FQ are two straight lines of the plane PQ, and if they are not parallel they will somewhere meet each other. But PG is on M, and FQ on N; hence, where the lines meet, the planes also must meet. But the planes nowhere can meet; hence, neither can the intersections PG, FQ.



THEOREM VI.

The segments of any two straight lines between parallel planes are proportional.

Let P, P′, P″ be parallel planes, and AA′, A′A″ the segments of any straight line; that is, AA′ limited by P and P′, and A′A″ by P′ and P″; let, also, BB′, B′B″ be the corresponding segments of any other line BB″.



Join A with B″, and let C′ be the point of P′ met by B″A. Join, also, A′ with C′, and A″ with B″, A′C′ is at once on the plane P′ and on that determined by AA″, AB″; hence, A′C′ is the common intersection of these two planes; and, in like manner, A″B″ is the common intersection of P″ and of the same plane A″AB″. The two intersections, therefore, A′C′, A″B″ are parallel lines, and, consequently,

$$AA' : A'A'' :: AC' : C'B'';$$

that is, $$\frac{AA'}{A'A''} = \frac{AC'}{C'B''}.$$

Join now C′ with B′, and A with B; we will have, in the same manner,

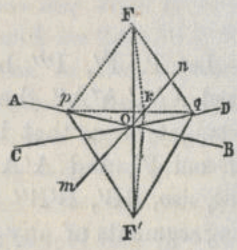
$$\frac{AC'}{C'B''} = \frac{BB'}{B'B''};$$

hence, $$\frac{AA'}{A'A''} = \frac{BB'}{B'B''}.$$

That is, $$AA' : A'A'' :: BB' : B'B''.$$

THEOREM VII.

When a straight line is perpendicular to two other lines intersecting each other, it is also perpendicular to the plane determined by them.



Let AB, CD be any two lines intersecting each other in O. If the straight line OF passing through their intersection is perpendicular to both of them, it is perpendicular, also, to any other straight line of the plane determined by AB, CD, and passing through O, and, consequently, perpendicular to the plane itself. In fact, produce FO to F′ in such a manner as to have $OF' = OF$. Draw, then, on the plane determined by AB, CD, any straight line mn passing through O; then from any point k of mn, draw pq any straight line reaching somewhere in p and in q the two given AB, CD. Join, moreover, F and F′ with the points p, k, and q: we will have

$$Fq = F'q,\ Fp = F'p.$$

Hence, the triangles Fpq, $F'pq$ are equal to each other, and, consequently, the angle

$$Fpq = F'pq;$$

hence, also, the triangles Fpk, $F'pk$ are equal to each other. Because, besides the common side pk and the side Fp of the one equal to the side $F'p$ of the other, the included angles Fpk, kpF' are also equal;

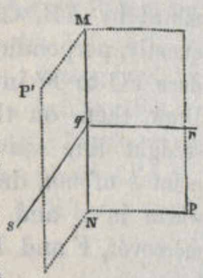
and, therefore, $$Fk = F'k.$$

Now, the triangles FOk, $F'Ok$, besides the common side Ok, have the side OF equal to the side OF′; and since, also, $Fk = F'k$, the triangles are equal,

and, consequently, $$FOk = F'Ok;$$

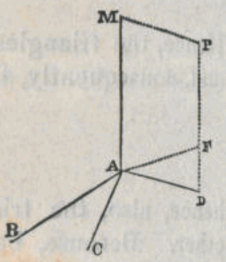
that is, FF′ is perpendicular to mn; and, since mn is any straight line on the plane determined by AB and CD, the same FF′ is therefore perpendicular to any straight line of the plane passing through O, and, consequently, to the plane itself.

Corollary I. Through any point of a given line a plane may pass perpendicular to this line.



Let q be any point of the straight line MN. A plane may pass through q to which MN is perpendicular. Let, in fact, P and P′ be two planes passing through MN. Draw on the first of these planes qr perpendicular to MN, and on the second qs perpendicular to the same MN. This MN will then be perpendicular to the plane determined by qr and qs.

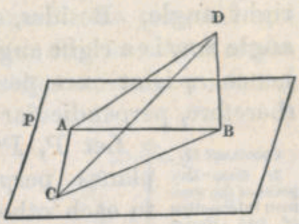
Corollary II. Three straight lines passing through the same point of another line and perpendicular to it are on the same plane.



Let AB, AC, AD be three straight lines perpendicular to MA and passing through the same point A of it: the three lines are on the same plane. Else, let P be the plane determined by AD and AM; if the plane on which AB and AC are does not pass through AD, it will cut the plane P along another line,—for instance, AF. But then, since AF is on the plane to which MA is perpendicular, it must make a right angle with MA; but, by supposition, MAD also is a right angle; hence, MAF = MAD, which being impossible, it is impossible also that AD be out of the plane determined by the other two perpendiculars AB, AC.

THEOREM VIII.

If two straight lines on a plane are perpendicular to each other, and one of them passes through the foot of a normal to the plane, this line, together with the normal, determines the plane to which the other straight line is perpendicular.



Let AB, AC be two straight lines of the plane P perpendicular to each other, and let AB pass through the foot of DB normal to P: the other line AC will be perpendicular to the plane determined by AB and BD.

Take, in fact, AC equal to BD, and join A with D and C with B: the two triangles ABD, ACB, besides the common side AB, and the side BD of the one equal to the side AC of the other, have the included angles ABD, BAC also equal; therefore,

$$AD = BC.$$

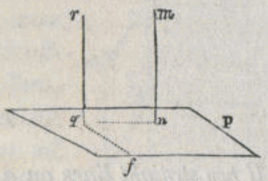
Join, now, C with D: the triangles CDA, CDB have the common side CD, and the two remaining sides of the one equal to the two remaining sides of the other; therefore, the angles also opposite to equal sides are equal; hence

$$CBD = CAD.$$

But CBD is a right angle; hence, CAD also is a right angle. But CAB is likewise a right angle; therefore AC is normal to the plane of the triangle DAB, which is that determined by AB and BD.

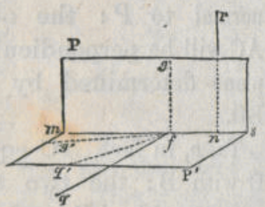
Corollary I. If one of two parallel lines is perpendicular to a plane, the other also is perpendicular to it.

Let *mn*, *rq* be parallel to each other, and consequently on the same plane. If *mn* is perpendicular to the plane P, *rq* also is perpendicular to the same P. Because, joining *n* with *q*, and drawing on P, *qf* perpendicular to *qn*, we will have *qf* perpendicular to the plane determined by *qn* and *nm*, which is the same as the plane of the parallel lines; hence, *rqf* is a right angle. Besides, since *rq*, *mn* are parallel and the angle *mnq* is a right angle, *rqn* also must be a right angle; hence, *rq* is at once perpendicular to *qn* and to *qf*, and, therefore, perpendicular to the plane P.



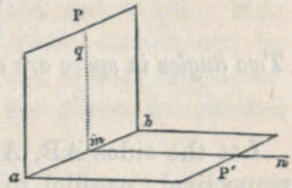
Corollary II. If from any point of the common intersection of two planes perpendicular to each other we draw a perpendicular to one of them, it will be on the other plane.

Let P, P′ be two planes perpendicular to each other, and let *ms* be their common intersection. The plane P cannot be perpendicular to P′ without passing through some line perpendicular to the same P′. Let *nr* be this line. Any other line *fg* on the plane P, parallel to *nr*, is likewise perpendicular to P′. Now, if from *f*, which is any point of the intersection of the planes, we draw *fq* perpendicular to P, *fq* must lie on P′, else it will be either above or below P′, and have, for example, the direction *fg′*. And let *fq′* be the intersection of the plane determined by *fg*, *fg′* with P′; then, since *gf* is perpendicular to P′, the angle *gfq′* is a right angle; and, since by supposition *fg′* is perpendicular to the plane P, the angle also *g′fg* is a right angle; we would have, therefore, the angles *q′fg*, *g′fg* equal to each other, which is impossible; hence, the perpendicular drawn from any



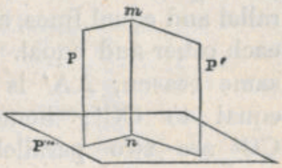
point of the common intersection to the plane P must lie on the other plane P′.

COROLLARY III. The perpendicular line to the common intersection, coinciding with one of two perpendicular planes, is a normal to the other plane.



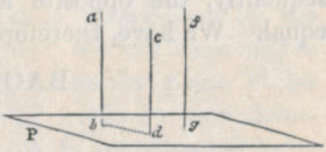
From the same theorem we infer that when the planes P and P′ are perpendicular to each other, and *mn* on the plane P′ is perpendicular to the intersection *ab*, the same *mn* is perpendicular to P; because, drawing *mq* perpendicular to P′ it will coincide with the plane P, and *qmn* is a right angle; and, since *bmn* also is a right angle, thus *mn* is normal to the plane determined by *mb*, *mq*, which is the plane P.

COROLLARY IV. When two planes are perpendicular to a third plane, their intersection is a normal to the same third plane.



Let *mn* be the intersection of two planes P and P′, both perpendicular to P″: *mn* must then be a normal to P″. In fact, if from *n* we erect a perpendicular to P″, it must be at once on P and on P′. But no other straight line is common to both planes except their intersection; hence, *mn* is a normal to P″.

COROLLARY V. When two lines in space are each parallel to another straight line, the two lines are parallel to each other.

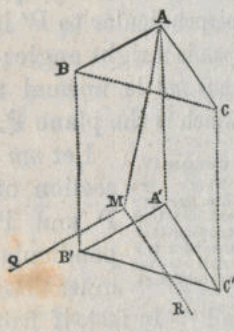


Let *ab* and *cd* be parallel to another straight line *fg* in space, and let P be a plane to which *fg* is perpendicular: then (COR. 1) *ab* and *cd* are also perpendicular to P. Join, now, *b* with *d*; the angles *abd*, *cdb* are both right angles, and *bd* represents the intersection of the plane perpendicular to P, determined by *cd* and *bd*, with P; hence, (COR. 2,) *ba* must coincide with the same plane, and therefore is parallel to *dc*.

THEOREM IX.

Two angles in space are equal when their sides are respectively parallel.

Let the sides AB, AC of the angle BAC in space be respectively parallel to the sides A′B′, A′C′ of the angle B′A′C′ on another plane.



Take AB = A′B′, and AC = A′C′, and then join A with A′, B with B′, C with C′, and also B with C and B′ with C′. Now, BB′, AA′, joining the extremities of two parallel and equal lines, are parallel to each other and equal. But, for the same reason, AA′ is parallel and equal to CC′; hence, BB′ and CC′ are two parallel and equal lines; therefore, BC, also, and B′C′, which join their extremities, are parallel and equal. Hence, the three sides of the triangle BAC are equal to the three sides of the triangle B′A′C′, and, consequently, the opposite angles to equal sides are also equal. We have, therefore,

$$BAC = B'A'C'.$$

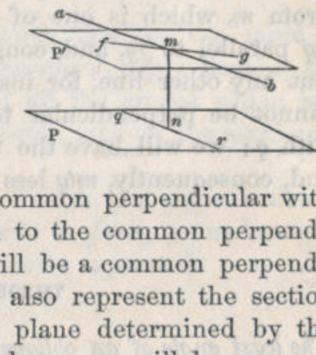
THEOREM X.

The planes determined by parallel lines forming angles in space are parallel.

Let AM be the perpendicular let fall from A to the plane determined by A′B′, A′C′, and draw from M, MR, MQ parallel to A′C′, A′B′, and, consequently, parallel also to AB and AC, (TH. 8, COR. 5.) But QMA and RMA are right angles; hence, also, BAM and CAM. MA, therefore, is perpendicular to the plane determined by AB, AC. But it is perpendicular also to the plane determined by A′B′, A′C′; hence, the two planes are parallel to each other.

THEOREM XI.

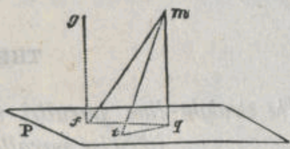
The straight line parallel to a given plane lies on a plane likewise parallel to the given plane.



Let the line ab be parallel to the plane P,—that is, such that it can never reach the plane, even indefinitely produced. Now, through any point m of ab, let a plane P′ pass parallel to P, having a common perpendicular with P: if we suppose mn parallel to the common perpendicular, mn also (TH. 8, COR. 1) will be a common perpendicular to both planes. Let qr also represent the section of the plane P made by the plane determined by the parallel line ab and the normal mn: we will have $mnq = mnr = 90°$. Now, if ab does not lie on the plane P′, let fg be the section of the plane P′ made by that determined by ab and mn, so that ab, rq, fg are supposed to be on the same plane. But, since mn is perpendicular to P′, we would have $nmf = nmg = 90°$, and, consequently, $bmn < 90°$; hence, ab and qr would not be parallel, and would somewhere meet each other. But ab cannot meet qr without meeting the plane P, which is against the supposition; therefore, ab lies on a plane P′ parallel to P.

THEOREM XII.

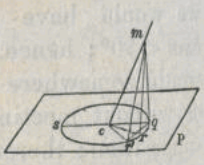
From any point out of the plane only one perpendicular may be drawn to the plane.



Let m be any point out of the plane P, and let mf be any straight line drawn from m to P; let also fg be a perpendicular to the plane P, and fq the intersection of the plane P made by the plane determined by gf and fm. From m, which is one of the points of this plane, draw mq parallel to fg, and, consequently, perpendicular to P. But any other line, for instance mt, drawn from m to P, cannot be perpendicular to the plane. Join, in fact, t with q: we will have the triangle mqt right-angled in q, and, consequently, mtq less than a right angle.

THEOREM XIII.

The least angle of an oblique line with the plane is that which it makes with the straight line joining the foot of the oblique line with that of the normal drawn to the plane from any point of the oblique line.



Let m be any point of the oblique line which reaches the plane P in C; let also mq be the perpendicular drawn from the same point, m, to the plane: join C with q, and with the centre C and radius Cq describe the circle qrs on P; draw also from q the chords qr, qr', and join r, r' with the centre of the circle and with m. From the right-angled triangles mqr, mqr' we have

$$\overline{mr}^2 = \overline{mq}^2 + \overline{qr}^2$$
$$\overline{mr'}^2 = \overline{mq}^2 + \overline{qr'}^2.$$

Now, $qr' > qr$, and, consequently, $mr' > mr$. Hence, from the triangles mCr, mCr', which have the common side mC, and the side Cr of the one equal to the side Cr' of the other, we have (B. I. TH. 6)

$$mcr' > mcr;$$

and, since the more the chords increase the more r' approaches s, and the more the same chords decrease the more r approaches q, thus, of all the angles which mC makes with the radii of the circle, mCs is the *maximum*, and mCq is the *minimum*.

We may here observe that when the angle which a straight line makes with a plane is given, we understand the minimum, unless it is otherwise expressed.

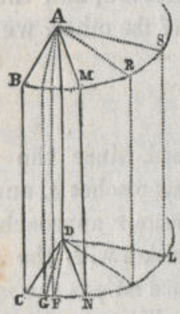
THEOREM XIV.

The angle formed by two planes is measured by that formed by two straight lines, one on each plane, and both perpendicular to the common intersection.

We had already occasion to observe that when two quantities m and m' are such that if, when m becomes $2m$, $3m$ ...., or $\frac{m}{2}, \frac{m}{3}$ ...., m' also becomes $2m'$, $3m'$

. . . ., or $\frac{m'}{2}$, $\frac{m'}{3}$ . . . ., one quantity is the measure of the other in all cases.

Again: an angle formed by two planes is double or triple, &c. the angle formed by two other planes, when this second is contained twice, three times, &c. in the first. Or, *vice versâ*, an angle formed by two planes is one-half, one-third, &c. of another, when the first is contained twice, three times, &c. in the second.



Let now ABCD, AMND be two planes forming an angle, and let AD be their common intersection. Let also AB, AM, and DC, DN be perpendicular to AD. Describe on the plane determined by AB, AM, with the centre A and radius AB, an arc BMS of a circle, and also on the plane determined by DC, DN, with the centre D and radius DC, describe an arc CNL of a circle. Now, if on BMS we take MR, RS . . . . equal to BM, and join A with R, with S . . . ., we will have

$$BAR = 2\,BAM,$$
$$BAS = 3\,BAM, \text{ \&c.},$$

and, since DA is perpendicular to the plane MAB, the radii AR, AS . . . are all perpendicular to AD. But AD and AR, AD and AS, &c. determine the positions of the planes DAR, DAS . . . ; and the angle which DAR makes with DAB is the double of the angle which MD makes with DAB; for, if we imagine the two planes DB, DM, preserving the same mutual inclination, to be turned about AD, when BD will take the place of MD, MD will take that of DR. In like manner, the angle formed by the planes BD, DS is three times the angle which DB makes with DM, &c.; hence,

$$\text{BDAR} = 2\,\text{BDAM},$$

$$\text{BDAS} = 3\,\text{BDAM}, \text{ \&c.}$$

Let us now take, on the arc CN, $\text{CF} = \frac{1}{2}\text{CN}$, $\text{GC} = \frac{1}{3}\text{CN}$, &c., so as to have the angles

$$\text{CDF} = \tfrac{1}{2}\text{CDN},$$

$$\text{CDG} = \tfrac{1}{3}\text{CDN}, \text{ \&c.}$$

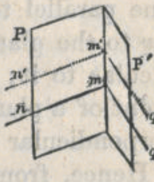
AD and DF, AD and DG, &c. determine the position of the planes AF, AG, &c. Now, in the same manner in which the angle formed by the two planes BD and DR contains twice the angle BDM, and the angle formed by BD and DS contains three times the same BDM, so BDM contains twice the angle formed by AC and AF, three times the angle formed by AC and AG, &c.; hence,

$$\text{BDAF} = \tfrac{1}{2}\text{BDAN},$$

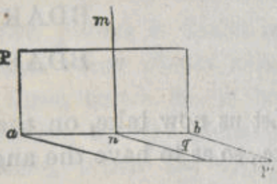
$$\text{BDAG} = \tfrac{1}{3}\text{BDAN}, \text{ \&c.}$$

The angles, therefore, formed by the perpendiculars drawn to the common intersection of two planes, change like the angles formed by the planes; hence, the one is the measure of the other.

Corollary. Hence, to measure the angle which two planes P, P′ make together, it is enough to draw from any point *m* of the intersection on the planes P and P′ the perpendiculars *mn*, *mq* to the same intersection: the angle *nmq* is the angle formed by the two planes. It is plain, also, (TH. 9,) that if from any other point *m′* of the intersection we draw on the planes the perpendiculars *m′n′*, *m′q′* to the



intersection, and, consequently, parallel to mn, mq, the angles $n'm'q'$, nmq are equal to each other.

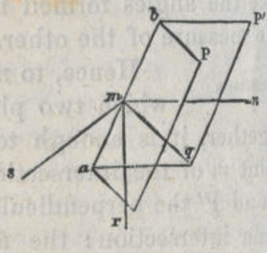


SCHOLIUM. Concerning planes perpendicular to each other.

We may now see the reason why we call a plane P perpendicular to another plane P′ when P passes through a straight line perpendicular to P′. For, let nm be the straight line perpendicular to P′ and through which P passes; now, since any line on P′ passing through n is perpendicular to mn, nq also perpendicular to the common intersection and ab itself, are perpendicular to mn, hence, mnq measures the angle formed by the planes; and, since mnq is a right angle, the planes also are perpendicular to each other.

THEOREM XV.

The perpendiculars to two planes inclined to each other form an angle equal to the angle of the planes.



We have seen that from any point out of the plane a perpendicular may be drawn to the plane, and also any straight line parallel to the perpendicular to the plane is also perpendicular to it; hence, from any point of a plane we may erect a perpendicular line to it.

Hence, from the point m of the common intersection of two planes P and P′, erect ms perpendicular to P, and mr perpendicular to P′. Since the intersection ab is on both planes, it is perpendicular to mr and to ms, and, consequently, to the plane determined by them. Let, now, mq, mn be the intersections of the plane determined by mr, ms with P and P′: we will have $amq = amn = 90°$; consequently, the angle qmn is the measure of the angle of the planes. But $qmn = smr$; because

$$smq = 90°, \; rmn = 90°,$$

and, consequently, $smq = rmn$,

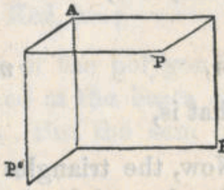
or, $smr + rmq = nmq + rmq$;

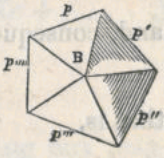
that is, $smr = nmq$.

The angle, therefore, formed by the perpendiculars to the planes is the same as that formed by the planes themselves.

SOLID ANGLES.

Although a plane intersecting another plane forms an angle, the angle formed by them is not a solid angle. To form solid angles three planes at least are required. The angle A, for instance, formed by the planes P, P′, P″, is a solid angle; the angle B also, formed by the planes p, p', p'', p''', p'''', is a solid angle, &c. A solid angle, then, is formed by more than two planes whose mutual intersections concur in the same point. Solid angles are called also *polyedral* angles, because they are formed by several plane angles.





THEOREM XVI.

When the polyedral angle is formed by three plane angles, the sum of two of them is always greater than the third.

Let BAC, BAD, CAD be three plane angles forming the solid angle A, and let BAD be greater than either of the other two. Draw from A, Af on the plane BAD, so as to have fAD = CAD, and take Af = AC. Join, also, D with f, and produce Df till it meets AB somewhere in m.

From the equal triangles CAD, fAD we have CD = Df; hence, joining C with m, since $m\text{D} < m\text{C} + \text{CD}$: we have, also,

$$m\text{D} < m\text{C} + \text{D}f,$$

or, $$mf + f\text{D} < m\text{C} + \text{D}f;$$

that is, $$mf < m\text{C}.$$

Now, the triangles fAm, CAm, besides the common side Am, have Af = AC; hence, from the last inequality,

$$m\text{A}f < m\text{AC},$$

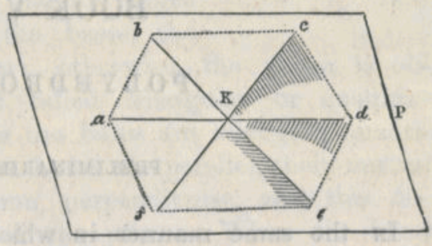
and, consequently,

$$\text{DA}f + m\text{A}f < \text{DA}f + m\text{AC};$$

that is, $$\text{DAB} < \text{DAC} + \text{BAC}.$$

THEOREM XVII.

The sum of the plane angles forming a polyedral angle is always less than four right angles.



Let K be any polyedral angle, and P a plane cutting the sides of the plane angles in a, b, c, &c. Join these points, so as to have the plane polygon abc . . . , and let n be the number of sides of the polygon, we will have the sum of the internal angles of the polygon equal to $(n-2)\,180°$. But, from the preceding theorem,

$$abc < ab\mathrm{K} + \mathrm{K}bc,\ bcd < bc\mathrm{K} + \mathrm{K}cd,\ \&c.;$$

hence, the sum of the internal angles of the polygon is less than the sum of the angles formed at the bases ab, bc . . . . of the triangles Kab, Kbc. . . . But the sum of the same angles is

$$n \cdot 180° - (a\mathrm{K}b + b\mathrm{K}c + \ldots .)$$

Hence, $(n-2)180° < n \cdot 180° - (a\mathrm{K}b + b\mathrm{K}c + \ldots .,)$ and, consequently,

$$a\mathrm{K}b + b\mathrm{K}c + \ldots . < 2 \cdot 180°.$$

That is, the sum of the plane angles forming any polyedral angle is less than four right angles.

BOOK VI.

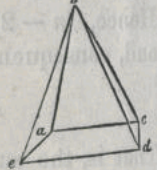
POLYEDRONS.

PRELIMINARIES.

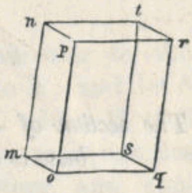
In the same manner in which surfaces are terminated either by straight or curve lines, so also solids are terminated either by plane or curve surfaces. Those solids that are terminated by plane surfaces are called *polyedrons*.

Polyedrons, like polygons, are either regular or not, according as their terminating planes are or are not regular and equal polygons. The terminating planes are called *faces*, and the straight lines in which adjacent faces meet each other are called *edges*.

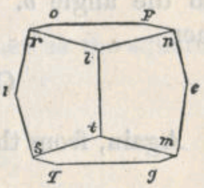
Now, the edges either meet together in one common point, as *ab*, *cb*, *db*, *eb*; or are parallel, as *mn*, *op*, *qr*, *st*; or, finally, variously inclined in different directions, as *lt*, *ts*, *tm*, *ln*, &c.



In the first case, and when the converging edges are terminated by one face, *aedc*, the solid is called a *pyramid;* the point *b* of concurrence is called the *vertex* of the pyramid, and the face *aedc* opposite to the vertex is called the *base* of the pyramid. A perpendicular, drawn from the vertex to the base, is called the *altitude*.



In the second case, and when the parallel edges are terminated by two faces *nprt* and *moqs*, the solid is called a *prism*, and the two terminating faces just mentioned are called the *bases* of the prism. When the parallel edges are perpendicular to the bases, the prism is a right prism: otherwise, the prism is oblique. The prism is called triangular or quadrangular, &c. according as the bases are triangles, quadrilaterals, &c. When the bases are parallel, their mutual distance is the common perpendicular, and this distance is the altitude of the prism. When the bases are not parallel, the altitude of the prism is evidently different for different points of the bases.

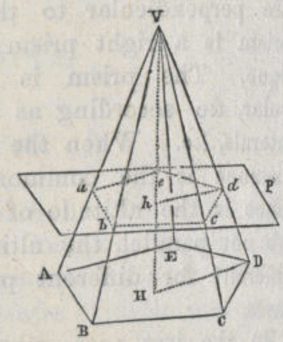


In the last case, when the edges are variously inclined, the solid preserves the general appellation of *polyedron*.

Any straight line joining two vertices that are not in the same face of any polyedron is called *diagonal*.

THEOREM I.

The section of a pyramid made by a plane parallel to the base is a polygon similar to that of the base.



Let P be a plane parallel to the base of the pyramid VABCDE, and let *abcde* be the section of the pyramid made by P: we will have AB and BC respectively parallel to *ab* and *bc*, and, consequently, the angle B is equal to the angle *b*. In like manner,

$$C = c, D = d, E = e, A = a.$$

Again, from the similar triangles *ab*V, ABV we have

$$ab : AB :: bV : BV.$$

And, from the similar triangles, *bc*V, BCV,

$$bc : BC :: bV : BV;$$

hence, $ab : AB :: bc : BC$,

or $ab : bc :: AB : BC$.

In like manner, we have

$$bc : cd :: BC : CD, \&c.$$

But two polygons having equal angles and the homologous sides proportional are similar; hence, the polygon produced by the section of P is similar to that of the base.

COROLLARY. The section is to the base as the square of the distance of the plane of the section from the vertex V is to the square of the distance of the base from the same vertex.

Draw from V, VH perpendicular to the base, and, consequently, also to P; and let h be the point of P met by VH. Join H with D, and h with d: we will have two triangles VHD, Vhd similar to each other, and, consequently,

$$Vh : VH :: Vd : VD;$$

and, also, $\overline{Vh}^2 : \overline{VH}^2 :: \overline{Vd}^2 : \overline{VD}^2$.

But $Vd : VD : de : DE$;

hence, $\overline{Vh}^2 : \overline{VH}^2 :: \overline{de}^2 : \overline{DE}^2$.

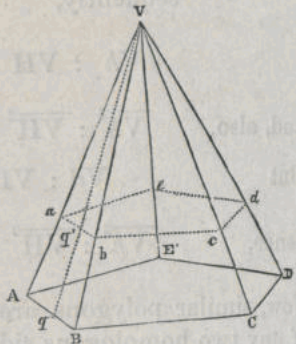
Now, similar polygons are to one another as the squares of any two homologous sides;

therefore, $\overline{de}^2 : \overline{DE}^2 :: ace : ACE$;

hence, $ace : ACE :: \overline{Vh}^2 : \overline{VH}^2$.

THEOREM II.

The surface of a pyramid having a regular polygon for base, and all the edges equal, is given by the product of the semi-perimeter of the base into the perpendicular let fall from the vertex to any side of the base.



Let the base of the pyramid VEABCD be a regular polygon, and let the edges VA, VB, &c. be all equal to one another. The faces of the pyramid will evidently be all isosceles and equal triangles, and the surface or area of the pyramid without the base will be as many times the area of one of these triangles as there are sides in the base. Now, the area of ABV, for example, is given by the product of Vq, the perpendicular drawn from the vertex to AB, into $\frac{1}{2}$AB. Hence, supposing n to be the number of the sides of the base, the surface of the pyramid will be expressed by

$$Vq \cdot \tfrac{1}{2}AB \cdot n;$$

that is, by

$$\tfrac{1}{2}(n \cdot AB) \cdot Vq.$$

SCHOLIUM. The surface of a truncated pyramid is given by the product of the perpendicular between the parallel sides of any face into half the sum of the perimeters of the bases.

Let now the pyramid be cut by a plane parallel to the base, and let *abcde* be the section. This section, together with the edges Va, Vb, &c., forms another pyramid, which, being taken from VABC . . . . , leaves a section of the given pyramid, called *truncated pyramid*, or *frustum* of a pyramid.

Calling M and m the perimeters of the two bases ABC . . . . . abc . . . ., observe that the faces of the frustum are all equal trapezoids; for the perimeters of the parallel bases are regular polygons, and the edges of the frustum, being the same differences of the sides of equal isosceles triangles, are also equal. Observe, also, that qq', the difference between the perpendiculars Vq, Vq', is the same for all. Now, the product of this perpendicular by half the sum of the parallel sides gives the area of the trapezoid; therefore, $qq' \cdot \frac{1}{2}(AB + ab)$ multiplied by the number n of the sides of the bases gives the surface of the frustum, the bases being excluded.

Now, $\quad qq' \cdot \frac{1}{2}(AB + ab)\, n = qq'\, \frac{1}{2}(n \cdot AB + n \cdot ab).$

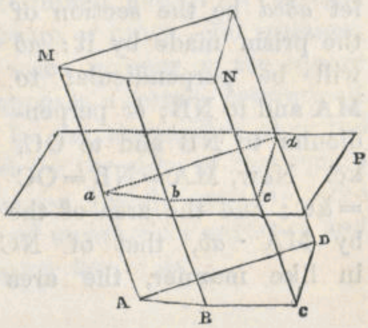
But $\quad n \cdot AB = M,\ n \cdot ab = m;$

hence, the surface of the frustum of our pyramid is expressed by

$$qq' \cdot \tfrac{1}{2}(M + m).$$

THEOREM III.

The section of a prism made by a plane parallel to the base is equal to the base.



Let the prism MNAC be cut by a plane, P, parallel to the base AC, and let $abcd$ be the section effected by the plane. Now, AB and ab are parallel between parallel lines, and, consequently, equal. In like manner, bc is equal and parallel to

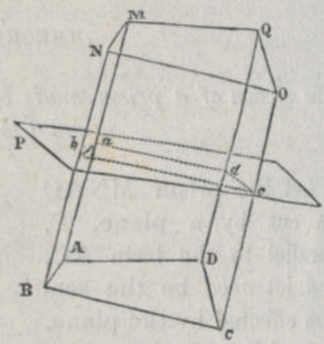
BC, &c.; hence, the polygons ABCD, *abcd* are equal to each other, and the faces *a*B, *b*C, &c. are all parallelograms.

Various kinds of prisms.

It is now plain that when the parallel bases of the prism are parallelograms, the prism is a polyedron having six parallelograms for faces, and each face is equal and parallel to its opposite. A prism of this kind is called a *parallelopipedon.* And if the parallelograms are rectangles, the prism is called a rectangular *parallelopipedon;* and when the parallelograms are all squares, the prism is then called a *cube.*

THEOREM IV.

The surface of a prism having parallel bases is expressed by the product of any one of its edges into the perimeter of a section made perpendicularly to the edges.



Let the bases MO, AC of the prism QB be parallel to each other; let also P be a plane cutting perpendicularly the edges, and let *abcd* be the section of the prism made by it: *ab* will be perpendicular to MA and to NB; *bc* perpendicular to NB and to OC, &c. Now, MA = NB = OC = &c.; and the area of the parallelogram MB is given by MA · *ab*, that of NC by NB · *bc*, or MA · *bc*: in like manner, the area of QC by MA · *dc*, &c.

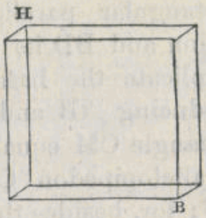
Hence, the surface of the prism, except the bases, is given by

$$MA\ (ab + bc + cd + \dots .),$$

the product, namely, of any one of the edges into the perimeter of the section vertical to them.

SOLIDITY OF BODIES.

We call, in geometry, the *solidity* of a body the amount of space occupied by it.



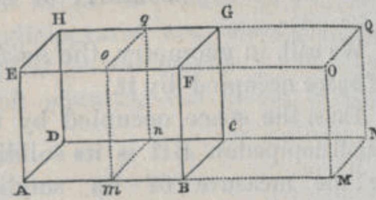
Thus, the space occupied by the parallelopipedon BH is its solidity, or the measure of its solidity. Hence, to measure and compare the solidities of bodies a certain space must be taken as their common measure, in the same manner in which a certain straight line is taken as unity of measure for linear lengths, and a certain area is taken as unity of measure for surfaces.

Now, the unity of measure for solids is a cube.

The cube has all its edges equal, but evidently it occupies a larger or smaller space, according to the length of these edges. Therefore, a cube occupying a determined space must have a determined length for its edge, which becomes a linear measure or unity,—for instance, an inch, a foot, &c.—in the same manner as the square used as unity of measure supposes a certain determined linear length for its side. In the supposition that the same linear unity is adopted for the sides of the square and for the edges of the cube,—a foot for example,—the square foot is then the unity of measure for surfaces, and a cubic foot the unity of measure for solids.

THEOREM V.

Two right-angled parallelopipedons having a common altitude are as their bases when the base of the one is a section of the base of the other.

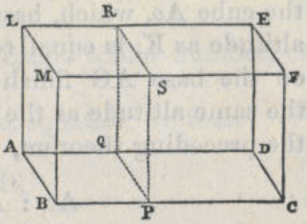


Whatever be the linear length and the unity of measure for solids, let BH be any rectangular parallelopipedon and BD its base. Duplicate the base by producing AB and DC to M and N, so as to have the rectangle CM equal to the rectangle DB, and finish the parallelopipedon CO. Now, CO is evidently equal to DF; for, besides the equal bases, the face FC is common, the faces ED, ON are equidistant from FC, parallel and equal to it. So that, if we imagine DB placed on CM so as to coincide exactly with it, DE will exactly coincide with CF, and CF with NO, and, consequently, HF, HC, EB will respectively coincide with GO, GN, FM. Hence, the space occupied by the parallelopipedon DO is twice that occupied by DF. Therefore, when the base of one of two right-angled parallelopipedons having the same altitude is twice the base of the other, its solidity also is twice the solidity of the other. It is plain that, if the base should become three times, four times, &c. the rectangle BD, the solidity of the corresponding parallelopipedon would likewise become three times, four times, &c. that of DF.

Vice versâ, if we divide the base BD into two equal parts by *mn* parallel to AD, and finish the parallelopipedon Do, we have, in like manner, the space $DF = 2Do$, or, $Do = \frac{1}{2}DF$, and, cutting off by lines parallel to AD one-third, one-fourth, &c. of the base DB, and finishing the right-angled parallelopipedons, their solidities will manifestly be one-third, one-fourth, &c. of the solidity of DF.

But it is well known that when one of two quantities becomes twice, three times, &c. as great, or one-third, one-fourth, &c., the other also increases and decreases in the same manner: whatever be the change effected with regard to one of these quantities, the same is the change of the other.

Hence, if out of the base AC of the right-angled parallelopipedon AF we cut off any rectangle AP and finish the parallelopipedon AS, whatever be the ratio between AP and AC, the same will be between AS and AF; that is, we shall always have



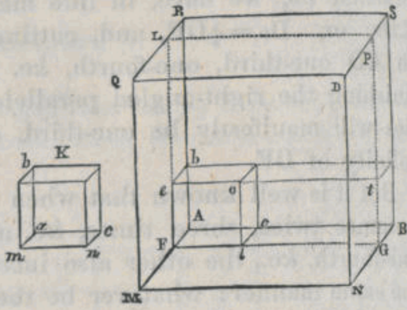
$$AP : AC :: AS : AF;$$

namely, *the solidities of two right-angled parallelopipedons having the same altitude, and one of them having for base a segment of the base of the other, are to each other as the areas of the bases.*

THEOREM VI.

The solidity of a cube is given by the product of the solid cube taken as unity of measure into the cube of the numerical value of the edge.

Let the edge mn of the cube K be the linear unity of measure, and K the unity of measure for solids, and let



MN (the edge of the cube AD) be equal to $\nu \cdot mn = \nu$; that is, let ν represent the numerical value of the edge of the cube AD, and, consequently, ν^3 represents the cube of the same number. Let us now take on AM $= \nu$, AF $= am = 1$, and on AR, Ab $= ab = 1$, and, finally, on AB take A$c = ac = 1$, and finish the cube Ao, which, having the same base and the same altitude as K, is equal to K. Produce, now, Fl to G, and on the base AG finish the parallelopipedon At, having the same altitude as the cube Ao: we will then have, from the preceding theorem,

$$Ao : At :: Al : AG;$$

hence, $$At = \frac{AG}{Al} Ao = \frac{AB \cdot AF}{Ac \cdot AF} \cdot K = \frac{AB}{Ac} \cdot K.$$

But $$\frac{AB}{Ac} = \frac{\nu}{1} = \nu;$$

hence, $$At = \nu \cdot K.$$

Produce, also, Fe and Gt to L and P, and finish the parallelopipedon AP: we will have

$$At : AP :: Ae : AL;$$

hence, $$AP = \frac{AL}{Ae} \cdot At = \frac{AR \cdot AF}{Ab \cdot AF} \cdot At = \frac{AR}{Ab} \cdot At.$$

But $$\frac{AR}{Ab} = \nu, \text{ and } At = \nu \cdot K;$$

hence, $$AP = \nu^2 K.$$

Compare, finally, AP with AD: we will have

$$AP : AD :: AL : AQ;$$

hence,

$$AD = \frac{AQ}{AL} \cdot AP = \frac{AM \cdot AR}{AF \cdot AR} \cdot AP = \frac{AM}{AF} \cdot AP.$$

Now, $\frac{AM}{AF} = \nu$, and $AP = \nu^2 K$;

hence, $AD = \nu^3 K$.

That is, the solidity of AD is the solidity of K taken as many times as there are units in the cube of the numerical value of the edge; that is, the cube of the number which indicates how many times the edge of K = 1 is contained in the edge of AD.

It is plain that ν may be an exact whole number, or with a fraction added to it.

Observe, also, that from the last equation we infer

$$K = \frac{1}{\nu^3} AD,$$

and, from $\frac{AB}{Ac} = \nu$,

$$Ac = = \frac{AB}{\nu}.$$

Now, supposing AD to be taken as unity of measure for solids, and, consequently, AB as unity of lengths,

$$Ac = \frac{1}{\nu}.$$

But $K = \frac{1}{\nu^3} AD$; hence, even when the cube taken as unity of measure has its edge greater than the edge of the cube to be measured, the solidity of the latter is expressed by the cube of the fraction representing the numerical value of its side into the solidity of the cube unity of measure.

Hence, generally, when the side S or edge of any cube is given, we may express the cube itself simply by S^3, the factor 1 or unity cube being understood.

SCHOLIUM. The solidity of the right angled parallelopipedon is given by the product of the numerical values of the edges into K = 1.

If the solid AD, instead of being a cube, would be any right-angled parallelopipedon, then the edge AB, for instance, measured by $mn=1$, may give ν for its numerical value; the edge AM would have δ, and the edge AR would have γ, for the numerical value. And then, following the same process as in the preceding demonstration, we will have

$$At = \nu \cdot K,$$
$$AP = \gamma \cdot At = \gamma \cdot \nu \cdot K,$$
$$AD = \delta \cdot AP = \delta \cdot \gamma \cdot \nu \cdot K,$$

or,

$$AD = \delta \cdot \gamma \cdot \nu.$$

The product, namely, of the numerical values of the three different edges of the right-angled parallelopipedon into K = 1 gives its solidity.

Important remark.

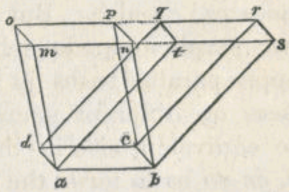
We may here remark that, since $\delta \cdot \gamma \cdot \nu$, or,

$$\nu \cdot \delta \cdot \gamma = AB \cdot AM \cdot AR,$$

and AB · AM is the expression of the base of the parallelopipedon and AR is its altitude, we generally also say that the solidity of any right-angled parallelopipedon is equal to the product of the base into the altitude, which is perfectly correct; for, if we conceive of the base ascending parallel to itself to the top of the altitude and leaving a trace, or multiplying itself continually while ascending, we will have the whole space of the solid exactly filled by the multiplied base. And if the ascent and multiplication of the base should stop at one-half, one third, &c. of the altitude, the solid thus effected would evidently be one-half, one-third, &c. of the whole.

THEOREM VII.

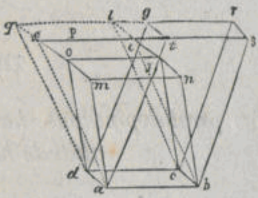
Two parallelopipedons having a common base and the same altitude have equal solidities.



Let *ac* be the base common to the parallelopipedons *bo* and *bq*, and let the upper parallel bases *no*, *tr* be equidistant from *bd*; that is, let both of them lie on the same plane; and let us suppose, also, on the same plane the faces *bm*, *bt*, and, consequently, in another common plane, the faces *co*, *cq*.

Thus, we have two triangular prisms *atqdom* and *bsrpcn* equal to each other, because the face *ao* of the one is equal to the face *bp* of the other, and the face *aq* of the first is equal to the face *br* of the second. Now, placing *bp* on *ao* so as to have *bc* coinciding with *ad* and *am* with *bn*, the two faces will exactly coincide with each other; and, since the angles *mat*, *nbs*, *odq*, *pcr* are all equal, we cannot have the face *cn* of one prism coinciding with the face *dm* of the other without the face *cs* coinciding with *dt*, and, consequently, the edge *sr* with *tq*; hence, also, *pr* with *oq* and *ns* with *mt*; that is, the two prisms may be made to coincide exactly; hence, their solidities are equal. Now, if from the whole solid *morsba* we take the prism *ato*, there will remain the parallelopipedon *aqs*; and if from the same solid we take the prism *bps*, there will remain the parallelopipedon *bo*. But the two triangular prisms taken from the common solid are equal in solidity; hence, the remainders also must be equal, and the two parallelopipedons *ar*, *ap* have the same solidity, or are equivalent.

SCHOLIUM. The same theorem extended to all cases.

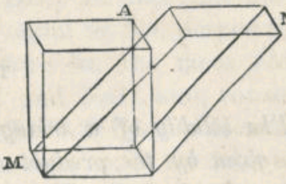


In the preceding demonstration we have supposed the faces *bm*, *bt* to be on one common plane, and, consequently, their opposite faces *co*, *cg* also. But let the two parallelopipedons have the common base *ac*, and the upper parallel bases on the same plane, but the remaining faces on different planes: the two parallelopipedons will be equivalent also in this case; for, produce *mo*, *np* and *st*, *gr* so as to form the parallelogram P on the common plane of the upper bases. Now, P is equal to the bases; and, finishing the parallelopipedon *bdfeql*, the faces *aq*, *ao* will be on the same plane determined by *qm*, *ma*, and the faces, *bl*, *bp* on the same plane determined by *ln*, *nb*; hence, the parallelopipedon *bo* is equivalent to *bq*. Again, the faces *fb*, *tb* are both on the plane determined by *fs*, *sb*, and the opposite faces on the plane determined by *qr*, *rc*; hence, also, the parallelopipedon *gb* is equivalent to *bq*. Therefore, the two parallelopipedons *gb* and *ob* are equivalent,—that is, have the same solidity.

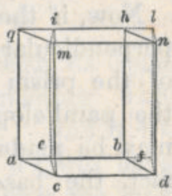
THEOREM VIII.

The solidity of any parallelopipedon is given by the product of the numerical values of the base and altitude into $K = 1$.

It is plain, from the preceding demonstration, that any parallelopipedon MN is equivalent to another parallelopipedon MA having the same base and the same altitude as MN. Suppose now the edges of the parallelopipedon MA to be perpendicular to the base. If the base is a rectangle, then MA is a right-angled parallelopipedon whose solidity is given by the product of the numerical values of the base and altitude into $K=1$. Now, the base and altitude and the solidity of MA are equal to the base and altitude and the solidity of MN; hence, to have, in our supposition, the solidity of MN, it is enough to multiply $K=1$ by the numerical values of its own base and altitude.



But if the base common to both parallelopipedons is not a rectangle, but any parallelogram *abdc*, produce, then, *ab*, and from *c* and *d* draw *ce*, *df* perpendicular to *af*, and on the rectangular base *cefd* finish the right-angled parallelopipedon *ceild*. Now, the two parallelopipedons *cdbqmh*, *cdfiml* are equivalent; because, if we take their common face *mndc* as base, they have the same base and equal altitude. But the solidity of the right-angled parallelopipedon *cmild* is given by the product of the numerical values of *ce*, *cd*, *cm* into $K=1$; hence, the solidity also of *dchqa* is expressed by

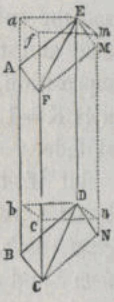


$$cd \cdot ce \cdot cm.$$

But $cd \cdot ce$ gives the numerical value of the base *abdc*; hence, whenever the edges are perpendicular to the bases, the solidity of the parallelopipedon is given by the product of the numerical values of the base and altitude into $K=1$, whether the parallel bases of the parallelopipedon be rectangular or not.

THEOREM IX.

The solidity of a triangular prism having parallel bases is given by the product of the numerical values of the base and altitude into K = 1.



Let ABCDEF be any triangular prism. Draw on the planes of the parallel bases AFE, BCD, EM, FM parallel to AF, AE, and DN, CN parallel to BC, BD, and finish the parallelopipedon BMNA having the same altitude as the prism and the bases equal to twice those of the prism.

Now, if the edges of the given prism are perpendicular to the bases, then the solidity of the prism is manifestly one-half that of the parallelopipedon, because the other prism DFMN may be made to coincide with the given DFAB. In fact, the base DCN placed on DBC may be made to coincide exactly with it, and, consequently, the perpendicular and equal edges with the corresponding edges and the prism with the prism.

But, if the prism DFAB is not a right prism, draw from the extremities D and E of the edge ED the planes E*afm*, D*bcn* perpendicular to the edges, which, produced, form the right prism D*fab* and the parallelopidon *mabn* = 2 D*fab*. Observe now that the solids E*a*AFM*mf*, D*b*BCN*nc* are equal to each other; because, first, the parallelogram *am* placed on *bn* may be made coincident with it, the point D with E and *a* with *b*, *f* with *c* and *m* with *n*. But, since *a*A, *f*F, perpendicular to the plane *am*, are respectively equal to *b*B and *c*C, perpendicular to *bn*, *am* cannot coincide with *bn* without the face *a*F coinciding with bC and AaE with bBD. In like manner, since Mm, perpendicular to am, is equal to Nn, perpendicular to bn, when am coincides with bn, the faces fM and MmE must coincide with cN and NnD, and, consequently, the solid $AFMEafm$ with $BCNDbcn$. Hence, the two polyedrons are equal; hence, also, if we add to the solid $bcnDEAFM$ either of the two equal polyedrons, the result will be the same. But when we add the first, the resulting solid is the parallelopipedon $bmna$, and when we add the second, the resulting solid is the parallelopipedon BMNA; hence, the two parallelopipedons are equivalent; that is,

$$BMNA = bmna,$$

and, since $$bmna = 2\,Dfab,$$

$$BMNA = 2\,Dfab.$$

Now, the polyedron $aFmE$ cannot coincide with its equal $bCnD$ without the pyramid $FEAfa$ coinciding with $CDBcb$; hence, the two pyramids are equal. And if we add to the solid $bcDEAF$ either of them, we will have the same result in solidity. But, by adding the first pyramid, we have the right prism $Dfab$; and, by adding the second, we have the given prism DFAB;

hence, $$DFAB = Dfab;$$

and, consequently,

$$2\,DFAB = 2\,Dfab.$$

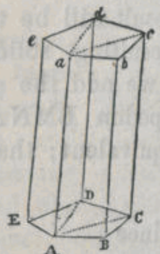
But $$2\,Dfab = BMNA;$$

hence, $$2\,DFAB = BMNA,$$

and, finally, $$DFAB = \tfrac{1}{2}\,BMNA.$$

Hence, the solidity of the triangular prism is one-half that of the corresponding parallelopipedon in all cases. But the solidity of the parallelopipedon is given by the product of the numerical values of the base and altitude into K = 1; hence, the solidity of the prism is expressed by one-half this same product; or, since the base of the triangular prism is one-half that of the parallelopipedon, and the altitude is the same for both, the solidity of any triangular prism having parallel bases is given by the product of the numerical values of its own base and altitude into K = 1.

COROLLARY. The solidity of any prism having parallel bases is given by the product of the numerical values of the base and altitude into K = 1.



Let, now, the parallel bases of the prism be any two polygons ABCDE, *abcde*. Draw through the edge A*a* and the opposite edges D*d*, C*c* the planes D*a*, C*a*: the polygonal prism will be thus divided into triangular prisms having parallel bases and a common altitude. Call s, s', s'' the solidities of these prisms, b, b', b'' the numerical values of their corresponding bases, and a the numerical value of their common altitude. Representing, also, by S the solidity of the polygonal prism, and by B the numerical value of its base, we will have

$$S = s + s' + s''.$$

But $s = b \cdot a,\ s' = b' \cdot a,\ s'' = b'' \cdot a;$

hence, $S = (b + b' + b'')a.$

But $b + b' + b'' = B;$

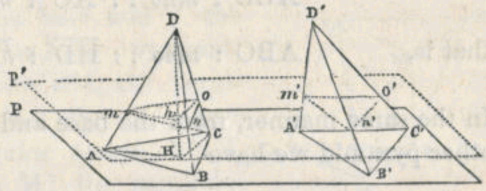
hence, $S = B \cdot a,$

or, $S = B \cdot a \cdot K.$

The solidity, namely, of any prism having parallel bases is given by the product of the numerical values of the base and altitude into K = 1.

THEOREM X.

Two triangular pyramids having equal altitudes and the areas of the bases also equal are equivalent in solidity.



Let P be the common plane of the bases of two triangular pyramids having the same altitude and the areas of the bases ABC, A′B′C′ equal.

Conceive another plane, first in coincidence with P, and then brought up to the summit of the pyramids, always parallel to P. Let P′ represent any of the positions of the movable plane, and *mno*, *m′n′o′* the sections of the pyramids made by it. Observe, now, that the spaces described by the parts of the movable plane contained within the faces of the pyramids, from the bases to the vertices, do not differ from the spaces filled by the pyramids themselves. Secondly, the spaces described by plane areas constantly equal to one another, and equally moved, are also equal to one another.

Let now DH be the common altitude of the pyramids DABC, D′A′B′C′, and D*h* the common altitude of the pyramids D*mno*, D′*m′n′o′*. Join H with C and A, and *h* with *o* and *m*: the similar triangles AHC, *mho* give

$$mo : AC :: ho : HC.$$

But $\quad ho : HC :: hD : HD;$

hence, $\quad mo : AC :: hD : HD,$

and $$\overline{mo}^2 : \overline{AC}^2 :: \overline{hD}^2 : \overline{HD}^2.$$

Now, the triangles ABC, *mno* also are similar to each other; hence, their areas are as the squares of the homologous sides. Representing, therefore, the areas by the triangles themselves, we will have

$$ABC : mno :: \overline{AC}^2 : \overline{mo}^2;$$

that is, $$ABC : mno :: \overline{HD}^2 : \overline{hD}^2.$$

In the same manner, from the base and the section of the other pyramid we have

$$A'B'C' : m'n'o' :: \overline{HD}^2 : \overline{hD}^2;$$

hence, $$ABC : mno :: A'B'C' : m'n'o'.$$

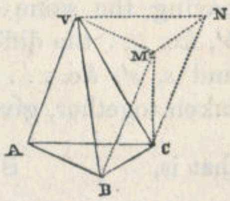
But $$ABC = A'B'C';$$

hence, also, $$mno = m'n'o'.$$

Now, *mno*, *m′n′o′* represent the areas of any two sections of the pyramids effected by the movable plane; therefore the areas of these sections are constantly equal, and, consequently, the spaces described by them from the bases to the vertices are likewise equal. But these spaces are the same as those occupied by the pyramids; hence, the two pyramids have equal solidities.

THEOREM XI.

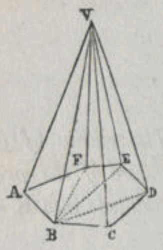
The solidity of any triangular pyramid is expressed by one-third of the product of the numerical values of the base and altitude into K = 1.



Let ABC be the base and V the vertex of any triangular pyramid. From B and C draw BM, CN parallel and equal to the edge AV, and finish the triangular prism AMN. Join then C with M; the parallelogram NCBM will thus be divided into two equal triangles, and the plane of the triangle VMC bisects the solid VBCMNV; for the sections VMCB, VMCN are two pyramids, having the bases CMB, CMN equal and a common altitude.

Take now VMN for the base of the pyramid VMCN, and, consequently, C for the vertex: we have a pyramid having a base equal to the base of the given pyramid, and for altitude the distance between the parallel planes VMN, ABC, which is the same as the altitude of the given pyramid; hence, the two pyramids ABCV, MNVC are equivalent. But MNVC is equivalent to CMBV; hence, the amount in solidity of the three pyramids is three times the solidity of the given pyramid. But the three pyramids form the prism; hence, a triangular prism having the same base and altitude of a given pyramid has a solidity three times that of the pyramid. But the solidity of the prism is the product of the numerical values of the base and altitude into K = 1; hence, the solidity of any triangular pyramid is one-third of the same product.

Corollary. The solidity of any pyramid is given by one-third of the product of the numerical values of the base and altitude into K = 1.



But let the base ABC .... of the given pyramid be any polygon. Draw from any of the angles—for instance, B—the diagonals BF, BE, &c. ...., the polygonal pyramid is thus cut into a number of triangular pyramids, all having the same altitude a. Call b, b', &c. ... the different bases of the triangular pyramids, and s, s', &c. ... the corresponding solidities, which, taken together, give the solidity S of the given pyramid;

that is, $$S = s + s' + s'' + \ldots.$$

But $$s = \tfrac{1}{3}a \cdot b \cdot K, \quad s' = \tfrac{1}{3}a \cdot b' \cdot K, \text{ \&c. } \ldots;$$

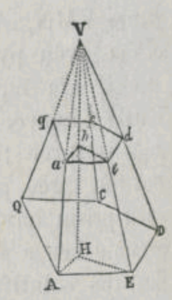
hence, $$S = \tfrac{1}{3}(b + b' + b'' + \ldots.)\, a \cdot K.$$

Now, the sum $b + b' + \ldots.$ is the base β of the given pyramid. Hence,

$$S = \tfrac{1}{3}\beta \cdot a \cdot K;$$

or, $$S = \tfrac{1}{3}\beta \cdot a.$$

Scholium. Concerning the solidity of a truncated pyramid.



Let AD, ad be the parallel bases of a truncated pyramid. Finish the pyramid, and let V be the vertex common to Vad and VAD: let also VH be the altitude of the whole pyramid and Vh that of the upper pyramid. Call, besides B and b, the bases AD, ad, and a the altitude Hh of the truncated pyramid. From the triangles VHA, Vha we have

$$VH : Vh :: AH : ah;$$

and, from the triangles AHE, *ahe*,

$$AH : Ah :: AE : ae;$$

hence, $$VH : Vh :: AE : ae;$$

and, consequently,

$$VH - Vh : Vh :: AE - ae : ae;$$

$$a : Vh :: AE - ae : ae;$$

from which $$Vh = \frac{a \cdot ae}{AE - ae}.$$

Now, since $$a = VH - Vh,$$

and, consequently, $$VH = a + Vh:$$

we will have, also,

$$VH = a + \frac{a \cdot ae}{AE - ae}$$

$$= a\left(1 + \frac{ae}{AE - ae}\right)$$

$$= \frac{a \cdot AE}{AE - ae}.$$

Call now S the solidity of the pyramid VAD, and *s* that of V*ad*, and call σ the solidity of the truncated pyramid:

we will have $$\sigma = S - s.$$

Now, $$S = \tfrac{1}{3}B \cdot VH = \tfrac{1}{3}B \cdot \frac{a \cdot AE}{AE - ae},$$

and $$s = \tfrac{1}{3}b \cdot Vh = \tfrac{1}{3}b \cdot \frac{a \cdot ae}{AE - ae};$$

hence, $$\sigma = \frac{a}{3}\left[\frac{B \cdot AE - b \cdot ae}{AE - ae}\right].$$

THEOREM XII.

The solidities of pyramids and prisms are as the bases when the altitudes are equal, and are as the altitudes when the bases are equal. The same solidities are equal when the bases are reciprocally as the altitudes.

Let us represent by P and P′ the solidities of two prisms, and by p and p' the solidities of two pyramids, and let A, B represent the altitude and base of P, and p and A′, B′ the altitude and base of P′ and p': we will have

$$P = A \cdot B, \quad P' = A' \cdot B',$$
$$p = \tfrac{1}{3} A \cdot B, \; p' = \tfrac{1}{3} A' \cdot B'.$$

Hence,
$$P : P' :: A \cdot B : A' \cdot B',$$
$$p : p' :: A \cdot B : A' \cdot B';$$

and, consequently, making $A = A'$,

$$P : P' :: B : B',$$
$$p : p' :: B : B';$$

and, making $B = B'$,

$$P : P' :: A : A',$$
$$p : p' :: A : A'.$$

But, if the bases are reciprocally as the altitudes; that is,

if $\quad A : A' :: B' : B,$

since then $\quad A \cdot B = A' \cdot B',$

we will have also $\quad P = P',$

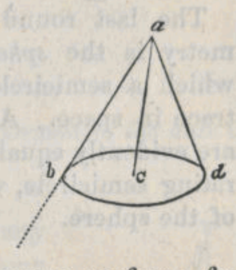
and $\quad p = p'.$

BOOK VII.

ROUND BODIES.

PRELIMINARIES.

The round bodies or solids of revolution belong to the class of bodies terminated by curve surfaces. They are called solids of revolution, because they are conceived as produced by the revolution of one line about another. Thus, for example, let a be any point out of the circular plane bd, and let ac be the straight line which joins a with the centre of the circle: conceive now another straight line ab, having one of its extremities constantly at a, and describing, with another extremity, the circle bd: the surface traced



by this line moved around ac is a curve surface; for each point of ab changes continually its direction. The solid terminated by the circular plane bd and the surface generated by ab, is called a *cone;* the point a, the *vertex*, and the circular plane bd the *base*, of the cone. The perpendicular let fall from the vertex to the plane of the base is the *altitude* of the cone; and when the perpendicular falls on the centre of the base the cone is called a *right* cone; otherwise, *oblique.* The straight line which joins the vertex with the centre of the base is, in all cases, called the *axis* of the cone.

If the straight line *ab*, keeping constantly one of its extremities *b* on the circle *bd*, is moved around the same circle, remaining always parallel to another straight line *cq* passing through the centre, it will trace a curve surface in space, and the solid terminated by it is called a *cylinder*. The line *cq*, passing through the centre and parallel to the generating line, is called the *axis* of the cylinder; the circle *bd* and another circle *am*, or, more generally, two plane surfaces terminating the cylinder, are called *bases*. When the axis is vertical to the base, the cylinder is a *right* cylinder; otherwise, it is *oblique*.

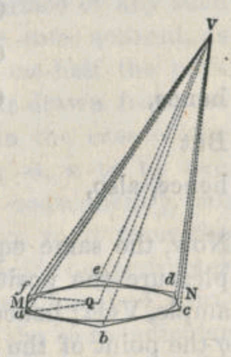
We may here observe that the cone and the cylinder bear analogy to the pyramid and the prism.

The last round body considered in elementary geometry is the *sphere*,—a solid terminated by a surface which a semicircle turned around its diameter would trace in space. All the points of the spherical surface are evidently equally distant from the centre of the generating semicircle, which is, consequently, also the centre of the sphere.

THEOREM I.

The cone may be considered as a pyramid whose base is a regular polygon of an infinite number of sides.

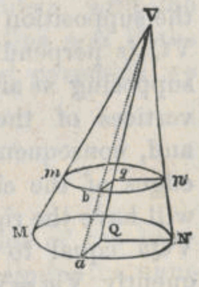
Let *abcd* . . . be any regular polygon circumscribed about the base of the cone VMN. A pyramid having *abcd* . . . for base and V for vertex will be also circumscribed about the cone. Now, by increasing indefinitely the number of the sides of the polygon, and, consequently, the number of the edges of the pyramid, since the more the number of the sides of the polygon increase the more the polygon approaches to coincidence with the circle, the pyramid also must approach to the inscribed cone. The cone, therefore, may be considered as the limit of a pyramid having the same vertex as the cone, and for base a circumscribed regular polygon (or also inscribed) about the base of the cone, with a continually-increasing number of sides. Or the cone may be considered as a pyramid whose base is a regular polygon having an infinite number of sides.



THEOREM II.

The section of the cone made by a plane parallel to the base is circular.

Let MN be the circular base of any cone MVN, and *mn* a section of the cone made by a plane parallel to the base: let, also, VQN, VQ*a* be any two planes passing through the axis of the cone. The intersections QN, Q*a* of the base made by these two planes are equal to each other, because both are radii of the same circle; hence, the intersections *qn*, *qb*, also, of *mn*, made by the same two planes, are equal to each other. In fact, from the similar triangles VQN, V*qn*, and VQ*a*, V*qb*, we have



$$QN : qn :: VQ : Vq,$$

$$Qa : qb :: VQ : Vq;$$

hence, $$QN : qn :: Qa : qb.$$

But $$QN = Qa;$$

hence, also, $$qn = qb.$$

Now, the same equality would be found by changing at pleasure the position of one of the two planes,—for example, VQa; hence, mbn is a circle having for its centre q the point of the plane mn met by the axis.

THEOREM III.

The surface of a right cone is given by the product of the semi-periphery of the base into the side of the cone.

Since all the angles of a regular polygon, either inscribed in the circle or circumscribed about it, are equidistant from the centre, the pyramid inscribed in the right cone or circumscribed about it, having a regular polygon for base, must have all its edges equal. For, in the supposition of the right cone, the axis VQ is perpendicular to the base; hence, supposing m and n to be any two of the vertices of the circumscribed polygon, and, consequently, Vm, Vn, two of the edges of the circumscribed pyramid, we will have the right-angled triangles VQm, VQn equal to each other, and, consequently, Vm = Vn. It is proved, in like manner, that any two edges of the inscribed pyramid are equal to each other; hence, the pyramid either inscribed or circumscribed about the cone, and having a regular polygon for

base, has all its edges equal. But the surface of any such regular pyramid, not taking the base into account, is given (B. VI. TH. 2) by the product of one-half the perimeter of the base into the perpendicular drawn from the vertex to any side of the base; hence, in the case of the circumscribed pyramid, and supposing m, n to be two contiguous vertices of the base, and, consequently, mn one of its sides, the vertical Vo, drawn to it from the vertex of the pyramid, multiplied by half the perimeter of the base, gives the surface of the circumscribed pyramid. Now, the point o of mn, met by the perpendicular Vo, is the point of contact of the side mn with the circular base of the cone. Because, since mQn is an isosceles triangle, having Qn = Qm, the perpendicular drawn from the centre Q to the tangent mn will bisect it. But the perpendicular Qo falls on the point of contact; hence the middle point o of mn is the point of contact; the triangle nVm, also, is an isosceles triangle, the perpendicular drawn from V, the vertex formed by the equal sides, to the opposite side mn, must bisect it, and, consequently, fall on the point of contact. Vo, therefore, having common with the cone the vertex and one of the points of the base, coincides with the generating line in one of its positions, and coincides, therefore, with the cone itself. Call, now, t the generating line and P the perimeter of the base of the circumscribed pyramid: we will have for its surface

$$S = \tfrac{1}{2} P \cdot t,$$

independently of the number of the sides of the base. But, by increasing beyond all assignable limits the number of the sides of the base, it becomes coincident with the base of the cone, and the pyramid with the cone itself. Hence, calling r the radius Qo of the base of the cone, and, consequently, $2r\pi$ its periphery, and calling σ the surface of the cone without the base, we will have

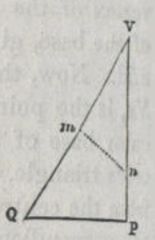
$$\sigma = \tfrac{1}{2} 2r\pi \cdot t;$$

that is, $$\sigma = r\pi \cdot t.$$

SCHOLIUM I. Another useful expression of the surface of the cone.

The surface of the right cone may be expressed, also, as follows:—

Let PQ be the radius r of the base, and VQ the generating side t. Divide VQ equally in m, and from m draw mn perpendicular to VQ: we will have



VQ, or $$t = 2\,\mathrm{V}m.$$

Hence, from the preceding equation,

$$\sigma = r \cdot \pi \cdot 2\,\mathrm{V}m;$$

or, since $$r = \mathrm{PQ},$$

$$\sigma = 2\pi \cdot \mathrm{PQ} \cdot \mathrm{V}m.$$

Now, the two triangles Vmn, VQP, right-angled the one in m, the other in P, and having, besides, the angle V common to both, are similar.

Hence, $$\mathrm{V}m : mn :: \mathrm{VP} : \mathrm{PQ},$$

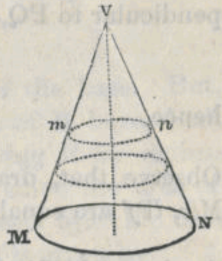
and $$\mathrm{V}m \cdot \mathrm{PQ} = mn \cdot \mathrm{VP},$$

and, consequently,

$$\sigma = 2\pi \cdot mn \cdot \mathrm{VP}.$$

SCHOLIUM II. Surface of the truncated right cone.

Let now the right cone VMN be cut by a plane mn parallel to the base. Let r be the radius of mn, and R the radius of MN: we will have (B. VI. TH. 2, SCH.) for the surface of the truncated cone mnNM, excluding the bases, (calling δ the surface of the truncated cone, and u the difference Mm between VM and Vm,)

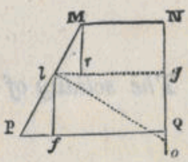


$$\delta = (\text{R} + r)\,\pi \cdot u.$$

Observe, in fact, that in the same manner in which VM represents the perpendicular to any of the sides of the base of the circumscribed pyramid, so Mm represents the common perpendicular to any two corresponding sides of the bases of the truncated pyramid, whatever be the number of the sides of the same bases. But the product of Mm into the semiperimeters of the bases gives the surface of the truncated pyramid; and, when the number of the sides of the bases is increased beyond all limits, the perimeters become the peripheries of the two circles, and the pyramid coincides with the truncated cone. Hence follows the preceding expression of the surface of the right truncated cone.

SCHOLIUM III. Another useful expression of the surface of the right truncated cone.

Let now MN be the radius r of the upper base of the truncated cone, PQ the radius R of the other base, and MP the generating side u: we will have



$$\delta = (\text{PQ} + \text{MN})\,\pi \cdot \text{MP}.$$

Divide MP equally in l, and from l draw lg perpendicular to NQ*o*, and *lo* perpendicular to MP; and, also, *lf* perpendicular to PQ,—that is, parallel to N*o*: we will have

$$MP = 2\,lP;$$

hence, $$\delta = 2\pi\,(PQ + MN)\,lP.$$

Observe that, drawing M*r* parallel to N*o*, the triangles M*lr*, *l*P*f* are equal to each other; hence, $lr = Pf$.

Again, $$MN = lg - lr,\ PQ = lg + Pf = lg + lr;$$
hence, $$PQ + MN = 2\,lg,$$
and $$\delta = 2\pi \cdot 2\,lg \cdot lP$$
$$= 4\pi \cdot lg \cdot lP.$$

Now, the triangles *lf*P, *lgo* are similar, because the sides of the one are perpendicular to the sides of the other;

hence, $$lP : lf :: lo : lg,$$
and $$lP \cdot lg = lf \cdot lo;$$
and, therefore, $$\delta = 4\pi\,lf \cdot lo.$$
But $$lf = gQ = \tfrac{1}{2}\,NQ;$$
hence, $$\delta = 2\pi NQ \cdot lo.$$

THEOREM IV.

The solidity of the cone is given by one-third of the product of the base into the altitude.

Let, again, R be the radius of the base of any cone, and let A be the altitude common to the cone and to a pyramid having for base any regular polygon B circumscribed about the circular base of the cone.

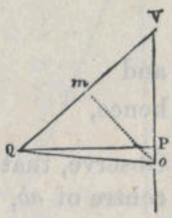
Now, the solidity of the pyramid is given by

$$\tfrac{1}{3}B \cdot A,$$

whatever be the number of the sides of the base. But, by increasing the number of the sides of B beyond all limits, B is changed into the circle, having R for radius; and the pyramid into the cone; and, since the area of the circle whose radius is R is expressed by $\pi \cdot R^2$, thus the solidity of any cone of circular base is given by

$$\tfrac{1}{3}\pi\overline{R}^2 \cdot A.$$

SCHOLIUM I. Another useful expression of the solidity of the right cone.



Let QP be the radius R of the base of any right cone. From m, the middle point of the generating side VQ, draw mo perpendicular to it, o being a point of the axis VP produced. Join, also, Q with o, and let Qo be the generating side of another right cone: we will have, for the value of the solid S generated by VQo about Vo,

$$\begin{aligned} S &= \tfrac{1}{3}\pi R^2 \cdot PV + \tfrac{1}{3}\pi R^2 \cdot Po \\ &= \tfrac{1}{3}\pi R^2 \cdot Vo \\ &= \tfrac{1}{3}\pi QP \cdot QP \cdot Vo. \end{aligned}$$

Now, from the similar triangles Vmo, VPQ we have

$$Vo : mo :: VQ : QP;$$

hence, $$QP \cdot Vo = mo \cdot VQ,$$

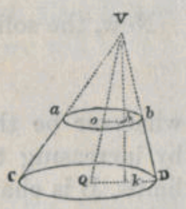
and, consequently,

$$S = \tfrac{1}{3}\pi \, QP \cdot mo \cdot VQ.$$

But (TH. 3, SCH. 1) $\pi \cdot QP \cdot 2\,Vm$, or $\pi \cdot QP \cdot VQ$, is the surface σ of the cone generated by VQ;

hence, $$S = \tfrac{1}{3}\sigma \cdot mo.$$

SCHOLIUM II. Solidity of the truncated cone obtained from the parallel bases and their distances.



Let $abDC$ be any truncated cone, and V the vertex of the cone finished; let, also, r be the radius of the upper base, and R the radius of the base CD: the segment hk of the altitude Vk affords the distance between the bases, which we will call d.

Now, the solidity of the truncated cone $abDC$ is equal to the solidity of the whole cone VCD minus that of Vab. But the solidity of VCD, or

$$VCD = \tfrac{1}{3}\pi R^2 \cdot Vk,$$

and $$Vab = \tfrac{1}{3}\pi \overline{r}^2 \cdot Vh;$$

hence, $$abDC = \tfrac{1}{3}\pi (R^2 \cdot Vk - r^2 \cdot Vh.)$$

Observe, that drawing the axis VoQ, and joining o, the centre of ab, with h, and Q, the centre of CD, with k, we have two similar triangles Voh, VQk; from which

$$Vk : Vh :: VQ : Vo;$$

and, from the similar triangles VQD, Vob,

$$VQ : Vo :: R : r;$$

hence, $$Vk : Vh :: R : r;$$

and, consequently, $$Vh = \frac{r}{R} Vk.$$

From $$Vk : Vh :: R : r,$$

we have also $$Vk - Vh : Vk :: R - r : R;$$

and, therefore, $$Vk = \frac{R\,(Vk - Vh)}{R - r} = \frac{Rhk}{R-r},$$

and $$Vh = \frac{r}{R}\,\frac{Rhk}{R-r} = \frac{r}{R-r}\,hk.$$

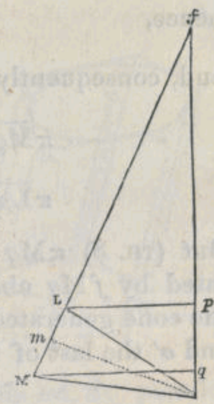
The expression, therefore, of the solidity of the truncated cone may be represented also as follows:

$$abDC = \tfrac{1}{3}\pi\Big[R^2\frac{R}{R-r}hk - r^2\frac{r}{R-r}hk\Big] =$$

$$\tfrac{1}{3}\pi hk\Big[\frac{R^3-r^3}{R-r}\Big].$$

SCHOLIUM III. Another useful expression of the solidity of the right truncated cone.

Let Mq be the radius of the lower base and Lp the radius of the upper base of a right cone, and let pq be the altitude of the truncated cone or the distance between the two bases, and LM the generating side. Bisect LM in m, and let mN be drawn perpendicularly to LM. Produce ML, qp till they meet together in f, and join N with M and with L: the solid generated by the surface fMN about fN is equal to the solid generated by the surface fLN plus that generated by LMN. Call S′ the solid generated by fLN, S″ that generated by fMN, and S‴ the solid generated by LMN: we will have



$$S'' = S' + S''',$$

and $$S''' = S'' - S'.$$

But one of the manners in which the solidities S″ and S′ may be expressed is (SCH. 1) by

$$S'' = \tfrac{1}{3}\pi\cdot\overline{Mq}^2\cdot fN,\ S' = \tfrac{1}{3}\pi\cdot\overline{Lp}^2\cdot fN;$$

hence, $$S''' = \tfrac{1}{3}\big[\pi\overline{Mq}^2\cdot fN - \pi\overline{Lp}^2\cdot fN\big].$$

Now, from the similar triangles $f\text{M}q$, $f m\text{N}$, we have

$$f\text{M} : \text{M}q :: f\text{N} : m\text{N};$$

hence, $$f\text{M} \cdot m\text{N} = \text{M}q \cdot f\text{N},$$

and, from the similar triangles $f\text{L}p$, $fm\text{N}$,

$$f\text{L} : \text{L}p :: f\text{N} : m\text{N};$$

hence, $$f\text{L} \cdot m\text{N} = \text{L}p \cdot f\text{N};$$

and, consequently,

$$\pi \overline{\text{M}q}^2 \cdot f\text{N} = \pi \text{M}q \cdot f\text{M} \cdot m\text{N},$$

$$\pi \overline{\text{L}p}^2 \cdot f\text{N} = \pi \text{L}p \cdot f\text{L} \cdot m\text{N}.$$

But (TH. 3) $\pi \text{M}q \cdot f\text{M}$ is the surface of the cone generated by $f\text{M}q$ about fq, and $\pi \text{L}p \cdot f\text{L}$ is the surface of the cone generated by $f\text{L}p$ about fp. Calling σ'' the first and σ' the last of these two surfaces, we will have

$$\pi \overline{\text{M}q}^2 \cdot f\text{N} = \sigma'' \cdot m\text{N},$$

$$\pi \overline{\text{L}p}^2 \cdot f\text{N} = \sigma' \cdot m\text{N};$$

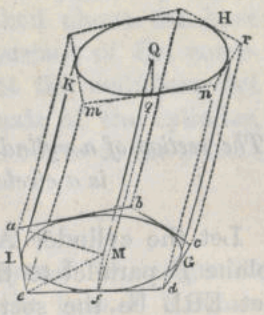
and, therefore, $$\text{S}''' = \tfrac{1}{3}(\sigma'' - \sigma')\, m\text{N}.$$

But the surface generated by LM alone is the difference $\sigma'' - \sigma'$; hence, calling σ''' the surface generated by LM about pq, we will have

$$\text{S}''' = \tfrac{1}{3}\sigma''' \cdot m\text{N}.$$

THEOREM V.

The cylinder is the limit of an inscribed or circumscribed prism having the sides of the bases indefinitely increasing in number.

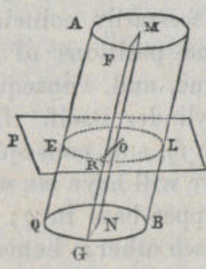


Let *abcd* be any regular polygon circumscribed about the base LG of the cylinder LGHK. From the point of contact *f* of any side *ed*, draw *fq* to the plane of the upper base and parallel to the axis of the cylinder: it must necessarily coincide with one of the positions of the generating line, and, consequently, with the cylinder itself. Draw also from *e* and *d*, *em*, *dn* parallel to *fq*, and, consequently, also, to the axis. Join *m* with *n*: we will have *mn* = *ed*, and parallel to it, and touching the upper base in *q*; for the radii Q*q*, M*f* are also parallel to each other; hence, the angle Q*qn* is equal to M*fd*. But M*fd* is a right angle; hence also Q*qn*; and, consequently, *mn* is a tangent of the circle KH in *q*. In like manner, if from *c* we draw *cr* to the plane of the upper base and parallel to the axis, and we join *n* with *r*, *nr* will be another tangent to the circle equal and parallel to *de*, &c. That is, if from all the angles of the circumscribed polygon *edc* . . . . to the plane of the upper base we draw straight lines parallel to the axis MQ and join their extremities, the upper base, also, of the cylinder will be circumscribed by a polygon equal to *edc* . . . ., and the resulting prism will be circumscribed about the cylinder.

Now, by increasing indefinitely the number of the sides of the circumscribed polygons, they approach more and more to the peripheries, and the edges of the prism to the surface of the cylinder. Hence, the cylinder is the limit of a circumscribed prism having the sides of the base increasing indefinitely in number. The same should be said with regard to the inscribed prism,—a prism, namely, having for bases regular polygons inscribed in the bases of the cylinder.

THEOREM VI.

The section of a cylinder made by a plane parallel to the base is a circle equal to that of the base.



Let the cylinder AB be cut by the plane P parallel to the base BQ, and let ERL be the section. From any point G of the base draw GF parallel to the axis MN, and let R be the point in which GF meets the section ERL; draw also the radius NG, and join the point O of the axis, met by P, with R: we will have OR and NG parallel to each other and between two other parallels ON, RG. Hence, OR is equal to the radius of the base. But R is any point of the section; hence, the distance of any point of the section from O is equal to the radius of the base; hence, the section itself is a circle equal to that of the base, and having its centre in the point O of the axis met by the intersecting plane.

THEOREM VII.

The surface of a right cylinder, not including the parallel bases, is given by the product of the periphery of the base into the axis of the cylinder.

Let P represent the perimeter of a regular polygon of any number of sides circumscribed about the base of the cylinder, and let S be the surface of the corresponding prism circumscribed about the cylinder; let also l represent the length of the axis of the cylinder, and, consequently, also, of any of the edges of the prism: we will have

$$S = P \cdot l.$$

But, increasing indefinitely and beyond all assignable limits the number of the sides of the base, P becomes the periphery of the base of the cylinder and S the surface of the cilinder, excluding the bases; hence, if r is the radius of the base, and, consequently, $2r\pi$ the periphery, calling σ the surface of the cylinder, we will have

$$\sigma = 2r\pi \cdot l.$$

THEOREM VIII.

The solidity of a cylinder having parallel bases is given by the product of the base into the altitude.

The solidity of the prism having parallel bases is given by the product of the base into the altitude, (B. VI. TH. 9, COR.;) hence, representing by s the solidity of the prism circumscribed about the cylinder having parallel bases, by a the value of the area of the regular base of the prism, and by A the altitude common to the prism and to the cylinder, we have

$$s = a \cdot A,$$

whatever be the number of the sides of the base. But, by increasing indefinitely the number of these sides, the perimeter of the prismatic base becomes the circular base of the cylinder, and the prism coincides with the cylinder itself; that is, r being the radius of the circular base, a will become $r^2\pi$, and, consequently, if S represents the solidity of the cylinder,

$$S = r^2\pi \cdot A.$$

From this and from the preceding theorems we infer the following corollaries:—

COROLLARY I. The solidity of the cone is one-third that of the cylinder having the same base and the same altitude.

The first of the two formulas,

$$s = \tfrac{1}{3}\pi \cdot r^2 \cdot A, \ s' = \pi \cdot r^2 \cdot A,$$

represents (TH. 4) the solidity of the cone having r for the radius of the base and A for the altitude. The second formula represents the solidity of the cylinder having also r for the radius of the base and A for altitude. But $s = \frac{1}{3}s'$; hence, the solidity of the cone is one-third that of the cylinder having the same base and the same altitude.

COROLLARY II. The cones and cylinders of equal altitude are as the bases. The cones and cylinders of equal bases are as the altitudes.

Let A, A′ be the different altitudes of two cones or of two cylinders having equal bases. For the solidities s, s' of the two cones, we will have

$$s = \tfrac{1}{3}\pi r^2 \cdot A, \ s' = \tfrac{1}{3}\pi r^2 \cdot A',$$

and for the solidities S, S′ of the two cylinders,

$$S = \pi r^2 \cdot A, \ S' = \pi r^2 \cdot A',$$

and, consequently, in both cases,

$$\frac{s}{s'}=\frac{A}{A'},\ \frac{S}{S'}=\frac{A}{A'};$$

that is,

$$s : s' :: A : A',$$
$$S : S' :: A : A'.$$

Let, now, R, R′ be the different radii of the bases of two cones or of two cylinders having the same altitude A: we will have, for the solidities of the cones,

$$s=\tfrac{1}{3}\pi R^2 \cdot A,\ s'=\tfrac{1}{3}\pi R'^2 \cdot A,$$

and, for those of the cylinders,

$$S=\pi R^2 \cdot A,\ S'=\pi R'^2 \cdot A;$$

hence, in both cases,

$$\frac{s}{s'}=\frac{\pi R^2}{\pi R'^2},\ \frac{S}{S'}=\frac{\pi R^2}{\pi R'^2};$$

that is,

$$s : s' :: \pi R^2 : \pi R'^2,$$
$$S : S' :: \pi R^2 : \pi R'^2,$$

or, representing simply by B, B′ the bases πR^2, $\pi R'^2$,

$$s : s' :: B : B',$$
$$S : S' :: B : B'.$$

COROLLARY III. Cones or cylinders of equal solidities have their bases reciprocally as their altitudes, and *vice versâ*.

Representing still by B and B′ the bases, the formulas

$$s=\tfrac{1}{3}B \cdot A,\ s'=\tfrac{1}{3}B' \cdot A'$$

represent the solidities of two cones having different bases and different altitudes; and the formulas

$$S=B \cdot A,\ S'=B' \cdot A'$$

represent the solidities of two cylinders having likewise different bases and different altitudes. Let, now, s be equal to s', and S to S′, in both cases: we will have

$$B \cdot A = B' \cdot A',$$

and, consequently,

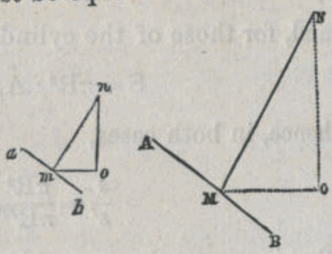
$$B : B' : A' : A.$$

The bases, namely, are reciprocally as the altitudes.

Vice versâ, if the bases of the two cones or cylinders are reciprocally as the altitudes, since then $B \cdot A = B' \cdot A'$, their solidities also must be equal.

COROLLARY IV. Similar cones or similar cylinders are as the cubes of the diameters of their bases.

Similar cones or similar cylinders are those which have the axes equally inclined to the bases and proportional to the diameters of the same bases. Thus, for example, let ab, AB be the diameters of two different bases, and mn, MN the axes of two cones or cylinders. Draw from n and from N the perpendiculars no, NO to the planes of the bases: we will have



$$no = A, \ NO = A'.$$

Joining, then, m with o, and M with O, since the axes are equally inclined to the bases,

$$nmo = NMO,$$

and the triangles nom, NOM are similar to each other;

hence, $$mn : MN :: A : A'.$$

But in similar cones or cylinders

$$mn : \text{MN} :: ab : \text{AB},$$

or,
$$mn : \text{MN} :: 2\text{R} : 2\text{R}';$$

hence,
$$\text{A} : \text{A}' :: 2\text{R} : 2\text{R}',$$

or,
$$\text{A} : \text{A}' :: \text{R} : \text{R}',$$

and
$$\frac{\text{A}}{\text{A}'} = \frac{\text{R}}{\text{R}'}.$$

But from the equations

$$s = \tfrac{1}{3}\pi\text{R}^2 \cdot \text{A},\ s' = \tfrac{1}{3}\pi\text{R}'^2 \cdot \text{A}',$$

which give the solidities of any two cones, and from the equations

$$\text{S} = \pi\text{R}^2 \cdot \text{A},\ \text{S}' = \pi\text{R}'^2 \cdot \text{A}',$$

which give the solidities of any two cylinders: we have

$$\frac{s}{s'} = \frac{\text{R}^2 \cdot \text{A}}{\text{R}'^2 \cdot \text{A}'} = \frac{\text{R}^2}{\text{R}'^2} \cdot \frac{\text{A}}{\text{A}'},$$

$$\frac{\text{S}}{\text{S}'} = \frac{\text{R}^2 \cdot \text{A}}{\text{R}'^2 \cdot \text{A}'} = \frac{\text{R}^2}{\text{R}'^2} \cdot \frac{\text{A}}{\text{A}'};$$

hence, in the supposition of two similar cones or cylinders, since $\frac{\text{A}}{\text{A}'} = \frac{\text{R}}{\text{R}'}$, we will have

$$\frac{s}{s'} = \frac{\text{R}^3}{\text{R}'^3},\ \text{and}\ \frac{\text{S}}{\text{S}'} = \frac{\text{R}^3}{\text{R}'^3};$$

that is,
$$s : s' :: \text{R}^3 : \text{R}'^3,$$

$$\text{S} : \text{S}' :: \text{R}^3 : \text{R}'^3.$$

THEOREM IX.

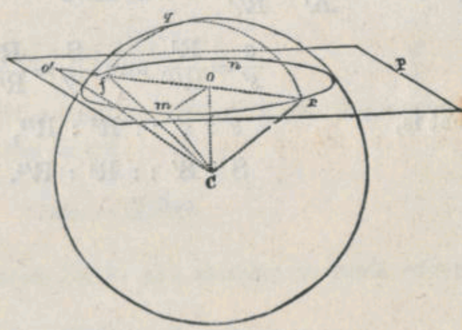
The sphere is the limit of a solid generated by the semi-perimeter of a polygon either circumscribed or inscribed in the circle and revolved about the diameter, and having the number of the sides constantly increasing.

In the same manner in which the periphery of a circle is the limit of an inscribed or circumscribed regular polygon the sides of which are constantly increasing in number, thus the solid generated by the rotation of the semi-periphery about the diameter is the limit of the solid generated by the semi-perimeter of the same polygon; hence, the values of the solidity and surface of the sphere will be easily obtained, provided we may determine the solidity and surface of the round body generated by the semi-perimeter of the inscribed or circumscribed polygon for any number of sides, as we will see better in some of the remaining theorems.

THEOREM X.

The section of the sphere made by a plane is circular.

If the plane cutting the sphere passes through its centre, our proposition is evident, because all the points of the sphere are equidistant from the centre.

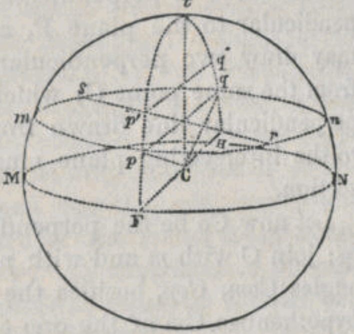


But let the section *mn* made by the plane P pass out of the centre: the perpendicular let fall from the centre on P must fall somewhere in *o* within the section *mn*. Else, let it fall out of the section; for instance, in *o′*. Draw, then, from *o′* any straight line *o′p* through the section *mn*: the plane determined by *o′p*, *o′*C passing through the centre, forms with the sphere a circular section *fqp*, of which C is the centre and *fp* the chord. But a straight line drawn from the centre to the middle point of the chord is perpendicular to it; hence, if C*o′* is perpendicular to the plane P, and, consequently, to *o′p*, we may draw two perpendiculars to the same straight line from the same point C; which is impossible. Hence, the perpendicular line drawn from the centre of the sphere to the intersecting plane must necessarily fall within the section.

Let now C*o* be the perpendicular, and draw from *o*, *om*, *op*; join C with *m* and with *p*. The two right-angled triangles C*om*, C*op*, besides the common side C*o*, have the hypothenuse C*m* of the one equal to the hypothenuse C*p* of the other, because *m* and *p*, two points of the spherical surface, are equidistant from the centre. Hence, the other two sides *om*, *op* of the triangles are also equal to each other. But, in the same manner in which we find *om* = *op*, we may have *om* equal to another line drawn from *o* to any other point of the section *mpnf*; hence, all the points of the section are equidistant from *o*; that is, *mpnf* is a circle having its centre in *o*, the point of the plane P met by the perpendicular drawn to it from the centre of the sphere.

THEOREM XI.

The greater the distance of the plane intersecting the sphere from the centre the smaller is the diameter of the section effected by it.



Let MN be the section of the sphere made by a plane passing through the centre.

We may here remark that, since the radius of this circular section can only be that of the sphere, all the circles the planes of which pass through the centre are equal, and are called *great circles*, because the radii of all the other circles (*small circles*) made by the sections of planes passing out of the centre are less than the radius of the sphere, and the more the centre of the small circle is distant from the centre of the sphere the smaller is its diameter.

In fact, let *mn* be the section of the sphere made by a plane parallel to MN. Draw in the circle *mn* any diameter *pq*, and let F*ptq*H be a great circle of the sphere having its plane in coincidence with *pq*, which is at once the diameter of *mn* and the chord of the arc *ptq*, less than FH, the diameter of F*t*H and MN. Now, by conceiving the plane of *mn* approaching to the extremity *t* of the radius of the sphere perpendicular to it, the sections or chords *p′q′*, &c. will be the diameters of the successive circles effected by the movable plane cutting the sphere. But the greater the distance of the chord from the centre of the circle the smaller is its length; hence, the greater the distance of the planes of the small circles from the centre of the sphere the smaller are their diameters.

SCHOLIUM I. The plane passing through the extremity of any radius of the sphere, and perpendicular to it, is a tangent plane.

Hence, since *pq* is any diameter, and the diameter *pq* becomes one single point when the plane of the section perpendicular to C*t* is brought to the extremity *t* of the radius, applying the same demonstration to any other diameter, *sr*, for instance, we see that the movable plane constantly perpendicular to the radius C*t* becomes a tangent plane when it is brought up to the extremity of the radius; for any straight line on that plane passing through *t* is a tangent to one of the great circles of the sphere, and, consequently, any other point of it besides *t* is at a greater distance from the centre of the sphere than *t*.

SCHOLIUM II. The great circles intersect mutually at the extremities of a diameter.

We may observe here, also, that the intersection of two planes passing through the centre of the sphere, must necessarily have one of its points in the same centre. But any straight line passing through the centre of the sphere must coincide with a diameter of the sphere; hence, the intersection of the planes of any two great circles is a diameter of the sphere, and a diameter also of the two circles at the extremities, of which their peripheries intersect each other; that is, two great circles have their points of intersection 180° apart from each other.

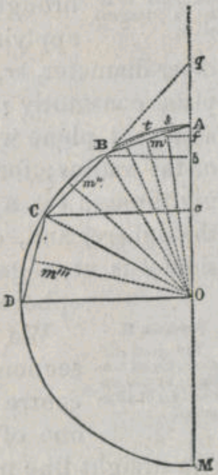
SCHOLIUM III. The plane of a great circle divides the sphere into two equal parts.

Conceive the two parts into which the plane of a great circle divides the sphere placed on the same plane as the circle, and with both convexities on the same side: the two surfaces, having all their points equidistant from the centre, must perfectly coincide with each other: hence, the plane of any great circle bisects the sphere.

THEOREM XII.

The surface of the sphere is four times that of the great circle.

Let AO = R be the radius of the generating semicircle ADM of the sphere, and, consequently, the radius of the sphere itself. Draw from O, OD perpendicular to the diameter AM, and divide the quadrant AD into three equal arcs AB, BC, CD: their corresponding chords will be three of the twelve sides of a regular polygon inscribed in the circle.



Conceive now the three sides revolved about the radius AO, and draw from B and C the perpendiculars Bb, Cc to AO, and from the centre O the perpendiculars Om', Om'', Om''' to the three sides.

It is plain that the side BA describes a right cone having Bb for the radius of the base and Ab for altitude. Now, m' is the middle point of BA; hence, calling a' the surface generated by BA about AO, we will have (TH. 3, SCH. 1)

$$a' = 2\pi \cdot Om' \cdot Ab.$$

The surface generated by BC is that of a truncated cone having bc for altitude; and, since m'' is the middle point of BC, hence, calling a'' the surface generated by BC, we will have (TH. 3, SCH. 3)

$$a'' = 2\pi \cdot Om'' \cdot bc,$$

and, since $$Om'' = Om',$$

$$a'' = 2\pi \cdot Om' \cdot bc.$$

So, also, the surface generated by DC is that of a right cone truncated having Oc for altitude; and, since m''' is the middle point of the generating side, calling a''' the surface, we will have

$$a''' = 2\pi \cdot Om''' \cdot cO.$$

But $$Om''' = Om'' = Om';$$

hence, $$a''' = 2\pi \cdot Om' \cdot cO.$$

Hence, the surface generated by the three sides together, or $a' + a'' + a'''$, is

$$2\pi \cdot Om' \, (Ab + bc + cO);$$

or $$2\pi \cdot Om' \cdot AO.$$

Dividing now each arc AB, BC, CD into two equal parts, and duplicating the number of the sides of the inscribed polygon, following the same process, we will find for the surface of the cone generated by the first side At, $2\pi \cdot sO \cdot Af$; sO being the perpendicular drawn from the centre to At, and Af the altitude of the cone. So, also, for the surface generated by the next side tB, we will have $2\pi \cdot sO \cdot fb$, &c. The surface, therefore, generated by the six sides will be

$$2\pi \cdot Os \cdot AO;$$

which does not differ from the preceding, except in the factor Os being nearer to the length of the radius than the preceding factor Om'. It is now evident that, duplicating constantly the number of the sides of the inscribed polygon, the variable factor approaches continually to AO = R, and becomes the radius itself when the number of sides surpasses all limits. But, then, the perimeter of the polygon coincides with the periphery of the circle;

hence, $$2\pi \cdot AO \cdot AO,$$

or $$2\pi \cdot R^2,$$

represents the surface generated by the quadrant ABCD about AO;

hence, also, $$4\pi \cdot R$$

gives the value of the surface generated by the semicircle ADM. But the surface generated by the semicircle turned about the diameter is the surface of the sphere; hence the surface of the sphere having R for radius is four times πR^2. Now, πR^2 is the area or value of the surface of the generating circle, or of any great circle of the sphere. The surface, therefore, of the sphere is four times that of the great circle.

THEOREM XIII.

The solidity of the sphere is given by the cube of its radius into $\frac{4}{3}\pi$.

Supposing the same inscribed polygon as in the preceding theorem, and calling μ' the solid generated by the triangle BAO revolved about AO, μ'' the solid generated by BCO, and μ''' the solid generated by CDO about the same AO, we will have (TH. 4, SCH. 1 and 3)

$$\mu' = \tfrac{1}{3}\alpha' \cdot \mathrm{O}m',$$
$$\mu'' = \tfrac{1}{3}\alpha'' \cdot \mathrm{O}m',$$
$$\mu''' = \tfrac{1}{3}\alpha''' \cdot \mathrm{O}m',$$

and, consequently, the solid generated by the whole surface embracing the three triangles, or $\mu' + \mu'' + \mu'''$, is given by

$$\tfrac{1}{3}(\alpha + \alpha' + \alpha'')\ \mathrm{O}m'.$$

But, from the preceding theorem,

$$\alpha' + \alpha'' + \alpha''' = 2\pi \cdot \mathrm{AO} \cdot \mathrm{O}m';$$

hence, the value of the same solid may be expressed also by

$$\tfrac{2}{3}\pi \cdot \mathrm{AO} \cdot \overline{\mathrm{O}m'}^2.$$

Duplicating the number of the sides of the inscribed polygon, and following the same process, we will obtain for the value of the corresponding solid

$$\tfrac{2}{3}\pi \cdot \mathrm{AO} \cdot \overline{\mathrm{O}s}^2;$$

and, duplicating constantly the number of the sides of the regular polygon, the last factor of the preceding expression approaches continually to the square of the radius $\mathrm{AO} = \mathrm{R}$, and becomes R^2 when the inscribed polygon coincides with the circle. Hence, the solid generated by the quadrant ABCD revolved about AO is given by

$$\tfrac{2}{3}\pi \cdot \mathrm{AO} \cdot \mathrm{R}^2;$$

that is,

$$\tfrac{2}{3}\pi \cdot \mathrm{R}^3.$$

COROLLARY. The surfaces of the spheres are as the squares, and the solidities as the cubes, of the respective radii.

Let r', r'' be the radii of two spheres: their corresponding surfaces σ', σ'' will be given by $4\pi r'^2$, $4\pi r''^2$, and the solidities S′, S″ by $\frac{4}{3}\pi r'^3$, $\frac{4}{3}\pi r''^3$;

hence,

$$\frac{\sigma'}{\sigma''}=\frac{4\pi r'^2}{4\pi r''^2}=\frac{r'^2}{r''^2},$$

$$\frac{S'}{S''}=\frac{\frac{4}{3}\pi r'^3}{\frac{4}{3}\pi r''^3}=\frac{r'^3}{r''^3};$$

that is

$$\sigma' : \sigma'' :: r'^2 : r''^2,$$

$$S' : S'' :: r'^3 : r''^3.$$

Plane and Spherical Trigonometry.

Plane Trigonometry.

PRELIMINARIES.

Object of plane trigonometry.

§ 1. The object of plane trigonometry is the *resolution* of this general problem:—To find the three unknown elements of a plane triangle when the other three are given.

Elements of triangles.

Sides and angles are the elements of any triangle.

The object proper to trigonometry cannot be obtained in all cases.

§ 2. The resolution of the problem is not possible in all cases; but, with the exception of the cases in which the given elements are the three angles, or two sides and the angle opposite to one of them, in all the other cases the problem is completely resolvable, as we will better see hereafter. In the first of the above-mentioned cases nothing more may be found than the ratio between the sides, and in the second the resolution is ambiguous.

Trigonometrical functions.

§ 3. In the resolution of the problem we make use of certain straight lines, called trigonometrical *functions*, or also functions of the arcs and of the angles having the same arcs for their measure.

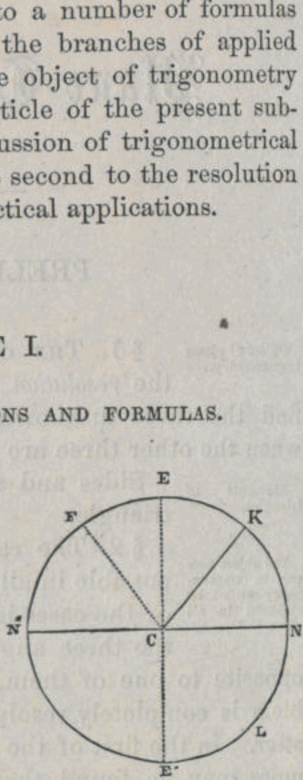
Their importance. Hence the necessity for the student to become quite familiar with them before proceeding to the resolution of the problem.

Division of the subject. § 4. For this reason, and also on account of the mutual dependence on one another of the various functions giving birth to a number of formulas of great use and utility in all the branches of applied mathematics no less than in the object of trigonometry itself, we will devote the first article of the present subject to the exposition and discussion of trigonometrical functions and formulas, and the second to the resolution of the problem and to some practical applications.

ARTICLE I.

TRIGONOMETRICAL FUNCTIONS AND FORMULAS.

Definitions. § 5. Let the diameters EE′, NN′ of the circle NEN′E′ be perpendicular to each other: we will have the circumference divided by them into four arcs, each of 90°. Let K be any point between the extremities N and E of one of these arcs: the two arcs EK and KN are called



Complements and supplements. *complements* of each other; and generally any two arcs a and b whose sum gives 90° are called complements of each other. But, from

$$a+b=90°,$$

we infer
$$b=90°-a;$$

hence, the arcs a and $90° - a$ are complements of each other.

Now, the arcs KN, KE measure the corresponding angles KCN, KCE, which are, accordingly, called complements of each other; and generally the complement of an arc signifies the same as the complement of the corresponding angle.

Again: let F be another point on the semi-periphery N′EN: the two arcs FN, FN′ taken from F to the extremities of the diameter NN′ are called *supplements* of each other; and more generally any of two arcs m and n whose sum is 180° is supplement of the other. And, since from

$$m + n = 180°,$$

we infer

$$n = 180° - m;$$

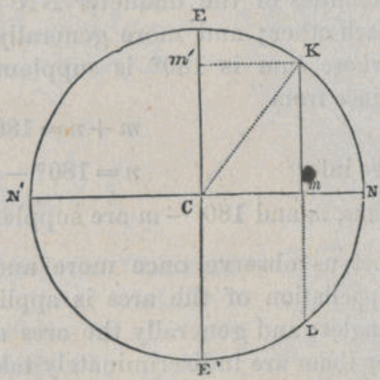
thus, m and $180° - m$ are supplements of each other.

Let us observe once more and forever that the same appellation of the arcs is applied to the corresponding angles; and generally the arcs and the angles measured by them are indiscriminately taken for one another.

Functions of the arcs and functions of their complements, or *co-functions.*

§ 6. The arcs and their functions are either positive or negative. Thus, for example, let N be the beginning of computation for the arcs NK, NE, &c., which we will consider as positive. If, instead of reckoning the arcs from N towards E, we take them from N towards E′, NL, NE′, &c., these arcs are to be considered as negative with reference to the upper arcs NK, NE, &c., or *vice versâ.* With regard to the functions, some, as we will presently see, are referred to the centre of the circle, and others to the diameters NN′, EE′,—the first drawn from the point of commencement of the arcs, and the second vertical to the first. It is enough to observe here that, taking as positive any segment of the radius CN from C towards N, any segment of the radius CN′ from C towards N′ is negative; and, taking as positive the functions referred to the diameter NN′ and above it, the same functions below the said diameter are negative; and, finally, taking as positive the functions referred to the diameter EE′ and on the side N, the functions referred to the same diameter and on the side N′ are negative.

Sine and cosine. § 7. Let now K be any point taken in the first quadrant NE. Draw from K, Km perpendicular to NN′: this perpendicular varies evidently with the arc NK; it is, consequently, a function (see Treat. on Alg., Introd. Art., § 16) of this arc, and it is called the *sine* of the arc KN. Now, the perpendicular Km, as well as any other drawn from the different points of the semicircle NEN′, is necessarily above the diameter NN′. Hence, the sines of the arcs from zero to 180° are all positive; but, if we take the arcs greater than 180°, since all the points of N′E′N from 180° to 360° are below the diameter NN′, the perpendiculars also drawn from the different points of N′E′N to the diameter NN′ are likewise below the same diameter, and consequently all negative; that is, the sines of the arcs from 180° to 360° are all negative. *Vice versâ*, if we take the negative arcs from 0° to 180°, then all the sines will be negative, and from 180° to 360° all the sines will be positive. Observe here, also, that the extremities of arcs of an equal number of degrees taken in the positive and negative direction



are equidistant from the extreme points N and N′ of the diameter NN′. Hence, supposing K and L to be the extremities of any two such arcs, we know from geometry (B. IV. TH. 2) that the chord KL is bisected by the radius CN and perpendicular to the same radius;

hence, $Km = Lm.$

But Km is the sine of KN, and Lm is the sine of LN = − KN: calling a the arc KN, the sine of this arc is indicated by $\sin a$; hence, the sine of LN by $\sin -a$. Now, from $Km = Lm$, we have that the length of $\sin a$ is equal to that of $\sin -a$. But the sign of $\sin -a$ is opposite to that of $\sin a$; hence,

$$\sin -a = -\sin a.$$

If the extremities of the arcs fall in the second and third quadrants, the chord which joins them will be perpendicular to the radius CN′ and bisected by it; and, consequently, the sines of the two arcs will be equal to each other in length, although affected with a different sign. Hence, in all cases, we have the same equality between $\sin -a$ and $-\sin a$.

It is plain that the sine must increase in length from 0° to 90°, and decrease, in like manner, from 90° to 180°; then, becoming negative, it increases again in length from 180° to 270°, and, finally, decreases with inverted order from 270° to 360°. But the sine of the arcs of 0° and of 180° are equal to zero. For the point N or arc 0° and the extreme point N′ of the arc of 180° fall on the diameter NN′, and no perpendicular of any length may be drawn from them to the same diameter. The sine of the arc of 90° or of the arc NE coincides with the radius EC, and the sine of the arc of 270° coincides with the radius EC′. These are the qualities of the sine with which the student must endeavor to become familiar. The rules we subjoin may help him for this purpose.

Sine's qualities. *The sine is positive from* 0° *to* 180°; *negative from* 180° to 360°.

$$Sin\ (0°) = sin\ (180°) = o.$$

The sine of the arc of 90° *and of the arc of* 270° *are equal to each other in length and equal to the radius; and, when the length of the radius is equal to* 1 *we have*

$$\text{Sin}\ (90°) = 1, \quad \sin\ (270°) = -1.$$

The sines of two equal arcs, the one taken in the positive, the other in the negative direction, are equal to each other in length, but affected with a different sign.

The segment Cm of the radius CN, or distance from the centre to the point in which the sine meets the radius, is called the *cosine* of the arc KN or of its corresponding angle KCN. Observe that, drawing from K, Km' perpendicular to CE, Km' is equal to the sine of the arc KE. But Km' = Cm; hence, the cosine of the arc KN is equal to the sine of its complement, and, for this reason, is called cosine, which means sine of the complement, or complement-sine.

The cosine of any arc a is expressed by *cos a*.

Observe now that, by increasing the arc NK, the complement diminishes in the same manner, and, consequently, its sine, until it becomes zero, when NK = 90°. Hence, the cosine, which for the arc of 0° is evidently equal to the radius CN, decreases continually by increasing the arc from 0° to 90°. But for the arcs between 90° and 180°, since any perpendicular drawn from the different points of the arc EN′ on the diameter NN′ necessarily falls on the radius CN′, and since we have as positive the cosines taken from C to N, we must take as negative the cosines of the arcs from 90° to 180°; and, in the same manner in which the positive cosines of the first quadrant decrease from the arc of 0° to that of 90°, those of the second quadrant increase in length from the arc of 90° to that of 180°, when the cosine becomes again equal to the radius. It is now plain that in the third quadrant, or for the arcs from 180° to 270°, the cosines, remaining still negative, decrease in length continually, and in the same manner as those of the first quadrant. In the fourth quadrant, or for the arcs between 270° and 360°, the cosines become positive again, and increase from zero to the length of the radius.

We have seen already that the sines of any two arcs which are equal, but one positive and the other negative, coincide with the chord which joins their extremities. Hence, the same arcs must have a common cosine. Thus, Cm is at once the cosine of the arc NK and of the negative arc NL: and, generally representing by a any arc, we will have

$$\cos a = \cos -a.$$

Briefly, the qualities of the cosine may oe expressed as follows:—

Qualities of the cosine.

The cosine is positive from 0° *to* 90°, *negative from* 90° *to* 270°, *and positive again from* 270° *to* 360°.

$$\cos (90°) = \cos (270°) = 0;$$

and, supposing the length of the radius to be 1,

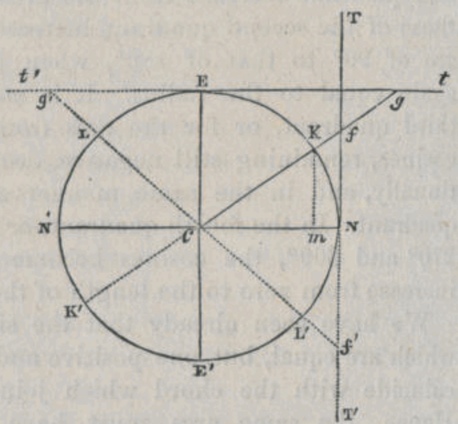
$$\cos (0°) = \cos (360°) = 1,$$

$$\cos (180°) = -1.$$

The same cosine is common to two arcs, the one positive and the other negative.

Tangent and cotangent.

§ 8. Supposing the rest as before, draw from N the geometrical tangent TT′, and also from E another tangent *tt′*. Draw to the point K of the arc NK the radius CK, and produce it to meet the tangents in *f* and *g*: the segment N*f* of NT is the trigonometrical *tangent* of the arc NK or of its corresponding angle KCN, and we express it by *tang* KN, or *tan* KN, or *tg* KN. The last expression seems to be preferred because of its simplicity.



The segment E*g* of E*t* is the *cotangent* of the arc KN and corresponding angle. Cotangent means the tangent of the complement. It is expressed by *cot* KN. Supposing, in fact, E to be the beginning of the arcs, E*g* would be the tangent of the arc EK, which is the complement of KN. The tangents and cotangents are functions of the arcs, because they vary with them. The manner in which they vary may be easily understood by the simple inspection of the figure. For, if we suppose the arc NK zero, then CK coincides with CN, and the length of the tangent is manifestly zero. But CK, coinciding with CN, is parallel to E*t*, and, consequently, can never meet it; hence, the cotangent of the arc zero is said to be infinite. It is also plain that by increasing the arc NK the tangent increases and the cotangent decreases till, when CK coinciding with CE, or the arc NK becoming an arc of 90°, the tangent becomes infinite and the cotangent zero.

The tangent is referred to the diameter NN′, and the cotangent to the other diameter EE′. The directions NT, Et are considered as positive, and, consequently, the directions NT′, Et' as negative; hence, the tangents and cotangents of the arcs from 0° to 90° are all positive.

Let us now take the arc NL between 90° and 180°. Join C with L, and produce CL in both directions so as to meet tt', TT′ in g' and in f': Nf' will be the tangent and Eg' the cotangent of the arc NEL. Eg,′ moreover, which, when NL is equal to NE, is zero, increases continually and becomes infinite when the point L of the arc NEL falls in N′. But Nf', which is infinite when LL′ coincides with EE′, decreases continually by increasing the arc NEL till it becomes zero, when LL′ coincides with NN′. The tangents, namely, and cotangents of the arcs taken from 90° to 180° are all negative, and the tangents decrease from infinite to zero, and the cotangents increase from zero to infinite.

The tangents and cotangents of the arcs taken from 180° to 270° have the same sign and follow the same order of increasing and decreasing as those of the arcs from 0° to 90°. In fact, let NEK′ be any arc taken from 180° to 270°: join K′ with C and produce it to f and g, Nf and Eg are the tangent and cotangent of the arc NEK′; the same, namely, as those of the arc NK in the first quadrant. In like manner, the tangents and cotangents of the arcs from 270° to 360° have the same sign and follow the same order of diminution and increase as those of the second quadrant; those, namely, of the arcs from 90° to 180°.

We may observe here, also, that, taking the arcs negative, that is, from N to E′, &c., the tangents and cotangents will keep the same order of increasing and decreasing as those of the positive arcs, but their sign will be opposite; so that, representing by a any arc, we will always have

$$\text{tg} -a = -\text{tg } a,$$
$$\cot -a = -\cot a.$$

Observe, also, that the infinite length of the tangent and cotangent, and generally all that which is commonly called infinite on account of being greater than all assignable limits, is frequently represented by the algorithm ∞, which is read infinite. Thus, we may briefly sum up the qualities of the tangent and cotangent as follows:—

Qualities of the tangent and of the cotangent.

The tangent and cotangent are positive from 0° *to* 90°, *and from* 180° *to* 270°, *and negative from* 90° *to* 180°, *and from* 270° *to* 360°, *when the arcs are positive, and vice versâ when the arcs are negative.*

$$\text{tg}\ (0°) = \text{tg}\ (180°) = \cot\ (90°) = \cot\ (270°) = 0,$$
$$\text{tg}\ (90°) = \cot\ (0°) = \infty,$$
$$\text{tg}\ (270°) = \cot\ (180°) = -\infty.$$

The tangent's length increases in the first and third quadrant, and the cotangent's in the second and fourth. The length of the tangents decreases in the second and fourth quadrant, and the length of the cotangents decreases in the first and third quadrant.

Secant and cosecant.

§ 9. The radius CK produced to the point f on the tangent is called the *secant* of the arc NK, and produced to the point g on the cotangent is called *cosecant* of the same arc NK; that is, secant of the complement of NK. These two functions of the arc are expressed by *sec* NK and *cosec* NK.

The secant is manifestly equal to the radius CN; for the arc zero, and, since CN is parallel to Et, the cosecant of the same arc is infinite. *Vice versâ*, the secant of the arc

NE of 90° is infinite, and the cosecant of the same arc is equal to the radius, and from 0° to 90° both secant and cosecant, like all the other functions, are considered as positive. But when we enter in the second quadrant, the secant Cf' must be taken in a direction opposite to that of the radius CL, while the cosecant Cg' is still taken in the same direction with the radius CL produced to g'; hence, the cosecant remains positive from 90° to 180°, at which point it becomes again infinite, and the secant is changed into negative from 90° to 180°, at which point it becomes again equal to the radius in length. For the arcs, also, from 180° to 270°, the secant is negative, because K′ being any point on the third quadrant, the radius CK′ cannot reach the tangent NT unless produced in a direction opposite to CK′. The same should be said of the cosecant with regard to Et; hence, for the arcs taken from 180° to 270°, both secant and cosecant are negative, and the first increases till it becomes infinite, the second decreases till it becomes equal to the radius. Lastly, for the arcs taken from 270° to 360°, the secant becomes again positive and the cosecant remains negative, and the first decreases until it becomes equal to the radius, and the second increases until it becomes infinite.

Qualities of the secant and cosecant.

Briefly: *The secant is positive from* 0° *to* 90°, *and from* 270° *to* 360°, *and negative from* 90° *to* 270°. *The cosecant is positive from* 0° *to* 180°, *and negative from* 180° *to* 360°.

Supposing the length of the radius to be 1,

$$\sec(0°) = \text{cosec}(90°) = 1,$$

$$\sec(180°) = \text{cosec}(270°) = -1,$$

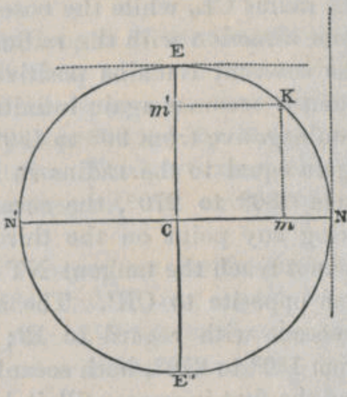
$$\left.\begin{array}{l}\sec(90°) = \text{cosec}(0°)\\ \sec(270°) = \text{cosec}(180°)\end{array}\right\} = \mp\infty.$$

The secant increases in the first and third quadrant, and de-

creases in the second and fourth. The cosecant increases in the second and fourth quadrant, and decreases in the first and third.

Versed-sine and versed-cosine.

§ 10. The perpendicular Km drawn to the diameter NN′ from any point K of the periphery is, as we have remarked, the sine of the arc NK; and Cm, or the perpendicular Km' let fall from K to the diameter EE′, is the cosine of the same arc. Now, in the same manner in which the points m, m' are unequally distant from C for different points of the periphery, thus are they also unequally distant from the extremities of the diameters on which the perpendiculars fall, and, consequently, Nm and Em' are functions of the arc NK; and the first of these functions is called *versed-sine*, and the second *versed-cosine*, or *coversed-sine*, and are expressed by *v. sin* NK, *v. cos* NK. The versed-sines, namely, are taken on the diameter NN′ from N, the beginning of arcs and tangents; and the versed cosines on the diameter EE′ from E, the beginning of cotangents.



Hence, the versed-sine increases for the whole semi-periphery NEN′, and decreases for the other N′E′N; is equal to zero, for the arc zero is equal to the radius for the arc of 90°; is equal to the diameter for the arc of 180°, and equal again to the radius for the arc of 270°, and always positive. The versed-cosine increases for the whole semi-periphery EN′E′, and decreases for the semi-periphery E′NE; is equal to the radius for the arc zero; is equal to zero for the arc of 90°; is equal to the radius again for the arc of 180°, and equal to the diameter for the arc of 270°, and it is always positive. Briefly:

Qualities of the versed-sine and cosine.

The versed-sines and versed-cosines are always positive.

Supposing the length of the radius to be 1,

$$\text{v. sin }(90°) = \text{v. sin }(270°) = \text{v. cos }(0°) = \text{v. cos }(180°) = 1,$$

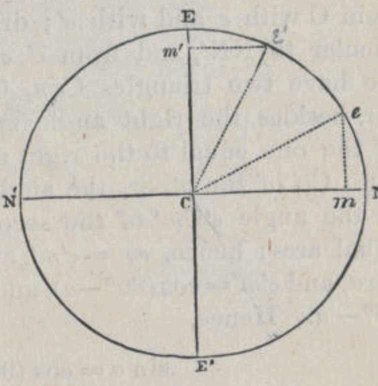
$$\text{v. sin }(180°) = \text{v. cos }(270°) = 2.$$

$$\text{v. sin }(0°) = \text{v. cos }(90°) = 0.$$

The versed-sine increases from 0° *to* 180°, *and decreases from* 180° *to* 360°. *The versed-cosine increases from* 90° *to* 270°, *and decreases from* 270° *to* 360°, *and from* 0° *to* 90°.

The sine of any angle is equal to the cosine of its complement, and *vice versâ.*

§ 11. Let a be any arc, either positive or negative. If positive, we will have the arc $90° - a$, by taking from E towards N an arc equal to a; hence, the extremities of the two arcs a and $90° - a$, the one reckoned from N and the other from E, are at an equal distance from those two points. And, therefore, if the first arc a terminates in the first quadrant, the other also must terminate in the same quadrant; and if the first terminates in the second quadrant, the other arc $90° - a$ must terminate in the fourth: finally, if the first arc terminates in the third quadrant, the second also must terminate in the same quadrant. Let the same be said when a is negative. For a negative must be taken from N towards E′, and with a negative, $90° - a$ becomes $90° + a$; hence, in this



case, also, the extremities of the two arcs a and $90°-a$, the one reckoned from N and the other from E, are equally distant from the two points, and are either both in the first or third quadrant, or the one in the second and the other in the fourth.

Now, both the sine and cosine of all the arcs terminating in the first quadrant are positive, and both the sine and cosine of all the arcs terminating in the third quadrant are negative. Again, the sine of any arc terminating in the second quadrant is positive, and the cosine of any arc terminating in the fourth quadrant is positive. But the cosine of any arc terminating in the second quadrant is negative, and the sine of any arc terminating in the fourth quadrant is negative. Hence, in all cases the two functions $\sin a$, $\cos (90°-a)$, or $\cos a$ and $\sin (90°-a)$, are affected with the same sign.

But they are, besides, equal to each other; for, let e and e' be the extremities of the two arcs a and $90°-a$. Join C with e and with e'; draw also from e, em perpendicular to NN′, and from e' $e'm'$ perpendicular to EE′: we have two triangles Cem, C$e'm'$ equal to each other; for, besides the right angle m and the hypothenuse Ce of the one equal to the right angle m' and the hypothenuse Ce' of the other, the angle eCm of the first is equal to the angle e'Cm' of the second, because measured by equal arcs; hence, $em = e'm'$, and Cm = Cm'. But $em = \sin a$, and $e'm' = \cos (90°-a)$, and C$m = \cos a$, and C$m' = \sin$ $90°-a$. Hence,

$$\sin a = \cos (90°-a)$$

$$\cos a = \sin (90°-a).$$

The same demonstration is applicable to all the cases, and a in the two equations represents any arc. Hence, generally, *The sine of any arc or angle is equal to the cosine of its complement, and vice versâ.*

The sine of any arc or angle is equal to the sine of its supplement, and the cosine of any arc or angle is equal to the negative cosine of its supplement.

§ 12. Since a represents any arc in the two last equations, we may take in them $a-90°$ instead of a. Thus, we will have

$$\sin(a-90°)=\cos(90°-(a-90°)),$$

$$\cos(a-90°)=\sin(90°-(a-90°)).$$

But $$90°-(a-90°)=180°-a;$$

hence, $$\sin(a-90°)=\cos(180°-a),$$

$$\cos(a-90°)=\sin(180°-a).$$

Again, we have seen (§ 7) that

$$\sin -a=-\sin a, \text{ and } \cos a=\cos -a.$$

But $$\sin(a-90°)=\sin -(90°-a)$$

$$\cos(a-90°)=\cos -(90°-a);$$

hence, $$\sin(a-90°)=-\sin(90°-a),$$

$$\cos(a-90°)=\cos(90°-a).$$

But, from the preceding number,

$$\sin(90°-a)=\cos a, \text{ and } \cos(90°-a)=\sin a.$$

Therefore, $$\sin(a-90°)=-\cos a,$$

$$\cos(a-90°)=\sin a;$$

and, consequently,

$$\sin a=\sin(180°-a),$$

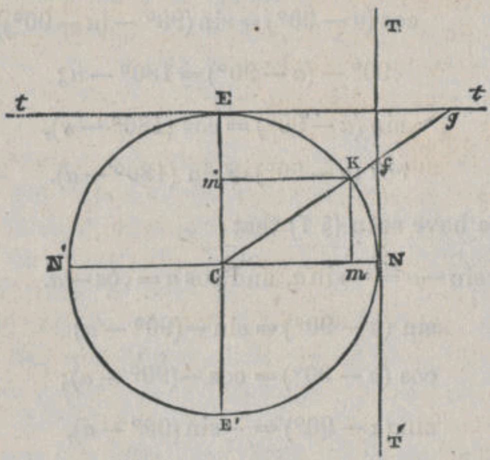
$$-\cos a=\cos(180°-a),$$

or $$\cos a=-\cos(180°-a).$$

That is, *the sine of any arc is equal to the sine of its supplement, and the cosine of any arc is equal to the negative cosine of its supplement.*

Trigonometrical formulas, or expressions of the mutual relations between the trigonometrical functions.

§ 13. Let NK be any arc a of the circle having r for radius. Drawing the diameters NN′, EE′ perpendicular to each other, and



the tangents TT′, tt' as above; drawing also the radius CK to the extremity K of the arc, and producing it to f and g, and, finally, Km perpendicular to NN′: we will have, with $\text{CK} = r$,

$$\text{K}m = \sin a,\ \text{N}f = \text{tg}\, a,\ \text{C}f = \sec a,$$

$$\text{C}m = \cos a,\ \text{E}g = \cot a,\ \text{C}g = \text{cosec}\ a.$$

Now, from the right-angled triangles CKm, CfN, CEg, we have

$$\overline{\text{CK}}^2 = \overline{\text{C}m}^2 + \overline{\text{K}m}^2,\ \ \overline{\text{C}f}^2 = \overline{\text{CN}}^2 + \overline{\text{N}f}^2,$$

$$\overline{\text{C}g}^2 = \overline{\text{E}g}^2 + \overline{\text{CE}}^2;$$

that is, $\left.\begin{array}{l} r^2 = \overline{\cos}^2\, a + \overline{\sin}^2\, a,\ \overline{\sec}^2\, a = r^2 + \overline{\mathrm{tg}}^2 \cdot a, \\ \overline{\mathrm{cosec}}^2\, a = \overline{\cot}^2\, a + \overline{r}^2. \end{array}\right\}$ (*i*)

Remark here, and once for all, that the powers of the functions of any arc are always expressed as in the preceding formulas, placing, namely, the exponent between the index of the function and the arc. Thus, the *m*th power of the tangent or of the sine of the arc *b* would be represented by $\mathrm{tg}^m\, b$, $\sin^m\, b$.

Observe also that if the radius *r* of the circle is made equal to 1, or if we consider the length of the radius as unity of measure, as is commonly the case, the preceding formulas will be changed into

$$\overline{\sin}^2\, a + \overline{\cos}^2\, a = 1$$

$$\overline{\sec}^2\, a = 1 + \overline{\mathrm{tg}}^2\, a,$$

$$\overline{\cos}^2\, a = 1 + \overline{\cot}^2\, a.$$

The similar triangles C K*m*, C*f*N give

$$\mathrm{N}f : \mathrm{CN} :: \mathrm{K}m : \mathrm{C}m;$$

The tangent of any arc divided by the radius is equal to the sine divided by the cosine of the same arc.

that is, $$\frac{\mathrm{N}f}{\mathrm{CN}} = \frac{\mathrm{K}m}{\mathrm{C}m};$$

or $$\frac{\mathrm{tg}\, a}{r} = \frac{\sin a}{\cos a},$$

and $$\mathrm{tg}\, a = r\,\frac{\sin a}{\cos a} \cdot (i');$$

and, when $r = 1$,

$$\mathrm{tg}\, a = \frac{\sin a}{\cos a} \cdot \cdot \cdot \cdot$$

From the same triangles we have also

$$\mathrm{C}f : \mathrm{CK} :: \mathrm{CN} : \mathrm{C}m;$$

that is,
$$\frac{Cf}{CK} = \frac{CN}{Cm};$$

The secant of any arc is equal to the square of the radius divided by the cosine of the same arc.

or,
$$\frac{\sec a}{r} = \frac{r}{\cos a},$$

and
$$\sec a = \frac{r^2}{\cos a} \; . \; . \; . \; . \; (i').$$

But, when $r = 1$,
$$\sec a = \frac{1}{\cos a}.$$

Draw now from K, Km' perpendicular to EE′: we will have $Km' = Cm = \cos a$, $Cm' = Km = \sin a$. Then, from the similar triangles CKm', CgE, we have

$$Eg : EC :: m'K : m'C;$$

that is,
$$\frac{Eg}{EC} = \frac{m'K}{m'C};$$

The cotangent divided by the radius is equal to the cosine divided by the sine of the same arc.

or,
$$\frac{\cot a}{r} = \frac{\cos a}{\sin a} \; . \; . \; . \; . \; (i');$$

and, when $r = 1$,
$$\cot a = \frac{\cos a}{\sin a}.$$

The same triangles give

$$Cg : CK :: CE : Cm';$$

that is,
$$\frac{Cg}{CK} = \frac{CE}{Cm'},$$

The cosecant is equal to the square of the radius divided by the sine of the same arc.

or,
$$\frac{\operatorname{cosec} a}{r} = \frac{r}{\sin a},$$

and
$$\operatorname{cosec} a = \frac{r^2}{\sin a} \; . \; . \; . \; . \; (i').$$

But, when $r = 1$,
$$\operatorname{cosec} a = \frac{1}{\sin a}.$$

With regard to the versed-sine and versed-cosine, we have

$$\mathrm{N}m = \mathrm{CN} - \mathrm{C}m,$$

$$\mathrm{E}m' = \mathrm{CE} - \mathrm{C}m',$$

in all cases; because, when Nm becomes greater than the radius, then Cm is negative, and $-$Cm is changed into $+$Cm; so also, when Em' is greater than the radius, Cm' becomes negative, and $-$Cm' is changed into $+$Cm'.

The versed-sine is equal to the radius minus the cosine of the arc. The versed-cosine is equal to the radius minus the sine of the arc.

Hence, generally,

$$\left.\begin{aligned} \text{v.}\sin a &= r - \cos a, \\ \text{v.}\cos a &= r - \sin a. \end{aligned}\right\} \quad (i'').$$

And, when $r = 1$,

$$\text{v.}\sin a = 1 - \cos a,$$

$$\text{v.}\cos a = 1 - \sin a.$$

The tangent of any arc or angle is equal to the cotangent of its complement.

§ 14. Substitute, in the first of the equations marked (i'), $90° - a$ instead of a: we will have

$$\operatorname{tg}(90° - a) = r\frac{\sin(90° - a)}{\cos(90° - a)};$$

that is, (§ 11,)

$$\operatorname{tg}(90° - a) = r\frac{\cos a}{\sin a}.$$

But, from the third equation marked (i'),

$$\frac{\cos a}{\sin a} = \frac{\cot a}{r},$$

therefore, $\operatorname{tg}(90° - a) = \cot a.$

The secant of any arc is equal to the cosecant of its complement.

Substitute, in the second equation marke (i'), $(90° - a)$ instead of a: we will have

$$\sec(90° - a) = \frac{r^2}{\cos(90° - a)}$$

$$= \frac{r^2}{\sin a.}$$

But, from the fourth (i'),

$$\frac{r^2}{\sin a} = \operatorname{cosec} a;$$

hence, $\sec(90° - a) = \cos a.$

We may, in like manner, obtain the same two formulas with the arcs inverted, viz.: $90° - a$ changed into a, and a into $90° - a$, by substituting, in the third and fourth formulas marked (i'), $90° - a$ instead of a. From the third we have

$$\cot(90° - a) = \operatorname{tg} a,$$

and, from the last,

$$\operatorname{cosec}(90° - a) = \sec a.$$

The versed-sine of any arc is equal to the versed-cosine of its complement, and *vice versâ*.

Substitute, now, in the equations (i''), $90° - a$ instead of a: we will have

$$\text{v.} \sin(90° - a) = r - \cos(90° - a),$$

$$\text{v.} \cos(90° - a) = r - \sin(90° - a),$$

and, consequently,

$$\text{v.} \sin(90° - a) = r - \sin a,$$

$$\text{v.} \cos(90° - a) = r - \cos a.$$

But, from the same equations (i''),

$$r - \sin a = \text{v.}\cos a,\ r - \cos a = \text{v.}\sin a;$$

hence,

$$\text{v.}\sin(90° - a) = \text{v.}\cos a,$$

$$\text{v.}\cos(90° - a) = \text{v.}\sin a.$$

The tangent, secant, and cotangent of any arc are respectively equal to the negative tangent, secant, and cotangent of the supplement. But the cosecant of any arc is equal to that of its supplement in every respect.

Substitute, in the first, second, and third equations marked (i'), $(180° - a)$ instead of a: we will have

$$\text{tg}\ (180° - a) = r\frac{\sin(180° - a)}{\cos(180° - a)},$$

$$\sec(180° - a) = \frac{r^2}{\cos(180° - a)},$$

$$\cot\ (180° - a) = r\frac{\cos(180° - a)}{\sin(180° - a)};$$

and, consequently, (§ 12,)

$$\text{tg}\ (180° - a) = -r\frac{\sin a}{\cos a},$$

$$\sec(180° - a) = -\frac{r^2}{\cos a}$$

$$\cot(180° - a) = -r\frac{\cos a}{\sin a}.$$

Now, from the same equations (i'),

$$r\frac{\sin a}{\cos a} = \text{tg}\ a,\ \frac{r^2}{\cos a} = \sec a,\ r\frac{\cos a}{\sin a} = \cot a;$$

hence,

$$\text{tg}\ (180° - a) = -\text{tg}\ a,$$

$$\sec(180° - a) = -\sec a,$$

$$\cot(180° - a) = -\cot a.$$

But when the same substitution of $180° - a$ instead of a is made in the last (i'), we have

$$\text{cosec}\,(180° - a) = \frac{r^2}{\sin\,(180° - a)}$$

$$= \frac{r^2}{\sin a},$$

and, consequently,

$$\text{cosec}\,(180° - a) = \text{cosec}\,a.$$

The versed-sine of any arc is equal to that of its supplement plus twice the cosine of the same arc.

The versed-cosine of any arc is exactly equal to that of its supplement.

Let us now make the substitution of $180° - a$ instead of a in the formulas (i''): we will have

$$\text{v. sin}\,(180° - a) = r - \cos\,(180° - a)$$

$$\text{v. cos}\,(180° - a) = r - \sin\,(180° - a);$$

and, consequently,

$$\text{v. sin}\,(180° - a) = r + \cos a,$$

$$\text{v. cos}\,(180° - a) = r - \sin a.$$

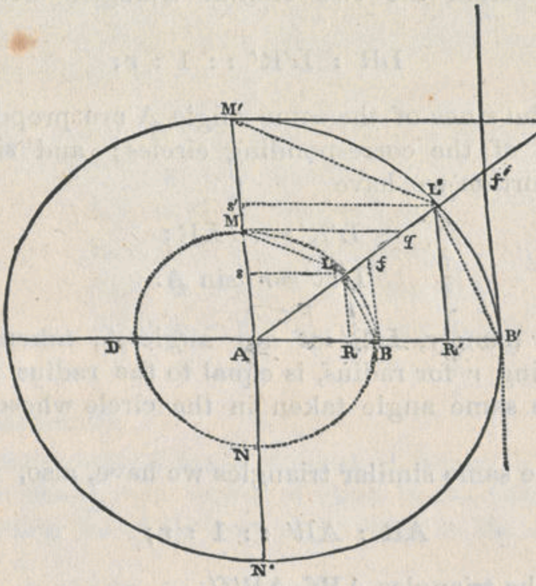
Now, $r + \cos a = r - \cos a + 2 \cos a$. But (i'') $r - \cos a =$ v. $\sin a$, and $r - \sin a =$ v. $\cos a$;

hence, $\text{v. sin}\,(180° - a) = \text{v. sin}\,a + 2 \cos a,$

$$\text{v. cos}\,(180° - a) = \text{v. cos}\,a.$$

The trigonometrical functions or lines are proportional to the radius of the corresponding circle.

§ 15. Let A be the common centre of two circles, the one having AB, and the other AB′, for radius. Draw the diameters D′B′, M′N′ perpendicular to each other; draw, also, any



other radius AL′: we will have at the same time the diameters DB, MN, and the radius AL of the internal circle. Now, the perpendiculars L′R′, LR, drawn from the extremities L′, L of the arcs B′L′, BL on the diameter D′B′, are both sines of the same angle L′AB′; but one evidently differs from the other. Let the same be said with regard to the tangents B*f*, B′*f*′; with regard to the secants A*f*, A*f*′; with regard to the cosines, cotangents, &c. Hence, the trigonometrical functions depend on the radius of the circle to which they are referred; hence, the radius must necessarily be taken into account with the function given or to be found.

To see now how the functions taken in a circle having r for radius are expressed, let us suppose the radius AB of the internal circle to be equal to the unity of measure for lengths, and the radius AB′ to be any radius r: from the two similar triangles LRA, L′R′A we have

$$\text{LR} : \text{L}'\text{R}' :: 1 : r;$$

that is, the sines of the same angle A are proportional to the radii of the corresponding circles; and since from the proportion we have

$$\text{L}'\text{R}' = r \cdot \text{LR};$$

hence, $$\text{L}'\text{R}' = r \cdot \sin \text{A}.$$

The sine, namely, L′R′ of any angle A, taken in the circle having r for radius, is equal to the radius into the sine of the same angle taken in the circle whose radius is 1.

From the same similar triangles we have, also,

$$\text{AR} : \text{AR}' :: 1 : r;$$

and from the triangles ABf, AB′f',

$$\text{B}f : \text{B}'f' :: 1 : r,$$
$$\text{A}f : \text{A}f' :: 1 : r.$$

From the triangles AMq, AM′q',

$$\text{M}q : \text{M}'q' :: 1 : r,$$
$$\text{A}q : \text{A}q' :: 1 : r.$$

Joining then L with B and with M, and L′ with B′ and M′, from the similar triangles LRB′, L′R′B′ we have

$$\text{RB} : \text{R}'\text{B}' :: \text{RL} : \text{R}'\text{L}' :: 1 : r;$$

and, from the triangles LMs, L′M′s',

$$Ms : M's' :: Ls : L's' :: 1 : r.$$

From which proportions we see that all the trigonometrical functions are proportional to the radii of the corresponding circles. Now from the same proportions we have the equations,

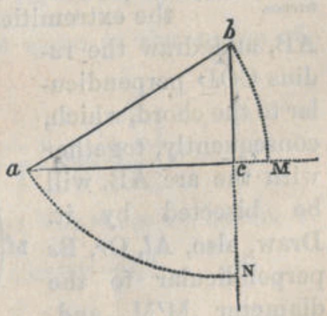
$$\begin{aligned} AR' &= r \cdot AR = r \cdot \cos A, \\ B'f' &= r \cdot Bf = r \cdot \text{tg } A, \\ Af' &= r \cdot Af = r \cdot \sec A; \\ M'q' &= r \cdot Mq = r \cdot \cot A, \\ Aq' &= r \cdot Aq = r \cdot \text{cosec } A; \\ R'B' &= r \cdot RB = r \cdot \text{v.-sin } A, \\ M's' &= r \cdot Ms = r \cdot \text{v.-cos } A. \end{aligned}$$

Hence, generally,

Any trigonometrical line or function of any angle A *taken in the circle having r for radius is equal to the radius into the corresponding line of the same angle taken in the circle whose radius is* 1.

The ratio of the sides about the right angle of any right-angled triangle is equal either to the tangent or to the cotangent of their opposite angles.

§ 16. Let abc be any right-angled triangle. Produce the sides bc, ac about the right angle, and, taking the hypothenuse for radius, and first a and then b for centres, describe the circular arcs bM, aN. Call h the hypothenuse, and s the side bc, and s' the side ac. Now, s is the sine and s' the cosine of the angle bac in the circle having h for radius; hence,



$$s = h \cdot \sin a,\ s' = h \cdot \cos a,\ (e'),$$

and, consequently,

$$\frac{s}{s'} = \frac{\sin a}{\cos a} = \operatorname{tg} a,$$

the tangent being that which is taken in the circle having 1 for radius. Now,

$$a + b = 90°;$$

hence, $$a = 90° - b,$$

and, consequently,

$$\operatorname{tg} a = \operatorname{tg}(90° - b) = \cot b,$$

and, therefore,

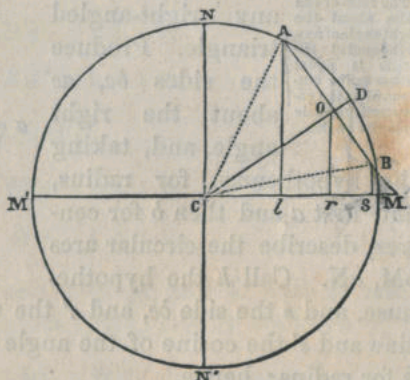
$$\frac{s}{s'} = \cot b.$$

In like manner, we will find

$$\frac{s'}{s} = \operatorname{tg} b = \cot a,\ (e'').$$

Relations between the functions of two different arcs and the functions of their sum or difference.

§ 17. Let the radius of the circle MNM′ be equal to 1, and take on the circle two arcs MA, MB, which we will call a and b. Join the extremities of the two arcs with the chord AB, and draw the radius COD perpendicular to the chord, which, consequently, together with the arc AB, will be bisected by it. Draw, also, Al, Or, Bs perpendicular to the diameter M′M, and, consequently, parallel to one another. And, since AO = OB, we



have, also, (GEOM., B. II. TH. 12,) $lr = rs$. From this construction we infer, first,

$$AB = MA - MB = a - b;$$

hence, $\frac{1}{2}AB$,

or, $$DB = DA = \frac{a-b}{2}.$$

Again, since MA + MB, or $a + b = MB + BA + MB = 2\,MB + 2\,BD = 2(MB + BD) = 2 \cdot MD$, we have also

$$MD = \frac{a+b}{2}.$$

In the same manner we have

$$Cr = \frac{Cs + Cl}{2}.$$

Now, the right-angled triangle OCr, according to the preceding number, gives

$$Cr = CO \cdot \cos OCM.$$

But CO is the cosine of the arc AD or DB; that is, the cosine of $\frac{a-b}{2}$; and the cosine of OCM, taken in the circle having 1 for radius, is the same as the cosine of the arc MD; that is, $\frac{a+b}{2}$;

hence, Cr, or $$\frac{Cs + Cl}{2} = \cos\frac{a-b}{2}\cos\frac{a+b}{2}.$$

But $Cs = \cos MB = \cos b$, $Cl = \cos MA = \cos a$; hence,

$$\cos a + \cos b = 2\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) \; .\;.\;.\;.\; (f).$$

From this formula, substituting in it $(180° - a)$ instead of a, we have

$$\cos(180° - a) + \cos b = 2\cos\left(90° - \frac{a-b}{2}\right)\cos\left(90° - \frac{a+b}{2}\right).$$

Now, $\cos(180° - a) = -\cos a$, $\cos\left(90° - \frac{a-b}{2}\right) = \sin \frac{a-b}{2}$, and $\cos\left(90° - \frac{a+b}{2}\right) = \sin\frac{a+b}{2}$; therefore

$$\cos b - \cos a = 2\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b) \;\ldots.\; (f').$$

Change, now, in (f) and (f'), a into $90° - a$, and b into $90° - b$: we will have

$$\cos(90° - a) + \cos(90° - b) = 2\cos(90° - \tfrac{1}{2}(a+b))\cos\tfrac{1}{2}(b-a),$$

$$\cos(90° - b) - \cos(90° - a) = 2\sin(90° - \tfrac{1}{2}(a+b))\sin\tfrac{1}{2}(b-a);$$

and, consequently,

$$\sin a + \sin b = 2\sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(b-a),$$
$$\sin b - \sin a = 2\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(b-a).$$

Observe, now, that $\cos\tfrac{1}{2}(b-a) = \cos - \tfrac{1}{2}(a-b)$, and $\sin\tfrac{1}{2}(b-a) = \sin - \tfrac{1}{2}(a-b)$. Now, (§ 7) $\cos - \tfrac{1}{2}(a-b) = \cos\tfrac{1}{2}(a-b)$; and $\sin - \tfrac{1}{2}(a-b) = -\sin\tfrac{1}{2}(a-b)$; hence, the last formulas are easily changed into the following:—

$$\sin a + \sin b = 2\sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) \;\ldots.\; (f''),$$
$$\sin a - \sin b = 2\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b) \;\ldots.\; (f''').$$

Other trigonometrical formulas.

§ 18. From the formulas of the preceding number (f), (f'), (f''), (f'''), we infer many others equally useful; and first make in each one of the said formulas $b = o$: we will have, from the first,

$$\cos a + 1 = 2\cos^2\tfrac{1}{2}a \;\ldots.\; (g);$$

from the second,

$$1 - \cos a = 2\sin^2\tfrac{1}{2}a \;\ldots.\; (g');$$

from the third and fourth,

$$\sin a = 2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a \;\ldots.\; (g'');$$

and, taking the difference between (g) and (g'),

$$\cos a = \cos^2 \tfrac{1}{2}a - \sin^2 \tfrac{1}{2}a \quad \ldots \quad (g''').$$

Dividing, now, (f') by (f), and then (f''') by (f''), finally, (g') by (g), since the radius of the circle to which our functions are referred is 1, we will easily obtain (§ 13)

$$\left.\begin{aligned} \frac{\cos b - \cos a}{\cos b + \cos a} &= \operatorname{tg} \tfrac{1}{2}(a+b) \operatorname{tg} \tfrac{1}{2}(a-b), \\ \frac{\sin a - \sin b}{\sin a + \sin b} &= \operatorname{tg} \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}(a+b) \\ &= \frac{\operatorname{tg} \tfrac{1}{2}(a-b)}{\operatorname{tg} \tfrac{1}{2}(a+b)}, \\ \frac{1-\cos a}{1+\cos a} &= \operatorname{tg}^2 \tfrac{1}{2}a, \end{aligned}\right\} (h).$$

Since the arcs a and b are any two arcs, change in (f), (f'), (f''), and (f'''), a into $a+b$, and b into $a-b$: we will obtain four more formulas, as follows:—

$$\left.\begin{aligned} \cos(a+b) + \cos(a-b) &= 2 \cos a \cos b, \\ \cos(a-b) - \cos(a+b) &= 2 \sin a \sin b, \\ \sin(a+b) + \sin(a-b) &= 2 \sin a \cos b, \\ \sin(a+b) - \sin(a-b) &= 2 \cos a \sin b, \end{aligned}\right\} (h').$$

Adding, now, together the two first (h'), and then taking their difference, and repeating the same operations on the two remaining (h'), we have

$$\left.\begin{aligned} \cos(a-b) &= \cos a \cos b + \sin a \sin b, \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b, \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b, \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b, \end{aligned}\right\} (h'').$$

Divide the third of these equations by the second, and the fourth by the first: we will have

$$\operatorname{tg}(a+b)=\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b-\sin a\sin b},$$

$$\operatorname{tg}(a-b)=\frac{\sin a\cos b-\cos a\sin b}{\cos a\cos b+\sin a\sin b}.$$

And dividing both numerator and denominator of the second members by $\cos a\cdot\cos b$, we will have

$$\left.\begin{aligned}\operatorname{tg}(a+b)&=\frac{\operatorname{tg}a+\operatorname{tg}b}{1-\operatorname{tg}a\operatorname{tg}b},\\ \operatorname{tg}(a-b)&=\frac{\operatorname{tg}a-\operatorname{tg}b}{1+\operatorname{tg}a\operatorname{tg}b},\end{aligned}\right\}(h''').$$

Let us now change in the first equation (h'), a into $a+c$, and then a into $a-c$: we will obtain

$$\cos(a+b+c)+\cos(a+c-b)=2\cos(a+c)\cos b,$$
$$\cos(a+b-c)+\cos(a-b-c)=2\cos(a-c)\cos b.$$

Observe that $\cos(a-b-c)=\cos-(a-b-c)=\cos(b+c-a)$, and $\cos(a+c)+\cos(a-c)=2\cos a\cos c$; hence, adding to each other the two last equations, we have

$$\left.\begin{aligned}\cos(a+b+c)+\cos(a+c-b)+\cos(a+b-c)+\\ \cos(b+c-a)=4\cos a\cos b\cos c,\end{aligned}\right\}(h'''').$$

And, in the supposition that

$$a+b+c=180°,$$

since then

$$\tfrac{1}{2}(a+b+c)=90°,$$
$$\tfrac{1}{2}(a+b-c)=90°-c,$$
$$\tfrac{1}{2}(a+c-b)=90°-b,$$
$$\tfrac{1}{2}(b+c-a)=90°-a,$$

changing, in (h''''), a, b, c into $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$, we will have, from (h''''),

$$\sin b + \sin c + \sin a = 4 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \ldots . (h''''').$$

Change now, in the first formula (h'''), b into $b + c$, and let us suppose again $a + b + c = 180°$: we will have

$$\text{tg}\,(180°) = \frac{\text{tg}\,a + \text{tg}\,(b + c)}{1 - \text{tg}\,a\,\text{tg}\,(b + c)}.$$

Now, $\text{tg}\,(180°) = 0$, and from the same, (h'''), we have

$$\text{tg}\,(b + c) = \frac{\text{tg}\,b + \text{tg}\,c}{1 - \text{tg}\,b \cdot \text{tg}\,c}.$$

Hence, we have, first,

$$\text{tg}\,a + \text{tg}\,(b + c) = 0,$$

and, consequently,

$$\text{tg}\,a + \frac{\text{tg}\,b + \text{tg}\,c}{1 - \text{tg}\,b \cdot \text{tg}\,c} = 0;$$

from which

$$\text{tg}\,a - \text{tg}\,a \cdot \text{tg}\,b \cdot \text{tg}\,c + \text{tg}\,b + \text{tg}\,c = 0.$$

Hence, when $a + b + c = 180°$,

$$\text{tg}\,a + \text{tg}\,b + \text{tg}\,c = \text{tg}\,a \cdot \text{tg}\,b \cdot \text{tg}\,c \ldots . (h'''''').$$

ARTICLE II.

RESOLUTION OF TRIANGLES, AND APPLICATIONS.

Numerical value of trigonometrical functions and their logarithms.

§ 19. All trigonometrical functions are, as we have seen in the preceding article, rectilinear, and, consequently, such as may be compared with the radius of the circle. Now, taking the radius as the common measure or unity of measure for trigonometrical functions, we may find them either less, or equal to, or greater than the radius; and in every one of these cases they may be expressed numerically,—that is, by that number which expresses to what part of the radius they are equal, or how many times they contain the radius in their length. This number is called the numerical value of the functions. We will presently see how this numerical value may be obtained. But let it be known, first, that not only the numerical values of the functions have been determined for a large number of arcs, but also the logarithms of the same value and tables have been constructed thereof. These tables will be better appreciated in the following pages. But observe, now, that, supposing n to be the numerical value of any function, its logarithm given by the tables is the exponent to be given to $a = 10$ to obtain n. (See Alg., § 122.) And, since the tables do not give the numerical value of the functions, but only the corresponding logarithm, when, for example, we wish to know what is the numerical value of sin 10° or tg 20°, &c., we will take from the tables of trigonometrical logarithms the logarithms of these values, and, finding then the corresponding numbers of these logarithms in the common tables having $a = 10$ for base, these numbers are the numerical values of the functions.

Let us now give an idea of the manner in which the numerical values of the various functions can be obtained. And, first, observe that it is enough for us to have the numerical value of one of these functions, for example, the sine of any arc a, because the other functions of the same arc are then given by the formulas (i), (i'), (i''), (§ 13.) Thus, for example, since from the first equation (i) we have

$$\cos^2 a = r^2 - \sin^2 a,$$

or, taking the radius 1,

$$\cos a = \sqrt{1 - \sin^2 a},$$

when the numerical value of $\sin a$ is known, that also of $\cos a$ is determined by the preceding formula. So likewise from the first (i'), taking in it $r = 1$, we have

$$\operatorname{tg} a = \frac{\sin a}{\cos a};$$

Hence, when the numerical values of $\sin a$ and $\cos a$ are known, the numerical value of the tangent of the same arc a is also known, &c. Hence, when the numerical values of the sines of the arcs a, a', a'', &c. are known, the numerical values of the other functions of the same arcs are given by the said formulas.

Observe, secondly, that from the formula (g''), (§ 18,) we have

$$\sin 2a = 2 \sin a \cdot \cos a,$$

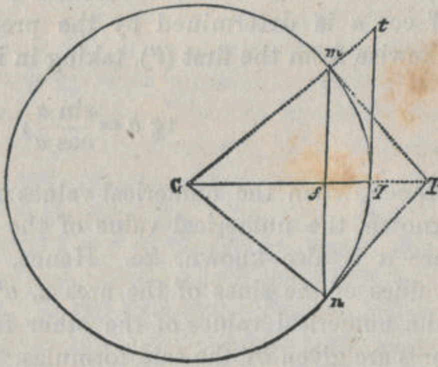
and, from the third (h'), changing in it a into $2a$ and b into a,

$$\sin 3a = 2 \sin 2a \cdot \cos a - \sin a;$$

and taking in the same formula, (h'), $3a$ instead of a, and a instead of b, we have

$$\sin 4a = 2 \sin 3a \cos a - \sin 2a, \text{ \&c.}$$

Therefore, when the numerical value of $\sin a$ is known, and from this the numerical value of $\cos a$ is inferred, the numerical value of $\sin 2a$ is immediately given by the first of the last formulas, and then that of $\sin 3a$ by the second, that of $\sin 4a$ by the next, &c. In the supposition, therefore, that we may find the numerical value of the sine of the arc of $10''$ or of $1'$, we will obtain, by means of the same formulas, the numerical values of the sines of the arcs of 20″, of 30″, &c. . . ., or of 2′, of 3′, &c. Now, we may obtain, with the greatest desirable accuracy, the numerical value of the sine of 10″, for instance, or 1′. For, let qm be any arc, and qn another arc equal to qm; join the centre of the circle with m, q, and n, and draw the chord mn, which is bisected by the radius Cq, and perpendicular to it. Hence,



$$mf = \sin mq,$$

or $$\tfrac{1}{2}mn = \sin mq.$$

Again: drawing the tangents qt, mp, since from the equal triangles tqC, pmC we have $mp = tq$, and tq is the trigonometrical tangent of mq, we will have

$$mp = \text{tg } mq,$$

and also the tangent np, which, on account of the equal triangles Cpn, Cpm, necessarily meets Cp in the same point p of mp, is the tangent of the arc qn; and, since $pm = pn$, we have also

$$mp = \tfrac{1}{2}(pm + pn);$$

hence, $$\tfrac{1}{2}(pm + pn) = \text{tg } mq$$

$$= \frac{\sin mq}{\cos mq}.$$

Now, $mfn < mqn$,

$pn + pm > mqn$,

and, consequently,

$\frac{1}{2}mfn < \frac{1}{2}mqn$,

or, $\frac{1}{2}mfn < mq$,

and $\frac{1}{2}(pn + pm) > mq$,

therefore, $\sin mq < mq$,

$\frac{\sin mq}{\cos mq} > mq$,

and $\sin mq > mq \cos mq$.

Multiplying both members of the last inequality by $2 \cos mq$, we have

$$2 \sin mq \cdot \cos mq > 2\, mq \cos^2 mq.$$

But, from (g''), (§ 18,) $2 \sin mq \cdot \cos mq = \sin 2\,mq$; and, from $(i,)$ (§ 13,) when $r = 1$, $\cos^2 mq = 1 - \sin^2 mq$;

hence, $\sin 2\,mq > 2\,mq\,(1 - \sin^2 mq)$.

Now, $2\,mq$ represents any arc; therefore the same inequality is also applicable to the arc mq;

that is, $\sin mq > mq\,(1 - \sin^2 \frac{1}{2}mq)$.

But we have seen that the sine of any arc mq is less than the arc itself;

hence, $\sin \frac{1}{2}mq < \frac{1}{2}mq$,

and, also, $\sin^2 \frac{1}{2}mq < \frac{\overline{mq}^2}{4}$.

Call d the difference between $\sin^2 \frac{1}{2}mq$ and $\frac{\overline{mq}^2}{4}$: we will have

$$\sin^2 \tfrac{1}{2}mq = \frac{\overline{mq}^2}{4} - d,$$

and, consequently,

$$\sin mq > mq \left(1 - \frac{\overline{mq}^2}{4} + d\right);$$

that is,

$$\sin mq > mq \left(1 - \frac{\overline{mq}^2}{4}\right) + mq \cdot d,$$

and, consequently, much more

$$\sin mq > mq \left(1 - \frac{\overline{mq}^2}{4}\right);$$

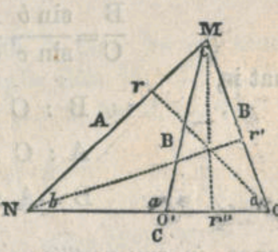
Hence, we have at once the sine of any arc mq less than the arc itself, and greater than the same arc multiplied

by $$1 - \frac{\overline{mq}^2}{4}.$$

Now, we have from geometry the ratio between the radius and the circumference numerically expressed; that is, the radius being 1, the circumference is (B. IV. TH. 14) 6·28 . . . . ; and, dividing this number by 360, we will have the linear value of one degree of the circumference numerically expressed in a part of the radius; and in a like manner we may obtain the numerical value of the arc of one minute, ten or twenty seconds, &c. Therefore, the arc mq, also, may be numerically given in a part of the radius, and, consequently, also, $mq\left(1 - \frac{\overline{mq}^2}{4}\right)$. But if we take the arc mq of one minute, the numerical values of mq, and of mq $\left(1 - \frac{\overline{mq}^2}{2}\right)$ do not differ from each other for many decimal figures; hence, the same number, as far as the figures of the two numerical values are equal, expresses necessarily the numerical value of the sine of the arc mq given in a part of the radius, since the numerical value of the sine is between the two numerical values of mq and $mq\left(1-\frac{\overline{mq}^2}{4}\right)$.

Equations for the resolution of triangles.

§ 20. Let MNO be any triangle. Call the side MN, A, and its opposite angle a. Call the sides MO, NO, B, C, and their respectively opposite angles b, c. Call, also, p, p', p'' the perpendiculars Or, Nr', Mr'' drawn from the vertices to the opposite sides. With each perpendicular we have two right-angled triangles: with p, the right-angled triangles OrM, OrN; with p', Nr'M, Nr'O; and with p'', Mr''O, Mr''N. Hence, also, (§ 16,) the equations,



$$p = \text{B}\sin c,\; p = \text{C}\sin b,$$
$$p' = \text{A}\sin c,\; p' = \text{C}\sin a,$$
$$p'' = \text{B}\sin a,\; p'' = \text{A}\sin b,$$

and, consequently,

$$\text{B}\cdot\sin c = \text{C}\cdot\sin b,$$
$$\text{A}\cdot\sin c = \text{C}\cdot\sin a,$$
$$\text{B}\cdot\sin a = \text{A}\cdot\sin b;$$

and
$$\frac{\text{B}}{\sin b} = \frac{\text{C}}{\sin c},\quad \frac{\text{A}}{\sin a} = \frac{\text{C}}{\sin c},$$
$$\frac{\text{B}}{\sin b} = \frac{\text{A}}{\sin a};$$

which last equations may be more simply expressed in one, as follows:—

$$\left.\frac{\mathrm{A}}{\sin a}=\frac{\mathrm{B}}{\sin b}=\frac{\mathrm{C}}{\sin c}\right\} (e_1).$$

From the same equations we have also

$$\frac{\mathrm{B}}{\mathrm{C}}=\frac{\sin b}{\sin c}, \frac{\mathrm{A}}{\mathrm{C}}=\frac{\sin a}{\sin c}, \frac{\mathrm{B}}{\mathrm{A}}=\frac{\sin b}{\sin a};$$

that is,

$$\mathrm{B} : \mathrm{C} :: \sin b : \sin c,$$

$$\mathrm{A} : \mathrm{C} :: \sin a : \sin c,$$

$$\mathrm{B} : \mathrm{A} :: \sin b : \sin a.$$

Hence, *in any rectilinear triangle the sides are as the sines of their opposite angles.*

It is well known from geometry that the sum of the three angles of any triangle is equal to two right angles;

hence, $a+b+c=180° \ldots . (e_2)$.

Now, from the last proportion, or from its equivalent, $\mathrm{A} : \mathrm{B} :: \sin a : \sin b$, we have (see Treat. on Alg., § 119)

$$\mathrm{A}+\mathrm{B} : \mathrm{A}-\mathrm{B} :: \sin a+\sin b : \sin a-\sin b,$$

or $\mathrm{A}-\mathrm{B} : \mathrm{A}+\mathrm{B} :: \sin a-\sin b : \sin a+\sin b;$

and, consequently, (§ 18,) (h),

$$\frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}+\mathrm{B}}=\frac{\sin a-\sin b}{\sin a+\sin b}=\frac{\operatorname{tg} \frac{1}{2}(a-b)}{\operatorname{tg} \frac{1}{2}(a+b)};$$

from which

$$\operatorname{tg} \tfrac{1}{2}(a-b)=\frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}+\mathrm{B}} \operatorname{tg} \tfrac{1}{2}(a+b).$$

But, from (e_2), $a+b=180°-c$, and $\frac{1}{2}(a+b)=90°-\frac{1}{2}c$;

hence, $$\text{tg}\,\tfrac{1}{2}(a+b)=\text{tg}\,(90°-\tfrac{1}{2}c).$$

Now, (§ 14,) $$\text{tg}\,(90°-\tfrac{1}{2}c)=\cot\tfrac{1}{2}c;$$

hence, $$\text{tg}\,\tfrac{1}{2}(a-b)=\frac{A-B}{A+B}\cot\tfrac{1}{2}c\ldots(e_3).$$

The angle opposite to the side A may be an acute angle, like MOr'', or an obtuse angle, like $MO'N$: in the first case, we have from Geometry (B. III. TH. 7, SC. 2)

$$A^2=B^2+C^2-2\,C\cdot Or'';$$

and, in the second, calling B, C the sides MO', NO',

$$A^2=B^2+C^2+2\,C\cdot O'r''.$$

Now, (§ 16,) $$Or''=B\cos a,$$

and $$O'r''=B\cos MO'r''$$
$$=B\cos(180°-MO'N)$$
$$=B\cos(180°-a)=-\cos a.$$

Hence, from both the preceding equations, we infer

$$A^2=B^2+C^2-2\,B\cdot C\cos a\ldots(e_4);$$

that is, whether the angle a be acute or obtuse, the square of its opposite side is equal to the sum of the squares of the other two sides minus the double product of the same two sides into the cosine of the angle a.

Now, from the formulas (g), (g'), (§ 18,) we have

$$\cos a = 2\cos^2 \tfrac{1}{2}a - 1,$$

$$\cos a = 1 - 2\sin^2 \tfrac{1}{2}a.$$

Substituting in succession these two values in (e_4), we will have

$$\begin{aligned} A^2 &= B^2 + C^2 - 2B\cdot C\,(2\cos^2 \tfrac{1}{2}a - 1) \\ &= B^2 + C^2 + 2B\cdot C - 4B\cdot C\cos^2 \tfrac{1}{2}a \\ &= (B+C)^2 - 4B\cdot C\cos^2 \tfrac{1}{2}a, \\ A^2 &= B^2 + C^2 - 2B\cdot C\,(1 - 2\sin^2 \tfrac{1}{2}a) \\ &= B^2 + C^2 - 2B\cdot C + 4B\cdot C\sin^2 \tfrac{1}{2}a \\ &= (B-C)^2 + 4B\cdot C\sin^2 \tfrac{1}{2}a. \end{aligned}$$

Hence,

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}a &= \frac{A^2 - (B-C)^2}{4B\cdot C}, \\ \cos^2 \tfrac{1}{2}a &= \frac{(B+C)^2 - A^2}{4B\cdot C}, \end{aligned}\right\} \quad (e_5).$$

These are the equations with which we may resolve the problem that (§ 1) forms the object of plane trigonometry. In fact, excluding the case of the given elements being the three angles, in which case the length of the sides cannot be determined, since any number of similar triangles may have different sides, with the exception of this case the given elements may be,

First—One angle and two sides;
Second—One side and two angles;
Third—Three sides.

In the first of the three cases the given angle is either formed by the given sides or opposite to one of them. If included, then by means of the formula (e_4) we may obtain the third side, because the second member of this equation contains two sides and the cosine of the included angle. Hence, substituting instead of B and C the values of the two given sides, and taking from the tables the cosine of the given angle, and placing it instead of $\cos a$, the whole second member becomes known, and, consequently also, the first, which is the square of the third side. Then, from the formulas (e_3) and (e_2) we may have the other two angles; because by means of (e_3), which contains, in the second member, two sides and the included angle, we may have the difference of the other two angles,—that is, knowing the tg $\frac{1}{2}(a-b)$, we may obtain from the tables $\frac{1}{2}(a-b)$, and from (e_2) we easily have $\frac{1}{2}(a+b)=90°-\frac{1}{2}c$. Hence, half the sum and half the difference of a and b, which represent our unknown angles, are thus known; but, adding $\frac{1}{2}(a-b)$ to $\frac{1}{2}(a+b)$, we have a, and subtracting $\frac{1}{2}(a-b)$ from $\frac{1}{2}(a+b)$, we have b; hence, a and b also become known. . . . We may also commence by finding first the angles and then the side by means of the equation (e_1). If the angle is not included but opposite to one of the two given sides, then from the equation (e_1) we may have the sine of the angle opposite to the other side, because from (e_1) we infer, for example,

$$\sin b = \frac{B}{A} \sin a,$$

and, consequently, substituting for A and B the values of the two given sides, and for $\sin a$ the sine of the angle opposite to A, we obtain evidently the sine of the angle b opposite to the side B. From $\sin b$, the tables will give b. But (§ 12) $\sin b = \sin (180° - b)$; hence, for $\sin b$ the tables will give two angles b and $(180° - b)$;

therefore, in this resolution there is ambiguity, which is frequently taken away by some conditions of the problem revealing which of the two angles is to be selected. When the second angle is found, the third angle is immediately given by (e_2), and the third side is obtained from the same (e_1), since from

$$\frac{A}{\sin a} = \frac{C}{\sin c},$$

we have $$C = \frac{\sin c}{\sin a} A.$$

In the second case,—namely, when the given elements are two angles and one side,—the third angle is immediately given by (e_2), and the two unknown sides by (e_1), as above.

In the last case,—namely, when the given elements are the three sides,—one of the angles is given by (e_5); then, knowing two sides and the included angle, we may find the other angle, as in the first case, or else the three angles may be all obtained from (e_5), by changing the disposition of the sides in the second members.

Resolution of right-angled triangles.

§ 21. The preceding formulas afford a means to resolve any rectilinear triangle whenever the resolution is possible, and, consequently also, when the triangle to be resolved is a right-angled triangle, we may use the same equations. But, in this case, the equations (e'), (e''), (§ 16,) render the resolution more speedy. For, when the two sides s, s' about the right angle are given, we may obtain the other two angles from (e''), and the hypothenuse h from (e'); when the hypothenuse is given with another side, we may find, first, one of the two acute angles, and then the other side from (e'), &c. But let us see some examples.

EXAMPLES. EXAMPLE I. When one side and two angles are given.

§ 22. Let the side A of a triangle be equal to 2301,82 either feet or yards, and let the angle b be equal to 26° 17′ 59″,4, and the angle c equal to 84° 56′ 24″,3. Find the other elements.

From (e_2) we have, first,

$$a = 180° - (c + b)$$

$$= 180° - 111° \; 14' \; 23'',7.$$

Hence, $$a = 68° \; 45' \; 36'',3.$$

With regard to the sides B and C, we have, from (e_1),

$$B = A\frac{\sin b}{\sin a}, \quad C = A\frac{\sin c}{\sin a};$$

hence,

$$B = 2301,82\frac{\sin(26° \; 17' \; 59'',4)}{\sin(68° \; 45' \; 36'',3)},$$

$$C = 2301,82\frac{\sin(84° \; 56' \; 24'',3)}{\sin(68° \; 45' \; 36'',3)}.$$

And, taking the logarithms, (see Treat. on Alg.,)

$$\text{l.B} = \text{l.}(2301,82) + \text{l.}\sin(26° \; 17' \; 59'',4)$$
$$- \text{l.}\sin(68° \; 45' \; 36'',3),$$

$$\text{l.C} = \text{l.}(2301,82) + \text{l.}\sin(84° \; 56' \; 24'',3),$$
$$- \text{l.}\sin(68° \; 45' \; 36'',3).$$

Now, from the common tables we have

$$\text{l.}\ 2301,82 = 3,362071,$$

and from the tables of trigonometrical functions,

$$\text{l. sin } 26^\circ\ 17'\ 59'',4 = 9{,}646471,$$

$$\text{l. sin } 68^\circ\ 45'\ 36'',3 = 9{,}969449,$$

$$\text{l. sin } 84^\circ\ 56'\ 24'',3 = 9{,}998304;^*$$

hence, $\text{l. B} = 3{,}362071 + 9{,}646471 - 9{,}969449,$

$$\text{l. C} = 3{,}362071 + 9{,}998304 - 9{,}969449;$$

that is, $\text{l. B} = 3{,}039093,$

$$\text{l. C} = 3{,}390926.$$

Now, from the common tables we have

$$3{,}039093 = \text{l. } 1094{,}2,$$

$$3{,}390926 = \text{l. } 2460{,}0;$$

hence, $\text{l. B} = \text{l. } 1094{,}2,$

$$\text{l. C} = \text{l. } 2460{,}0;$$

that is, $\text{B} = 1094{,}2,$

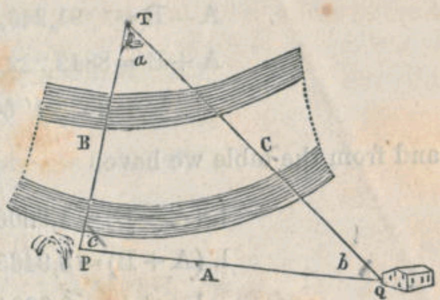
$$\text{C} = 2460{,}2.$$

Application. The student may immediately appreciate the practical profit which can be derived from the resolution of triangles by the following application.

Let T represent an inaccessible point, the distance of which from the points P and Q on the opposite side of a river or of a ravine is to be determined. Measure the rectilinear distance of the two points P and Q, and suppose it to be the length of our given side A; that is,

* The student will find at the end of the book some tables of logarithms. But for exact and laborious calculations, tables much more voluminous are unquestionably required. Those of Callet are excellent. The direction of the teacher, however, and the use of the small tables here added, will render easy the use of other tables more complete.

2301,83 yards. Then, by means of a graduated instrument, measure the angle TQP which the visual rays directed to P and T form in Q, and let the angle formed by these rays or lines be our given angle b; that is, $26° \; 17' \; 59'',4$: measure in like manner the angle TPQ formed in P by the visual rays directed from P to Q and to T, and let this angle be the above-given angle $c = 84° \; 56' \; 24'',3$.



Now, the visual rays with the base form, evidently, a triangle of which we know one side and the two adjacent angles, and which, resolved, gives for the length of the side B or distance of P from the inaccessible point T, 1094,2, and for the length of C or distance of Q from the same T, 2460,2 yards.

EXAMPLE II. When the given elements are two sides and the included angle.

Let, now, the given elements be two sides and the included angle; that is, let

$$A = 4466,784,$$

$$B = 4375,438,$$

$$c = 46° \; 49' \; 40'',4.$$

We will first find half the difference between the two remaining angles a and b, with the equation (e_3); for, applying the logarithms to this equation, we have

$$\text{l. tg} \; \tfrac{1}{2}(a - b) = \text{l.} \; (A - B) + \text{l. cot} \; \tfrac{1}{2}c - \text{l.} \; (A + B).$$

Now, from the given elements we easily infer

$$A - B = \quad 91{,}346,$$
$$A + B = 8842{,}222,$$
$$\tfrac{1}{2}c = 23^\circ\ 24'\ 50'',2\,;$$

and from the table we have

$$\text{l. } (A - B) = 1{,}9606895,$$
$$\text{l. } (A + B) = 3{,}9465614,$$
$$\text{l. cot } \tfrac{1}{2}c = 0{,}3634844\,;$$

hence, $\quad \text{l. tg } \tfrac{1}{2}(a - b) = 8{,}3776125.$

And, again, from the tables,

$$\tfrac{1}{2}(a - b) = 1^\circ\ 21'\ 59'',9.$$

Now, from the equation (e_2) we have, in our case,

$$a + b = 180^\circ - 46^\circ\ 49'\ 40'',4,$$
$$= 133^\circ\ 10'\ 19'',8,$$

and, consequently,

$$\tfrac{1}{2}(a + b) = 66^\circ\ 35'\ 9'',9\,;$$

hence, adding first and then subtracting from the value of $\frac{1}{2}(a + b)$ the preceding one of $\frac{1}{2}(a - b)$, we will have

$$a = 67^\circ\ 57'\ 9'',8,$$
$$b = 65^\circ\ 13'\ 10'',0.$$

Thus we have obtained the two unknown angles. To obtain the unknown side C, the equation (e_1) gives us

$$C = \frac{\sin c}{\sin a} A,$$

and, consequently,

$$\text{l. } C = \text{l. sin } c + \text{l. } A - \text{l. sin } a.$$

Now, from the tables,

$$\text{l. } \sin c = 9{,}8629065,$$

$$\text{l. } A = 3{,}6499950,$$

$$\text{l. } \sin a = 9{,}9670211;$$

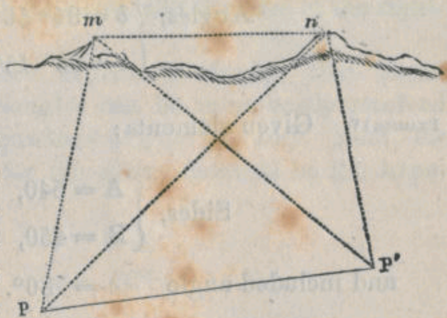
hence,

$$\text{l. } C = 3{,}5458804,$$

and, from the tables,

$$C = 3514{,}6.$$

Application. This case, also, may be applied to some geodetical purposes. Thus, for example, let mn be the unknown rectilinear distance of the summit of the mountain m from that of the mountain n to be determined. In the supposition that the summits of the two mountains are visible from P and from P′, and the rectilinear distance of P from P′ is known, we may, drawing the visual rays Pn, P′n, Pm, P′m, resolve the two triangles nPP′, mPP′, as in the application to the first example. Knowing, then, the sides mP, nP of the triangle mnP, and measuring the angle mPn, we have the case of the given elements of the triangle, being two sides and the included angle. Applying, therefore, to this case the preceding resolution, we may know what is the distance mn.



Other examples. The method of resolving the triangles is the same in all cases; and the two preceding examples, fully developed, give a sufficient direction to resolve others, which we subjoin here, with the values of the unknown elements to be found, for the exercise of the student.

EXAMPLE III. Given elements:

$$\text{Sides,}\ \begin{cases} A = 2301{,}82, \\ B = 5174{,}93, \\ C = 4842{,}28. \end{cases}$$

Elements to be found:

$$\text{Angles,}\ \begin{cases} a = 26°\ 17'\ 59'', \\ b = 84°\ 56'\ 40'', \\ c = 68°\ 45'\ 21''. \end{cases}$$

EXAMPLE IV. Given elements:

$$\text{Sides,}\ \begin{cases} A = 540, \\ B = 450, \end{cases}$$

and included angle $c = 80°$.

Elements to be found:

$$\text{Angles,}\ \begin{cases} a = 33°\ 34'\ 39'', \\ b = 18°\ 21'\ 21'', \end{cases}$$

Side $C = 2400$.

EXAMPLE V. Given elements:

$$\text{Sides,}\ \begin{cases} A = 390, \\ B = 651, \end{cases}$$

Angle $b = 55°\ 41'\ 57''$.

Elements to be found:

$$\text{Angles,}\begin{cases} a = 29^\circ\ 39'\ 46'', \\ c = 94^\circ\ 38'\ 17'', \end{cases}$$

$$\text{Side}\quad C = 700.$$

With the same elements of the preceding examples other examples may be formed. Thus, for instance, in the last, we may suppose the side A to be known, and two angles, or the three sides, or the two sides with the included angle, &c., and find the other elements; so that, without adding more examples, the preceding can be multiplied at pleasure. We will, however, add the case of the right-angled triangle.

Example of the right-angled triangle when the hypothenuse is given, and one of the acute angles.

We have remarked already that right-angled triangles can be more easily resolved by the equations (e'), (e''), (§ 16.) Thus, for example, let the given elements be the hypothenuse

$$h = 875,$$

and the angle $a = 57^\circ$:

we will immediately have tne other acute angle b;

because $a + b = 90^\circ$,

and, consequently, $b = 90^\circ - a = 90^\circ - 57^\circ = 33^\circ$.

With regard to the sides s, s', they are easily obtained from the equations (e'),

or $s = h \sin a,\ s' = h \cos a$;

for, applying the logarithms, we have

$$\text{l. } s = \text{l. } h + \text{l. } \sin a,\ \text{l. } s' = \text{l. } h + \text{l. } \cos a.$$

Now, $$\text{l. } h = \text{l. } 875 = 2{,}9420081,$$

$$\text{l. } \sin a = \text{l. } \sin 57° = 9{,}9235914,$$

$$\text{l. } \cos a = \text{l. } \cos 57° = 9{,}7361088;$$

hence, $$\text{l. } s = 2{,}8655995,$$

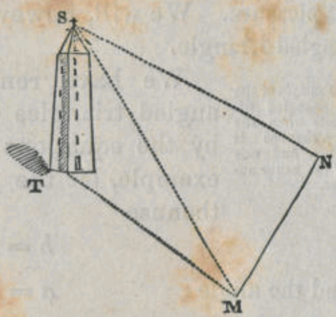
$$\text{l. } s' = 2{,}6781169;$$

and, finding the corresponding numbers,

$$s = 633{,}83,$$

$$s' = 476{,}56.$$

Application. If, for example, the height of a tower TS is to be determined: measure on the plane on which the tower is built a base or straight line MN; then, drawing the visual rays MS, NS to the top of the tower, we will have a triangle SMN resolvable, as we have seen in the preceding examples. Thus the side MS becomes known.



Imagine now a vertical or plumb-line ST from the top to the foot of the tower,—a vertical, namely, to the plane NMT. Drawing, then, a visual ray from M to the foot of the tower, and measuring the angle SMT, we will have in the right-angled triangle STM, besides the hypothenuse SM, the acute angle SMT also known. The triangle, therefore, can be resolved as above; and we may thus know the height of the tower.

Spherical Trigonometry.

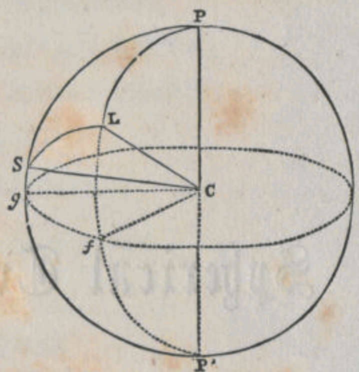
PRELIMINARIES.

Object of Spherical Trigonometry. § 23. THE object of Spherical Trigonometry is the resolution of spherical triangles; that is, to find the unknown elements of a spherical triangle when three of them are given.

Spherical triangles. A spherical triangle is any triangle traced on the surface of a sphere. But not all the triangles which can be described on a spherical surface are considered in trigonometry, but those only which are formed by arcs of great circles; that is, by the arcs of those circles the planes of which pass through the centre of the sphere.

Their elements. The elements of a spherical triangle are the same as those of a rectilinear triangle,—three sides and three angles.

Value of the elements; how estimated. Let SPL be any spherical triangle, formed, as it is understood, by arcs of great circles. Draw from the vertices the radii PC, LC, SC to the centre of the sphere, which is the common centre to the arcs or sides of the triangle: the three radii, with the



planes determined by them, form a solid angle in C. Now, the measure of the angles of the spherical triangle is the same as that of the angles formed by their respective planes. For instance, the measure of the angle P or SPL is the same as that of the planes PCL, PCS on which lie the arcs PL, PS. But the measure of the angle formed by two planes is given (GEOM., B. V.) by the angle formed by two perpendiculars drawn to the common intersection of the two planes,—the one lying on one plane, the second on the other. Drawing, therefore, from C, C*f*, C*g* both perpendicular to CP, and the first on the plane of the circle PL, the second on that of the circle PS, the angle *g*C*f* is the measure of the angle P. Now, the angle *g*C*f* is measured by the arc *gf* of the great circle whose plane is perpendicular to PC; and, since *f* is on the plane of PL, and *g* on the plane of PS, the arc *gf* is the arc contained between the sides of the same angle. Hence, the measure of the angle P is the arc of the great circle, the plane of which is perpendicular to the diameter passing through the vertex of the angle and determined by the sides of the same angle, produced if necessary; or, more briefly, since the extremities of the diameters perpendicular to great circles are called *poles* of the same circles, the measure of the angle P, and generally of any spherical angle, is the arc of the great circle (of which P is the pole) contained within the sides of the same angle. The angles are always taken less than 180°.

The measure or value of the sides is taken or estimated in the same manner as the measure of any other arc. We may still remark that, since the arc, for instance, PL, and the angle PCL, are mutually a measure of each other, so the measure of any side of the spherical triangle is the same as that of the angle formed by the radii drawn from its extremities to the centre of the sphere, and these sides or arcs are always taken less than 180°.

In any spherical triangle the sum of two sides is always greater than the third side, and the three sides together cannot amount to 360°.

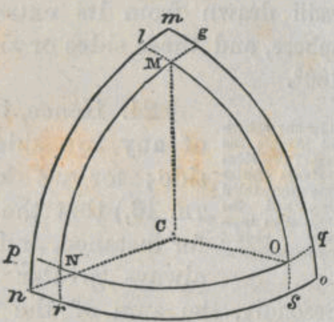
§ 24. Hence, it follows, first, that the sum of any two sides is greater than the third side; for we know, from geometry, (B. V. TH. 16,) that the sum of any two angles PCL, for instance, and LCS, of the solid angle C, is always greater than the third angle SCP. Secondly, the sum of the three sides of any spherical triangle can never reach 360°; for we know also, from geometry, (B. V. TH. 17,) that the sum of the plane angles forming a solid angle is always less than 360°.

Observe here, also, that the diameter perpendicular to the plane of a great circle is called the *axis* of the same circle. Hence, the poles of any circle or part of circle considered in spherical trigonometry are the extremities of its axis.

ARTICLE I.

FORMULAS AND EQUATIONS FOR THE RESOLUTION OF TRIANGLES.

When the vertices of one triangle are poles of the sides of another triangle, the vertices of the second triangle are reciprocally poles of the sides of the first.



§ 25. Let the vertices M, N, O of any spherical triangle be respectively poles of the sides *no*, *om*, *mn* of another triangle. C being the centre of the sphere, MC will be perpendicular to the plane of the circle *no*, and, consequently, to the radius C*n* on that plane. Also, OC must be perpendicular to the plane of the circle *mn*, and, consequently, to C*n*, which is on the plane of the same circle. Hence, C*n*, being at once perpendicular to CM and to CO, is perpendicular to the plane determined by them. But the plane of the radii CM, CO is the plane of the circle MO; hence, C*n* coincides with the axis of the same circle, and *n* is the pole of MO. In like manner we prove that *o* is the pole of MN, and *m* the pole of NO.

COROLLARY I. The sides of one triangle are supplements of the opposite angles of the other triangle, or *vice versâ*.

Hence, it follows that, producing the arcs or sides of the triangle MNO till they meet the sides of the other triangle in *r*, *s*, &c., we will have

$$ns = 90°,\ \ or = 90°,$$

$$Nq = 90°,\ \ Op = 90°,$$

and, consequently,

$$ns + or = 180°,$$

$$Nq + Op = 180°.$$

Now, $ns + or = nr + rs + os + sr = no + sr,$

and $Nq + Op = NO + Oq + pN + NO = pq + NO;$

hence, $no + rs = 180°$

$$NO + pq = 180°;$$

that is, *no* and *rs* are supplements of each other, and also NO and *pq*. Now, *rs* is the measure of the angle M; hence, the side *no* of the external triangle is supplement of the opposite angle of the internal triangle, and *vice versâ*. Again, *pq* is the measure of the angle *m*; hence, the side NO of the internal triangle is supplement of the opposite angle of the external triangle, and *vice versâ*. The same demonstration is applicable to the remaining sides and angles of the two triangles; hence, calling, for the sake of brevity, A, B, C the sides NO, MO, MN of the triangle MON, and *a*, *b*, *c* the respective opposite angles of the same triangle, and calling A′, B′, C′, *a*′, *b*′, *c*′ the corresponding sides and angles of the other triangles, we will have

$$A' + a = 180°,\ B' + b = 180°,\ C' + c = 180°,$$

$$A + a' = 180°,\ B + b' = 180°,\ C + c' = 180°;$$

or, $$\left.\begin{aligned} a = 180° - A',\ b = 180° - B',\ c = 180° - C', \\ A = 180° - a',\ B = 180° - b',\ C = 180° - c'. \end{aligned}\right\} \text{(I)}.$$

Corollary II. The three angles of any spherical triangle, taken together, amount to less than six, and more than two, right angles.

Now, the first three equations (I) give

$$a+b+c=3\cdot 180^\circ-(A'+B'+C')$$
$$=180^\circ+360^\circ-(A'+B'+C').$$

But we have seen above (§ 24) that $A'+B'+C'<360^\circ$; hence, $360^\circ-(A'+B'+C')$ give a positive result, and, consequently,

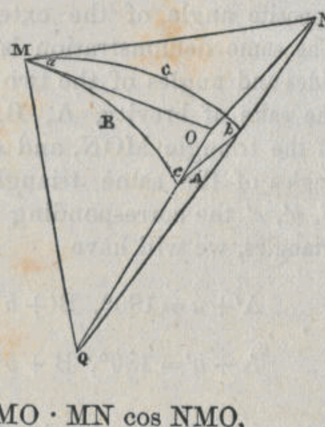
$$a+b+c>180^\circ.$$

That is, *The sum of the three angles of any spherical triangle is greater than two right angles.*

Again, since $a+b+c$ is less than $3\cdot 180^\circ$, as the preceding equation evidently shows, and $3\cdot 180^\circ$ amounts to six right angles, hence, *The sum of the angles of any spherical triangle cannot amount to six right angles.*

Useful formulas.

§ 26. Let MBAC be any spherical triangle; join the vertices with Q, the centre of the sphere; draw also the tangents MN, MO, the first to the arc or side C, the second to the arc B, and produce the radii Qc, Qb to O and N; the planes of the triangles MOQ, MNQ coincide with those of the great circles to which the sides B and C belong; join, lastly, O with N. From the triangle ONM we have (§ 20) (e_4)



$$\overline{ON}^2=\overline{OM}^2+\overline{MN}^2-2\,MO\cdot MN\cos NMO,$$

and from the triangle NOQ

$$\overline{ON}^2=\overline{OQ}^2+\overline{NQ}^2-2\,OQ\cdot NQ\cos OQN.$$

Now, supposing the value of the radius of the sphere to be r, we will have (§ 18)

$$OM = r \cdot \text{tg } B,\ \ MN = r \cdot \text{tg } C,$$

$$OQ = r \cdot \sec B,\ \ NQ = r \cdot \sec C.$$

Observe that the tangents MO, MN are both perpendicular to the radius MQ, which is the common intersection of the two planes on which are the arcs B and C; hence, the angle NMO is the measure of the angle a of the spherical triangle: the angle OQN is the measure of the arc or side A. Hence,

$$\cos NMO = \cos a,$$

$$\cos OQN = \cos A.$$

Making now the substitution of these and of the other values in the preceding equations, we will have

$$\overline{ON}^2 = r^2 \text{ tg}^2 B + r^2 \text{ tg}^2 C - 2 r^2 \text{ tg } B \cdot \text{tg } C \cos a,$$

$$\overline{ON}^2 = r^2 \sec^2 B + r^2 \sec^2 C - 2 r^2 \sec B \sec C \cos A;$$

from which

$$r^2 \text{ tg}^2 B + r^2 \text{ tg}^2 C - 2 r^2 \text{ tg } B \cdot \text{tg } C \cdot \cos a = r^2 \sec^2 B + r^2 \sec^2 C - 2 r^2 \sec B \sec C \cdot \cos A,$$

and, consequently,

$$2 r^2 \sec B \cdot \sec C \cos A = r^2 \sec^2 B + r^2 \sec^2 C - r^2 \text{ tg}^2 B - r^2 \text{ tg}^2 C + 2 r^2 \text{ tg } B \cdot \text{tg } C \cos a =$$

$$r^2 (\sec^2 B - \text{tg}^2 B) + r^2 (\sec^2 C - \text{tg}^2 C) + 2 r^2 \text{ tg } B \cdot \text{tg } C \cos a.$$

Now, when $r = 1$, as is the case for the simple functions of our arcs and angle, (§ 15,) from the second (i) (§ 13) we have

$$\sec^2 B = 1 + \text{tg}^2 B, \text{ and } \sec^2 C = 1 + \text{tg}^2 C;$$

therefore,

$$\sec^2 B - tg^2 B = 1 + tg^2 B - tg^2 B = 1,$$
$$\sec^2 C - tg^2 C = 1 + tg^2 C - tg^2 C = 1;$$

hence, the last equation can be simplified as follows:—

$$2r^2 (\sec B \sec C \cos A) = 2r^2 + 2r^2 \text{ tg } B \text{ tg } C \cos a;$$

and, dividing both members by the common factor $2r^2$,

$$\sec B \sec C \cos A = 1 + \text{tg } B \text{ tg } C \cos a.$$

But (§ 13) (i') $\sec B = \frac{1}{\cos B}$, $\sec C = \frac{1}{\cos C}$;

hence, $\sec B \sec C = \frac{1}{\cos B \cos C}$.

Substituting this value in the last equation, and multiplying then both members by cos B cos C, and observing that (§ 13) $\text{tg } B \text{ tg } C = \frac{\sin B}{\cos B} \cdot \frac{\sin C}{\cos C}$, we will finally obtain

$$\cos A = \cos B \cos C + \sin B \sin C \cos a. \quad (\text{II}).$$

Drawing from the vertices c and b tangents to the sides B, A, C of the triangle like MO, MN, with the same process we have

$$\left.\begin{aligned} \cos B &= \cos A \cos C + \sin A \sin C \cos b, \\ \cos C &= \cos A \cos B + \sin A \sin B \cos c, \end{aligned}\right\} (\text{II}).$$

Substituting in these formulas the values given by the equations (I), we will have (§ 12, 14) $-\cos a' = \cos b' \cos c' - \sin b' \sin c' \cos A'$, $-\cos b' = \cos a' \cos c' - \sin a' \sin c' \cos B'$, $-\cos c' = \cos a' \cos b' - \sin a' \sin b \cos C'$. Observe that the accents used in the formulas marked (I) are introduced to distinguish the angles and sides of one triangle from the angles and sides of the other; but, since each of the two triangles represents any spherical triangle, and here we make no comparison, we may use the angles and sides without accents as usually, and thus, from the equations last obtained, we have

$$\left.\begin{aligned}\cos a &= \sin b \sin c \cos A - \cos b \cos c,\\ \cos b &= \sin a \sin c \cos B - \cos a \cos c,\\ \cos c &= \sin a \sin b \cos C - \cos a \cos b.\end{aligned}\right\}\text{(III)}.$$

Other formulas. § 27. Observe that (§ 13) $\sin^2 a = 1 - \cos^2 a$, and, since a is any angle, $\sin^2 b = 1 - \cos^2 b$, and $\sin^2 c = 1 - \cos^2 c$. Now, from the formulas (II) we have

$$\cos^2 a = \frac{1}{\sin^2 B \sin^2 C}(\cos^2 A - 2\cos A \cos B \cos C + \cos^2 B \cos^2 C),$$

$$\cos^2 b = \frac{1}{\sin^2 A \sin^2 C}(\cos^2 B - 2\cos A \cos B \cos C + \cos^2 A \cos^2 C),$$

$$\cos^2 c = \frac{1}{\sin^2 A \sin^2 B}(\cos^2 C - 2\cos A \cos B \cos C + \cos^2 A \cos^2 B);$$

hence,

$$\sin^2 a = \frac{1}{\sin^2 B \sin^2 C}(\sin^2 B \sin^2 C - \cos^2 A + 2\cos A \cos B \cos C - \cos^2 B \cos^2 C),$$

$$\sin^2 b = \frac{1}{\sin^2 A \sin^2 C}(\sin^2 A \sin^2 C - \cos^2 B + 2\cos A \cos B \cos C - \cos^2 A \cos^2 C),$$

$$\sin^2 c = \frac{1}{\sin^2 A \sin^2 B}(\sin^2 A \sin^2 B - \cos^2 C + 2\cos A \cos B \cos C - \cos^2 A \cos^2 B).$$

But $\sin^2 B \sin^2 C = (1 - \cos^2 B)(1 - \cos^2 C) = 1 - \cos^2 B - \cos^2 C + \cos^2 B \cos^2 C$, and $\sin^2 A \sin^2 C = \cdot \cdot \cdot = 1 - \cos^2 A - \cos^2 C + \cos^2 A \cos^2 C$,

$\sin^2 A \sin^2 B = \ldots = 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B$;

hence, substituting these values in the numerators of the last equations, we have

$$\sin^2 a = \frac{1}{\sin^2 B \sin^2 C}(1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C),$$

$$\sin^2 b = \frac{1}{\sin^2 A \sin^2 C}(1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C),$$

$$\sin^2 c = \frac{1}{\sin^2 A \sin^2 B}(1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C).$$

Now, the last factor is the same in each equation. Calling, for the sake of brevity, F this factor, we will have

$$\sin^2 a = \frac{F}{\sin^2 B \sin^2 C} = \frac{F \sin^2 A}{\sin^2 A \sin^2 B \sin^2 C},$$

$$\sin^2 b = \frac{F}{\sin^2 A \sin^2 C} = \frac{F \sin^2 B}{\sin^2 A \sin^2 B \sin^2 C},$$

$$\sin^2 c = \frac{F}{\sin^2 A \sin^2 B} = \frac{F \sin^2 C}{\sin^2 A \sin^2 B \sin^2 C};$$

hence, $$\frac{\sin^2 a}{\sin^2 A} = \frac{\sin^2 b}{\sin^2 B} = \frac{\sin^2 c}{\sin^2 C},$$

or, $$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c},$$

and, consequently,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad \text{(IV)}.$$

In the spherical triangle the sines of the sides are as those of the opposite angles.

Now, from this equation are easily inferred the proportions

$$\sin A : \sin B :: \sin a : \sin b,$$

$$\sin A : \sin C :: \sin a : \sin c,$$

$$\sin B : \sin C :: \sin b : \sin c;$$

that is, *In any spherical triangle the sines of the sides are to one another as the sines of their opposite angles.*

The first equation marked (II) gives

$$\cos a = \frac{\cos A - \cos B \cos C}{\sin B \sin C};$$

and from the equation (IV) we have

$$\sin a = \frac{\sin A \sin b}{\sin B};$$

hence,
$$\frac{\cos a}{\sin a} = \frac{\cos A - \cos B \cos C}{\sin A \sin C \sin b};$$

that is,
$$\cot a = \frac{1}{\sin A \sin C \sin b}(\cos A - \cos B \cos C),$$

and, substituting instead of cos B its value given by the second (II),

$$\cot a = \frac{1}{\sin A \sin C \sin b}(\cos A - \cos A \cos^2 C - \sin A \sin C \cos C \cos b),$$

$$= \frac{1}{\sin A \sin C \sin b}(\cos A \sin^2 C - \sin A \sin C \cos C \cos b),$$

$$= \frac{1}{\sin b}(\cot A \sin C - \cos C \cos b);$$

hence,

$$\cot a \sin b = \cot A \sin C - \cos C \cos b. \quad (\text{v}).$$

We would have obtained, in like manner, from the second (II),

$$\cot b \sin a = \cot B \sin C - \cos C \cos a. \quad (\text{v}).$$

From (g), (g'), (§ 18,) we have

$$2 \sin^2 \tfrac{1}{2}a = 1 - \cos a,$$

$$2 \cos^2 \tfrac{1}{2}a = 1 + \cos a.$$

Now, from the first (II), as we have already seen,

$$\cos a = \frac{\cos A - \cos B \cos C}{\sin B \sin C};$$

hence, $2 \sin^2 \tfrac{1}{2}a = 1 - \dfrac{\cos A - \cos B \cos C}{\sin B \sin C}$

$$= \frac{\sin B \sin C + \cos B \cos C - \cos A}{\sin B \sin C},$$

$$2 \cos^2 \tfrac{1}{2}a = 1 + \frac{\cos A - \cos B \cos C}{\sin B \sin C}$$

$$= \frac{\sin B \sin C - \cos B \cos C + \cos A}{\sin B \sin C}.$$

But from (h'') (§ 18)

$$\sin B \sin C + \cos B \cos C = \cos (B - C),$$

$$\sin B \sin C - \cos B \cos C = -\cos (B + C);$$

hence, $2 \sin^2 \tfrac{1}{2}a = \dfrac{\cos (B - C) - \cos A}{\sin B \sin C},$

$$2 \cos^2 \tfrac{1}{2}a = \frac{\cos A - \cos (B + C)}{\sin B \sin C}.$$

Now, from (f') (§ 17) we have

$$\cos(B-C)-\cos A = 2\sin\tfrac{1}{2}(A+B-C)\sin\tfrac{1}{2}(A+C-B),$$
$$\cos A-\cos(B+C) = 2\sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(B+C-A);$$

hence,

$$\left.\begin{aligned}\sin^2\tfrac{1}{2}a &= \frac{\sin\tfrac{1}{2}(A+B-C)\sin\tfrac{1}{2}(A+C-B)}{\sin B\sin C},\\ \cos^2\tfrac{1}{2}a &= \frac{\sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(B+C-A)}{\sin B\sin C},\end{aligned}\right\}(\text{VI}).$$

From the formulas marked (I) we have

$$\sin^2\tfrac{1}{2}a = \sin^2(90°-\tfrac{1}{2}A') = \cos^2\tfrac{1}{2}A',$$
$$\cos^2\tfrac{1}{2}a = \cos^2(90°-\tfrac{1}{2}A') = \sin^2\tfrac{1}{2}A',$$
$$\sin\tfrac{1}{2}(A+B-C) = \sin\left(90°-\tfrac{1}{2}(a'+b'-c')\right) = \cos\tfrac{1}{2}(a'+b'-c'),$$
$$\sin\tfrac{1}{2}(A+C-B) = \sin\left(90°-\tfrac{1}{2}(a'+c'-b')\right) = \cos\tfrac{1}{2}(a'+c'-b'),$$
$$\sin\tfrac{1}{2}(A+B+C) = \sin\left(270°-\tfrac{1}{2}(a'+b'+c')\right) = -\cos\tfrac{1}{2}(a'+b'+c'),$$
$$\sin(270°-d) = \sin\left(180°-(d-90°)\right) = \sin(d-90°) = -\sin(90°-d) = -\cos d,$$
$$\sin\tfrac{1}{2}(B+C-A) = \sin\left(90°-\tfrac{1}{2}(b'+c'-a')\right) = \cos\tfrac{1}{2}(b'+c'-a'),$$
$$\sin B = \sin(180°-b') = \sin b' \quad \sin C = \sin(180°-c') = \sin c'.$$

Making, now, the substitution in the preceding formulas (VI), commencing with the second and writing the arcs and angles without accents, as in a similar case of the preceding number, we will have

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}\mathrm{A} &= -\frac{\cos \frac{1}{2}(a+b+c) \cos \frac{1}{2}(b+c-a)}{\sin b \sin c}, \\ \cos^2 \tfrac{1}{2}\mathrm{A} &= \frac{\cos \frac{1}{2}(a+b-c) \cos \frac{1}{2}(a+c-b)}{\sin b \sin c}, \end{aligned}\right\} \text{(VII)}.$$

We have seen already (§ 24) that the sum of the three sides of any spherical triangle is always less than 360°;

hence, $\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C}) < 180°$.

Again, (§ 25,) $a+b+c < 540°$ and $> 180°$;

hence, $\frac{1}{2}(a+b+c) < 270°$ and $> 90°$.

Also, since (I

$$\mathrm{C}' = 180° - c,\ \mathrm{B}' = 180° - b,\ \mathrm{A}' = 180° - a,$$

and (§ 24) $\mathrm{C}' < \mathrm{A}' + \mathrm{B}'$,

we will have $180° - c < 360° - (a+b)$,

and, consequently, $a+b-c < 180°$,

and $\frac{1}{2}(a+b-c) < 90°$

We infer from these remarks that the second members of the equations (VI) and (VII) must be positive in all cases.

Formulas of Gauss and Napier's Analogies.

§ 28. Changing in the formulas (VI) a into b and b into a, and, consequently also, A into B, and *vice versâ*, we will have

$$\sin^2 \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\mathrm{B}+\mathrm{A}-\mathrm{C}) \sin \frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}{\sin \mathrm{A} \sin \mathrm{C}},$$

$$\cos^2 \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\mathrm{B}+\mathrm{A}+\mathrm{C}) \sin \frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})}{\sin \mathrm{A} \sin \mathrm{C}}.$$

and, from these and the same equations (VI),

$$\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}b=\sqrt{\frac{\sin\frac{1}{2}(A+B-C)\ \sin\frac{1}{2}(A+C-B)}{\sin B\ \sin C}}\times$$

$$\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(A+C-B)}{\sin A\ \sin C}}=$$

$$\frac{\sin\frac{1}{2}(A+C-B)}{\sin C}\sqrt{\frac{\sin\frac{1}{2}(A+B-C)\sin\frac{1}{2}(A+B+C)}{\sin A\ \sin B}},$$

$$\cos\tfrac{1}{2}a\ \sin\tfrac{1}{2}b\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(B+C-A)}{\sin B\ \sin C}}\times$$

$$\sqrt{\frac{\sin\frac{1}{2}(B+A-C)\ \sin\frac{1}{2}(B+C-A)}{\sin A\ \sin C}}=$$

$$\frac{\sin\frac{1}{2}(B+C-A)}{\sin C}\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(B+A-C)}{\sin A\ \sin B}}$$

Hence, also,

$$\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}b-\cos\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\frac{1}{\sin C}\times$$

$$[\sin\tfrac{1}{2}(A+C-B)-\sin\tfrac{1}{2}(B+C-A)]$$

$$\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(A+B-C)}{\sin A\ \sin B}}.$$

Now, from the fourth (h''), (§ 18,) we have

$$\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}b-\cos\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\sin\tfrac{1}{2}(a-b),$$

and from (f'''), (§ 17,)

$$\sin\tfrac{1}{2}(A+C-B)-\sin\tfrac{1}{2}(B+C-A)=2\cos C\sin\tfrac{1}{2}(A-B),$$

and, consequently,

$$\frac{1}{\sin C}[\sin \tfrac{1}{2}(A+C-B)-\sin \tfrac{1}{2}(B+C-A)]=$$
$$\frac{2\cos \tfrac{1}{2}C}{\sin C}\sin \tfrac{1}{2}(A-B).$$

Now, (g''), (§ 18,)

$$2\cos \tfrac{1}{2}C=\frac{\sin C}{\sin \tfrac{1}{2}C};$$

hence, $$\frac{1}{\sin C}[\sin \tfrac{1}{2}(A+C-B)-\sin \tfrac{1}{2}(B+C-A)]=$$
$$\frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(C)}.$$

Finally, from the second (VI),

$$\frac{\sin \tfrac{1}{2}(A+B+C)\ \sin \tfrac{1}{2}(A+B-C)}{\sin A\ \sin B}=\cos^2 \tfrac{1}{2}c;$$

hence, $$\sqrt{\frac{\sin \tfrac{1}{2}(A+B+C)\ \sin \tfrac{1}{2}(A+B-C)}{\sin A\ \sin B}}=\cos \tfrac{1}{2}c.$$

Substituting, now, all these values in the preceding equation, we will have

$$\sin \tfrac{1}{2}(a-b)=\frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}C}\cos \tfrac{1}{2}c. \quad (k).$$

In like manner, from the formulas (VI) and the values of $\sin^2 \tfrac{1}{2}b$, $\cos^2 \tfrac{1}{2}b$ inferred from them, we have

$$\cos \tfrac{1}{2}a\ \cos \tfrac{1}{2}b+\sin \tfrac{1}{2}a\ \sin \tfrac{1}{2}b=\frac{1}{\sin C}\times$$
$$[\sin \tfrac{1}{2}(A+B+C)+\sin \tfrac{1}{2}(A+B-C)]$$
$$\sqrt{\frac{\sin \tfrac{1}{2}(B+C-A)\ \sin \tfrac{1}{2}(A+C-B)}{\sin A\ \sin B}};$$

$$\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}b+\cos\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\frac{1}{\sin C}\times$$

$$[\sin\tfrac{1}{2}(A+C-B)+\sin\tfrac{1}{2}(B+C-A)]$$

$$\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(A+B-C)}{\sin A\ \sin B}},$$

$$\cos\tfrac{1}{2}a\ \cos\tfrac{1}{2}b-\sin\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\frac{1}{\sin C}\times$$

$$[\sin\tfrac{1}{2}(A+B+C)-\sin\tfrac{1}{2}(A+B-C)]$$

$$\sqrt{\frac{\sin\frac{1}{2}(B+C-A)\ \sin\frac{1}{2}(A+C-B)}{\sin A\ \sin B}};$$

Now, from (h''), (§ 18,)

$$\cos\tfrac{1}{2}a\ \cos\tfrac{1}{2}b+\sin\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\cos\tfrac{1}{2}(a-b),$$

$$\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}b+\cos\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\sin\tfrac{1}{2}\,(a+b),$$

$$\cos\tfrac{1}{2}a\ \cos\tfrac{1}{2}b-\sin\tfrac{1}{2}a\ \sin\tfrac{1}{2}b=\cos\tfrac{1}{2}(a+b),$$

and, from (f''), (f'''), (§ 17,)

$$\sin\tfrac{1}{2}(A+B+C)+\sin\tfrac{1}{2}(A+B-C)=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C,$$

$$\sin\tfrac{1}{2}(A+C-B)+\sin\tfrac{1}{2}(B+C-A)=2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B),$$

$$\sin\tfrac{1}{2}(A+B+C)-\sin\tfrac{1}{2}(A+B-C)=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C.$$

Again, besides the value of

$$\sqrt{\frac{\sin\frac{1}{2}(A+B+C)\ \sin\frac{1}{2}(A+B-C)}{\sin A\ \sin B}},$$

already found from the first (VI), we have

$$\sqrt{\frac{\sin\frac{1}{2}(B+C-A)\ \sin\frac{1}{2}(A+C-B)}{\sin A\ \sin B}}=\sin\tfrac{1}{2}c.$$

Observe also, that, from the same (g''), (§ 18,) besides

$$2\cos\tfrac{1}{2}C = \frac{\sin C}{\sin\frac{1}{2}C},$$

we have also
$$2\sin\tfrac{1}{2}C = \frac{\sin C}{\cos\frac{1}{2}C};$$

therefore, making the substitutions, we will have

$$\cos\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}\sin\tfrac{1}{2}c,$$

$$\sin\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}\cos\tfrac{1}{2}c,$$

$$\cos\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}\sin\tfrac{1}{2}c.$$

Now, from the preceding analogous formula (k), and from the last, we easily obtain

$$\left.\begin{aligned}\sin\tfrac{1}{2}(a-b)\ \sin\tfrac{1}{2}C &= \sin\tfrac{1}{2}(A-B)\ \cos\tfrac{1}{2}c,\\ \cos\tfrac{1}{2}(a-b)\ \sin\tfrac{1}{2}C &= \sin\tfrac{1}{2}(A+B)\ \sin\tfrac{1}{2}c,\\ \sin\tfrac{1}{2}(a+b)\ \cos\tfrac{1}{2}C &= \cos\tfrac{1}{2}(A-B)\ \cos\tfrac{1}{2}c,\\ \cos\tfrac{1}{2}(a+b)\ \cos\tfrac{1}{2}C &= \cos\tfrac{1}{2}(A+B)\ \sin\tfrac{1}{2}c.\end{aligned}\right\}\ (\text{VIII}).$$

These equations are called the formulas of *Gauss*,—the name of their illustrious inventor. We may infer from them, immediately, other formulas, first detected by *Napier*, and commonly known under the name of *Napier's analogies*. Dividing, in fact, the first by the second, and the third by the fourth, and then the first by the third, and the second by the fourth, we have immediately (§ 13) (i'),

$$\left.\begin{aligned}
\operatorname{tg}\tfrac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\cot\tfrac{1}{2}c,\\
\operatorname{tg}\tfrac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\cot\tfrac{1}{2}c,\\
\operatorname{tg}\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\operatorname{tg}\tfrac{1}{2}C,\\
\operatorname{tg}\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\operatorname{tg}\tfrac{1}{2}C,
\end{aligned}\right\}\text{(IX)}.$$

Observe, now, that from the equations marked (I), we have

$$a+b=360^\circ-(A'+B');$$

hence, $$\tfrac{1}{2}(a+b)=180^\circ-\tfrac{1}{2}(A'+B')$$

and $$\sin\tfrac{1}{2}(a+b)=\sin\tfrac{1}{2}(A'+B').$$

Now, since $\frac{1}{2}(A'+B'+C')$ cannot amount to 180°, sin $\frac{1}{2}(A'+B')$ is certainly positive, and consequently also the denominator of the third (IX) is always positive. But tg $\frac{1}{2}$C, also, is always positive; for, since C (§ 23) cannot reach 180°, $\frac{1}{2}$C is always less than 90°; hence, $\dfrac{\operatorname{tg}\frac{1}{2}C}{\sin\frac{1}{2}(a+b)}$ is essentially positive, and, for this reason, the equivalent ratio inferred from the third (IX), that is, $\dfrac{\operatorname{tg}\frac{1}{2}(A-B)}{\sin\frac{1}{2}(a-b)}$, is always positive, which necessarily supposes tg $(\frac{1}{2}A-B)$ and sin $\frac{1}{2}(a-b)$ to have the same sign; and, since $\frac{1}{2}(A-B)$, $\frac{1}{2}(a-b)$, either positive or negative, are both less than 90°, tg $\frac{1}{2}(A-B)$ cannot have the same sign as sin $\frac{1}{2}(a-b)$, unless with $A>B$, also $a>b$; and with $A<B$ also $a<b$. Hence, in any spherical triangle, *The greater side is opposite to the greater angle, and the less side to the less angle.*

Formulas containing an auxiliary angle.

The resolution of spherical triangles is, in some cases, rendered easier by using some angles, called *auxiliary angles*, introduced in some of the preceding formulas as follows:—

From the first formula marked (II) we have

$$\cos A = \cos B \cos C + \sin B \cos C \operatorname{tg} C \cos a;$$

for $$\cos C \operatorname{tg} C = \sin C.$$

Take now an angle φ, such that we may have

$$\operatorname{tg} C \cos a = \operatorname{tg} \varphi:$$

the preceding equation is then easily changed into

$$\cos A = \cos B \cos C + \sin B \cos C \frac{\sin \varphi}{\cos \varphi}$$

$$= \frac{\cos C}{\cos \varphi} [\cos B \cos \varphi + \sin B \sin \varphi].$$

Now (§ 18) (h''), $\cos B \cos \varphi + \sin B \sin \varphi = \cos (B - \varphi)$;

$$\left.\begin{aligned} &\text{hence,} \quad \cos A = \frac{\cos C}{\cos \varphi} \cos (B - \varphi), \\ &\text{and} \quad \cos (B - \varphi) = \frac{\cos \varphi}{\cos C} \cos A, \end{aligned}\right\} \text{(x)}.$$

From the first formula marked (III) we infer

$$\cos a = \sin b \cdot \cos c \operatorname{tg} c \cos A - \cos b \cos c$$

Make in it $\qquad \operatorname{tg} c \cos A = \cot \varphi;$

we will have

$$\cos a = \sin b \cos c \frac{\cos \varphi}{\sin \varphi} - \cos b \cos c$$

$$= \frac{\cos c}{\sin \varphi}(\sin b \cos \varphi - \cos b \sin \varphi);$$

hence, (§ 18) (h''),

$$\left.\begin{aligned} \cos a &= \frac{\cos c}{\sin \varphi} \sin (b - \varphi), \\ \sin (b - \varphi) &= \frac{\sin \varphi}{\cos c} \cos a, \end{aligned}\right\} \text{(XI)}.$$

We have from the first formula marked (V)

$$\cot a = \frac{\cot A \sin C}{\sin b} - \cos C \cot b$$

$$= \cot b \left[\frac{\cot A \sin C}{\cos b} - \cos C\right].$$

Make in it $\qquad \dfrac{\cot A}{\cos b} = \cot \varphi;$

we will have

$$\cot a = \cot b \left[\sin C \frac{\cos \varphi}{\sin \varphi} - \cos C\right]$$

$$= \frac{\cot b}{\sin \varphi}\left[\sin C \cos \varphi - \cos C \sin \varphi\right];$$

hence, (§ 18) (h''),

$$\left.\begin{aligned} \cot a &= \frac{\cot b}{\sin\varphi}\sin(C-\varphi), \\ \text{and}\qquad \sin(C-\varphi) &= \frac{\sin\varphi}{\cot b}\cot a, \end{aligned}\right\}\text{(XII)}.$$

We obtain, also, from the same formula (V),

$$\cot A = \frac{\cot a \sin b}{\sin C} + \cot C \cos b$$

$$= \cot C\left[\frac{\cot a \sin b}{\cos C} + \cos b\right];$$

and, making $$\frac{\cot a}{\cos C} = \operatorname{tg}\varphi,$$

$$\cot A = \cot C\left[\frac{\sin b \sin\varphi}{\cos\varphi} + \cos b\right]$$

$$= \frac{\cot C}{\cos\varphi}\left[\sin b \sin\varphi + \cos b\cos\varphi\right],$$

and, consequently,

$$\left.\begin{aligned} \cot A &= \frac{\cot C}{\cos\varphi}\cos(b-\varphi), \\ \text{and}\qquad \cos(b-\varphi) &= \frac{\cos\varphi}{\cot C}\cot A, \end{aligned}\right\}\text{(XIII)}.$$

Formulas for the resolution of right-angled spherical triangles.

§ 30. The preceding formulas are apt to resolve all sorts of spherical triangles. Some of them, however, may be considerably simplified for right-angled triangles. Let, in fact, the angle a be equal to 90°: the first formula marked (II) becomes

$$\cos A = \cos B \cos C, \Big\} \text{(XIV)}.$$

Hence, $\quad 1 : \cos B :: \cos C : \cos A;$

that is, in the right-angled spherical triangle *The radius* 1 *is to the cosine of one of the sides about the right angle as the cosine of the other side is to the cosine of the hypothenuse.*

With the same $a = 90°$, the first formula marked (III) becomes

$$\cos b \cos c = \sin b \sin c \cos A;$$

from which $\quad \cot b = \operatorname{tg} c \cdot \cos A, \Big\} \text{(XV)};$

that is, $\quad 1 : \operatorname{tg} c :: \cos A : \cot b.$

Hence, *The radius is to the tangent of one of the angles as the cosine of the hypothenuse is to the cotangent of the other angle.*

In the same supposition we have, from the second (III),

$$\cos b = \sin c \cos B, \Big\} \text{(XVI)};$$

from the equation (IV),

$$\sin B = \sin b \sin A, \Big\} \text{(XVII)};$$

from the first (V),

$$\cot A \sin C = \cos C \cos b;$$

or the equivalent,

$$\operatorname{tg} C = \operatorname{tg} A \cos b, \Big\} \text{(XVIII)};$$

and from the second (v),

$$\cot b = \cot B \sin C,$$

or the equivalent,

$$\left.\operatorname{tg} B = \operatorname{tg} b \sin C,\right\}\text{(xix)}.$$

Hence,

$$1 : \sin c :: \cos B : \cos b,$$
$$1 : \sin b :: \sin A : \sin B,$$
$$1 : \operatorname{tg} A :: \cos b : \operatorname{tg} C,$$
$$1 : \operatorname{tg} b :: \sin C : \operatorname{tg} B.$$

Observe that from the last equation (xix) we infer

$$\sin C = \frac{\operatorname{tg} B}{\operatorname{tg} b};$$

and, since C, as well as any side of the spherical triangle, is taken less than 180°, sin C, and, consequently, the ratio $\frac{\operatorname{tg} B}{\operatorname{tg} b}$, is always positive, which necessarily supposes the tangent of any one of the sides about the right angle to be affected with the same sign as the tangent of its opposite angle. In other words, when the side B is $<90°$ the angle b also is $<90°$; and when B is $>90°$ b also is $>90°$.

The angles and arcs which, like the preceding, terminate in the same quadrant,—either first or second,—are said to be of the same kind; else, of a different kind. Hence, since cos A is positive when in (xiv) we suppose both B and C of the same kind; and cos A is negative when B and C are of different kind; and cos A is positive when A is between 0° and 90°, negative when A is between 90° and 180°; it follows that the hypothenuse of a spherical right-angled triangle is greater than a quadrant only when the sides about the right angle are of a different kind.

But from the equation (XV) we have

$$\cos A = \frac{\cot b}{\operatorname{tg} c}.$$

And here, also, cos A will be either positive or negative according as b and c are of the same or of a different kind; hence, the hypothenuse A cannot be greater than 90°, unless the angles b and c adjacent to it are of a different kind.

Finally, from the equation (XVIII) we have

$$\operatorname{tg} A = \frac{\operatorname{tg} C}{\cos b};$$

from which it follows that tg A will be positive when C and b are of the same kind, and negative when C and b are of a different kind; which is the same as to say, the hypothenuse is greater than 90° only when one of the adjacent angles and the corresponding side are of a different kind.

ARTICLE II.

RESOLUTION OF THE SPHERICAL TRIANGLES.

Remarks. § 31. The formulas of the preceding article resolve the spherico-trigonometrical problem in all cases, whenever the resolution is possible; and the practical utility of this resolution belongs chiefly to astronomy. We need not to say that the material work of the resolution, when three of the elements are given to find out the other three, does not differ from that of plane triangles: it consists, namely, in applying the logarithms to the formulas adapted for the various cases, and finding then the corresponding arcs or angles.

The cases are six in number for common triangles, and six also for right-angled triangles. We will here point out the different cases, and what formulas are to be used in every one of them to resolve the problem.

Different cases for common spherical triangles. § 32. For common spherical triangles the given elements may be

(1.) The three sides.
(2.) The three angles.
(3.) Two sides and included angle.
(4.) One side and its adjacent angles.
(5.) Two sides and the angle opposite to one of them.
(6.) Two angles and the side opposite to one of them.

The first case (1) is resolved by the equations marked (VI); for, although the angle a only is expressed in the first member, it may be changed into b and c by a simple change of the disposition of the sides in the second members.

The second case (2) is resolved by the equations (VII), in like manner as the first.

The third (3) is resolved either by the formulas (IX) and (X), or (XII) and (X); for, from the first and second, (IX), we have the sum and difference of the two unknown angles, and consequently the angles themselves; from (X) we obtain the unknown side when the two remaining sides and the angles are given. The two angles may be obtained also from (XII).

The fourth case (4) is resolved by the last two formulas marked (IX), and by (XI), in like manner as the preceding; the two sides, however, may be obtained also from (XIII).

In the fifth case (5), we find the angle included by the given sides by means of the second equation (XIII), the third side by the second (X), and the third angle by the equation (IV): the last element, however, being given by means of the sine, is ambiguous.

In the last case (6), we have a resolution analogous to that of the preceding; by the second (XII), which gives the included side; by the second (XI), which gives the third angle; and by the same equation (IV), which ambiguously gives the last side.

Different cases for right-angled spherical triangles.

§ 33. The given elements for right-angled spherical triangles may be as follows:—

(1.) The hypothenuse and another side.
(2.) The hypothenuse and one of its adjacent angles.
(3.) The sides about the right angle.
(4.) The angles adjacent to the hypothenuse.
(5.) One of the sides about the right angle, and the angle opposite to the other.

(6.) One of the sides about the right angle, and its opposite angle.

In the first of these cases (1), we find the third side by means of the equation (XIV), the two remaining angles by (XVII) and (XVIII).

In the second case (2), we find the third angle by (XV), the two remaining sides by (XVII) and (XVIII).

In the third (3), we find the hypothenuse by (XIV), the angles by (XIX).

In the fourth (4), we find the hypothenuse by (XV), the sides by (XVI).

In the fifth (5), we find the hypothenuse by (XVIII), the remaining side by (XIX), and the remaining angle by (XVI).

In the last case (6), we find ambiguously the hypothenuse by means of (XVII), the third side by (XIX), and the third angle by (XVI).

Examples. § 34. We subjoin here the elements of some triangles, angles, and sides, so that, taking three of the elements as given, and the other three to be found in the different manners above mentioned, each triangle will afford six examples. The angles will be expressed by the small letters a, b, c, and their respectively opposite sides by the capital letters A, B, C.

$$\text{1st Triangle.}\begin{cases}\text{Angles.}\begin{cases}a = 62^\circ\ 39'\ 42'',\\ b = 124^\circ\ 50'\ 50'',\\ c = 50^\circ\ 31'\ 42''.\end{cases}\\[2ex] \text{Sides.}\begin{cases}A = 81^\circ\ 17',\\ B = 114^\circ\ \ 3',\\ C = 59^\circ\ 12'.\end{cases}\end{cases}$$

2d Triangle.

- Angles.
  - $a = 44° 18'$,
  - $b = 136° 40'$,
  - $c = 48° 48'$.
- Sides.
  - $A = 62° 42'$,
  - $B = 119° 5'$,
  - $C = 73° 13'$.

3d Triangle.

- Angles.
  - $a = 71° 42'$,
  - $b = 125° 37'$,
  - $c = 49° 32'$.
- Sides.
  - $A = 95° 56' 10''$,
  - $B = 121° 36' 31''$,
  - $C = 52° 50' 44''$.

4th Triangle.

- Angles.
  - $a = 121° 21'$,
  - $b = 34° 33'$,
  - $c = 42° 57'$.
- Sides.
  - $A = 77° 39' 31''$,
  - $B = 40° 26' 45''$,
  - $C = 51° 12' 21''$.

A TABLE

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

Remark I.—The points or dots •• are introduced instead of 0's, to indicate at a first glance that from thence the two figures of the second column stand in the line below. For example, let 1014, 1024 be two numbers the logarithms of which are to be found. The first part of the number until the last figure 4 is to be looked for in the column marked N, and the last figure 4 in the first line at the top or in the last at the bottom of the page. Now, the logarithm, or rather the decimal part of the logarithm, of the number 1014 is given by the figures in which the line opposite to 101 and the column below or above 4 intersect each other; but not the whole of it, for the first two ciphers are to be taken from the column below or above 0, and are the first two ciphers of the same column. Now, the first two ciphers of the column 0 in our examples are 00 and 01,—that is, 00 for the number 1014, and 01 for the number 1024, and the 01 instead of 00 is indicated by the dot •.

Remark II.—In the trigonometrical tables the first and the last column are the columns of minutes; the first belonging to the degrees at the top and the last to the degrees below. The voice of the teacher, and practice, will facilitate the use of the tables.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806181	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	•207	•602	•998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	•380	•766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	•320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	••38	•407	•776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	•266	•626	•987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	•258	•611	•963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	••26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	•253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	•245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	••12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	•194	•508	•822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	•142	•449	•756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	•168	•469	•769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	280
151	8977	9264	9552	9839	•126	•413	•699	•985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	••51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	••29	•303	•577	•850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
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N.	0	1	2	3	4	5	6	7	8	9	D
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	••50	•300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	•176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7580	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	•123	•351	•578	•806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	•161	•378	•595	•813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	••56	•268	•481	•693	•906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	•••8	•211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	••47	•246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	••54	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	••25	•215	•404	•593	•783	•972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	•143	•328	•518	•698	•883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	••30	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	••51	•228	•405	•582	•759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	••20	•192	•365	•538	•711	•883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	•102	•271	•440	•609	•777	•946	1114	1283	1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	•121	•286	•451	•616	•781	•945	1110	1275	1439	165
264	421604	1788	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
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594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
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596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
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706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
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937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	••28	••72	•117	•161	•206	•250	•294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

A TABLE

OF

LOGARITHMIC SINES AND TANGENTS

FOR EVERY

DEGREE AND MINUTE

OF

THE QUADRANT.

M.	Sine	D.	Cosine.	D.	Tang.	D.	Cotang.	
0	0·000000		10·000000		0·000000		Infinite.	60
1	6·463726	5017·17	000000	·00	6·463726	5017·17	13·536274	59
2	764756	2934·85	000000	·00	764756	2934·83	235244	58
3	940847	2082·31	000000	·00	940847	2082·31	059153	57
4	7·065786	1615·17	000000	·00	7·065786	1615·17	12·934214	56
5	162696	1319·68	000000	·00	162696	1319·69	837304	55
6	241877	1115·75	9·999999	·01	241878	1115·78	758122	54
7	308824	966·53	999999	·01	308825	996·53	691175	53
8	366816	852·54	999999	·01	366817	852·54	633183	52
9	417968	762·63	999999	·01	417970	762·63	582030	51
10	463725	689·88	999998	·01	463727	689·88	536273	50
11	7·505118	629·81	9·999998	·01	7·505120	629·81	12·494880	49
12	542906	579·36	999997	·01	542909	579·33	457091	48
13	577668	536·41	999997	·01	577672	536·42	422328	47
14	609853	499·38	999996	·01	609857	499·39	390143	46
15	639816	467·14	999996	·01	639820	467·15	360180	45
16	667845	438·81	999995	·01	667849	438·82	332151	44
17	694173	413·72	999995	·01	694179	413·73	305821	43
18	718997	391·35	999994	·01	719004	391·36	280997	42
19	742477	371·27	999993	·01	742484	371·28	257516	41
20	764754	353·15	999993	·01	764761	351·36	235239	40
21	7·785943	336·72	9·999992	·01	7·785951	336·73	12·214049	39
22	806146	321·75	999991	·01	806155	321·76	193845	38
23	825451	308·05	999990	·01	825460	308·06	174540	37
24	843934	295·47	999989	·02	843944	295·49	156056	36
25	861662	283·88	999988	·02	861674	283·90	138326	35
26	878695	273·17	999988	·02	878708	273·18	121292	34
27	895085	263·23	999987	·02	895099	263·25	104901	33
28	910879	253·99	999986	·02	910894	254·01	089106	32
29	926119	245·38	999985	·02	926134	245·40	073866	31
30	940842	237·33	999983	·02	940858	237·35	059142	30
31	7·955082	229·80	9·999982	·02	7·955100	229·81	12·044900	29
32	968870	222·73	999981	·02	968880	222·75	031111	28
33	982233	216·08	999980	·02	982253	216·10	017747	27
34	995198	209·81	999979	·02	995219	209·83	004781	26
35	8·007787	203·90	999977	·02	8·007809	203·92	11·992191	25
36	020021	198·31	999976	·02	020045	198·33	979955	24
37	031919	193·02	999975	·02	031945	193·05	968055	23
38	043501	188·01	999973	·02	043527	188·03	956473	22
39	054781	183·25	999972	·02	054809	183·27	945191	21
40	065776	178·72	999971	·02	065806	178·74	934194	20
41	8·076500	174·41	9·999969	·02	8·076531	174·44	11·923469	19
42	086965	170·31	999968	·02	086997	170·34	913003	18
43	097183	166·39	999966	·02	097217	166·42	902783	17
44	107167	162·65	999964	·03	107202	162·68	892797	16
45	116926	159·08	999963	·03	116963	159·10	883037	15
46	126471	155·66	999961	·03	126510	155·68	873490	14
47	135810	152·38	999959	·03	135851	152·41	864149	13
48	144953	149·24	999958	·03	144996	149·27	855004	12
49	153907	146·22	999956	·03	153952	146·27	846048	11
50	162681	143·33	999954	·03	162727	143·36	837273	10
51	8·171280	140·54	9·999952	·03	8·171328	140·57	11·828672	9
52	179713	137·86	999950	·03	179763	137·90	820237	8
53	187985	135·29	999948	·03	188036	135·32	811964	7
54	196102	132·80	999946	·03	196156	132·84	803844	6
55	204070	130·41	999944	·03	204126	130·44	795874	5
56	211895	128·10	999942	·04	211953	128·14	788047	4
57	219581	125·87	999940	·04	219641	125·90	780359	3
58	227134	123·72	999938	·04	227195	123·76	772805	2
59	234557	121·64	999936	·04	234621	121·68	765379	1
60	241855	119·63	999934	·04	241921	119·67	758079	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·241855	119·63	9·999934	·04	8·241921	119·67	11·758079	60
1	249033	117·68	999932	·04	249102	117·72	750898	59
2	256094	115·80	999929	·04	256165	115·84	743835	58
3	263042	113·98	999927	·04	263115	114·02	736885	57
4	269881	112·21	999925	·04	269956	112·25	730044	56
5	276614	110·50	999922	·04	276691	110·54	723309	55
6	283243	108·83	999920	·04	283323	108·87	716677	54
7	289773	107·21	999918	·04	289856	107·26	710144	53
8	296207	105·65	999915	·04	296292	105·70	703708	52
9	302546	104·13	999913	·04	302634	104·18	697366	51
10	308794	102·66	999910	·04	308884	102·70	691116	50
11	8·314904	101·22	9·999907	·04	8·315046	101·26	11·684954	49
12	321027	99·82	999905	·04	321122	99·87	678878	48
13	327016	98·47	999902	·04	327114	98·51	672886	47
14	332924	97·14	999899	·05	333025	97·19	666975	46
15	338753	95·86	999897	·05	338856	95·90	661144	45
16	344504	94·60	999894	·05	344610	94·65	655390	44
17	350181	93·38	999891	·05	350289	93·43	649711	43
18	355783	92·19	999888	·05	355895	92·24	644105	42
19	361315	91·03	999885	·05	361430	91·08	638570	41
20	366777	89·90	999882	·05	366895	89·95	633105	40
21	8·372171	88·80	9·999879	·05	8·372292	88·85	11·627708	39
22	377499	87·72	999876	·05	377622	87·77	622378	38
23	382762	86·67	999873	·05	382889	86·72	617111	37
24	387962	85·64	999870	·05	388092	85·70	611908	36
25	393101	84·64	999867	·05	393234	84·70	606766	35
26	398179	83·66	999864	·05	398315	83·71	601685	34
27	403199	82·71	999861	·05	403338	82·76	596662	33
28	408161	81·77	999858	·05	408304	81·82	591696	32
29	413068	80·86	999854	·05	413213	80·91	586787	31
30	417919	79·96	999851	·06	418068	80·02	581932	30
31	8·422717	79·09	9·999848	·06	8·422869	79·14	11·577131	29
32	427462	78·23	999844	·06	427618	78·30	572382	28
33	432156	77·40	999841	·06	432315	77·45	567685	27
34	436800	76·57	999838	·06	436962	76·63	563038	26
35	441394	75·77	999834	·06	441560	75·83	558440	25
36	445941	74·99	999831	·06	446110	75·05	553890	24
37	450440	74·22	999827	·06	450613	74·28	549387	23
38	454893	73·46	999823	·06	455070	73·52	544930	22
39	459301	72·73	999820	·06	459481	72·79	540519	21
40	463665	72·00	999816	·06	463849	72·06	536151	20
41	8·467985	71·29	9·999812	·06	8·468172	71·35	11·531828	19
42	472263	70·60	999809	·06	472454	70·66	527546	18
43	476498	69·91	999805	·06	476693	69·98	523307	17
44	480693	69·24	999801	·06	480892	69·31	519108	16
45	484848	68·59	999797	·07	485050	68·65	514950	15
46	488963	67·94	999793	·07	489170	68·01	510830	14
47	493040	67·31	999790	·07	493250	67·38	506750	13
48	497078	66·69	999786	·07	497293	66·76	502707	12
49	501080	66·08	999782	·07	501298	66·15	498702	11
50	505045	65·48	999778	·07	505267	65·55	494733	10
51	8·508974	64·89	9·999774	·07	8·509200	64·96	11·490800	9
52	512867	64·31	999769	·07	513098	64·39	486902	8
53	516726	63·75	999765	·07	516961	63·82	483039	7
54	520551	63·19	999761	·07	520790	63·26	479210	6
55	524343	62·64	999757	·07	524586	62·72	475414	5
56	528102	62·11	999753	·07	528349	62·18	471651	4
57	531828	61·58	999748	·07	532080	61·65	467920	3
58	535523	61·06	999744	·07	535779	61·13	464221	2
59	539186	60·55	999740	·07	539447	60·62	460553	1
60	542819	60·04	999735	·07	543084	60·12	456916	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

(88 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·542819	60·04	9·999735	·07	8·543084	60·12	11·456916	60
1	546422	59·55	999731	·07	546691	59·62	453309	59
2	549995	59·06	999726	·07	550268	59·14	449732	58
3	553539	58·58	999722	·08	553817	58·66	446183	57
4	557054	58·11	999717	·08	557336	58·19	442664	56
5	560540	57·65	999713	·08	560828	57·73	439172	55
6	563999	57·19	999708	·08	564291	57·27	435709	54
7	567431	56·74	999704	·08	567727	56·82	432273	53
8	570836	56·30	999699	·08	571137	56·38	428863	52
9	574214	55·87	999694	·08	574520	55·95	425480	51
10	577566	55·44	999689	·08	577877	55·52	422123	50
11	8·580892	55·02	9·999685	·08	8·581208	55·10	11·418792	49
12	584193	54·60	999680	·08	584514	54·68	415486	48
13	587469	54·19	999675	·08	587795	54·27	412205	47
14	590721	53·79	999670	·08	591051	53·87	408949	46
15	593948	53·39	999665	·08	594283	53·47	405717	45
16	597152	53·00	999660	·08	597492	53·08	402508	44
17	600332	52·61	999655	·08	600677	52·70	399323	43
18	603489	52·23	999650	·08	603839	52·32	396161	42
19	606623	51·86	999645	·09	606978	51·94	393022	41
20	609734	51·49	999640	·09	610094	51·58	389906	40
21	8·612823	51·12	9·999635	·09	8·613189	51·21	11·386811	39
22	615891	50·76	999629	·09	616262	50·85	383738	38
23	618937	50·41	999624	·09	619313	50·50	380687	37
24	621962	50·06	999619	·09	622343	50·15	377657	36
25	624965	49·72	999614	·09	625352	49·81	374648	35
26	627948	49·38	999608	·09	628340	49·47	371660	34
27	630911	49·04	999603	·09	631308	49·13	368692	33
28	633854	48·71	999597	·09	634256	48·80	365744	32
29	636776	48·39	999592	·09	637184	48·48	362816	31
30	639680	48·06	999586	·09	640093	48·16	359907	30
31	8·642563	47·75	9·999581	·09	8·642982	47·84	11·357018	29
32	645428	47·43	999575	·09	645853	47·53	354147	28
33	648274	47·12	999570	·09	648704	47·22	351296	27
34	651102	46·82	999564	·09	651537	46·91	348463	26
35	653911	46·52	999558	·10	654352	46·61	345648	25
36	656702	46·22	999553	·10	657149	46·31	342851	24
37	659475	45·92	999547	·10	659928	46·02	340072	23
38	662230	45·63	999541	·10	662689	45·73	337311	22
39	664968	45·35	999535	·10	665433	45·44	334567	21
40	667689	45·06	999529	·10	668160	45·26	331840	20
41	8·670393	44·79	9·999524	·10	8·670870	44·88	11·329130	19
42	673080	44·51	999518	·10	673563	44·61	326437	18
43	675751	44·24	999512	·10	676239	44·34	323761	17
44	678405	43·97	999506	·10	678900	44·17	321100	16
45	681043	43·70	999500	·10	681544	43·80	318456	15
46	683665	43·44	999493	·10	684172	43·54	315828	14
47	686272	43·18	999487	·10	686784	43·28	313216	13
48	688863	42·92	999481	·10	689381	43·03	310619	12
49	691438	42·67	999475	·10	691963	42·77	308037	11
50	693998	42·42	999469	·10	694529	42·52	305471	10
51	8·696543	42·17	9·999463	·11	8·697081	42·28	11·302919	9
52	699073	41·92	999456	·11	699617	42·03	300383	8
53	701589	41·68	999450	·11	702139	41·79	297861	7
54	704090	41·44	999443	·11	704646	41·55	295354	6
55	706577	41·21	999437	·11	707140	41·32	292860	5
56	709049	40·97	999431	·11	709618	41·08	290382	4
57	711507	40·74	999424	·11	712083	40·85	287917	3
58	713952	40·51	999418	·11	714534	40·62	285465	2
59	716383	40·29	999411	·11	716972	40·40	283028	1
60	718800	40·06	999404	·11	719396	40·17	280604	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·718800	40·06	9·999404	·11	8·719396	40·17	11·280604	60
1	721204	39·84	999398	·11	721806	39·95	278194	59
2	723595	39·62	999391	·11	724204	39·74	275796	58
3	725972	39·41	999384	·11	726588	39·52	273412	57
4	728337	39·19	999378	·11	728959	39·30	271041	56
5	730688	38·98	999371	·11	731317	39·09	268683	55
6	733027	38·77	999364	·12	733663	38·89	266337	54
7	735354	38·57	999357	·12	735996	38·68	264004	53
8	737667	38·36	999350	·12	738317	38·48	261683	52
9	739969	38·16	999343	·12	740626	38·27	259374	51
10	742259	37·96	999336	·12	742922	38·07	257078	50
11	8·744536	37·76	9·999329	·12	8·745207	37·87	11·254793	49
12	746802	37·56	999322	·12	747479	37·68	252521	48
13	749055	37·37	999315	·12	749740	37·49	250260	47
14	751297	37·17	999308	·12	751989	37·29	248011	46
15	753528	36·98	999301	·12	754227	37·10	245773	45
16	755747	36·79	999294	·12	756453	36·92	243547	44
17	757955	36·61	999286	·12	758668	36·73	241332	43
18	760151	36·42	999279	·12	760872	36·55	239128	42
19	762337	36·24	999272	·12	763065	36·36	236935	41
20	764511	36·06	999265	·12	765246	36·18	234754	40
21	8·766675	35·88	9·999257	·12	8·767417	36·00	11·232583	39
22	768828	35·70	999250	·13	769578	35·83	230422	38
23	770970	35·53	999242	·13	771727	35·65	228273	37
24	773101	35·35	999235	·13	773866	35·48	226134	36
25	775223	35·18	999227	·13	775995	35·31	224005	35
26	777333	35·01	999220	·13	778114	35·14	221886	34
27	779434	34·84	999212	·13	780222	34·97	219778	33
28	781524	34·67	999205	·13	782320	34·80	217680	32
29	783605	34·51	999197	·13	784408	34·64	215592	31
30	785675	34·31	999189	·13	786486	34·47	213514	30
31	8·787736	34·18	9·999181	·13	8·788554	34·31	11·211446	29
32	789787	34·02	999174	·13	790613	34·15	209387	28
33	791828	33·86	999166	·13	792662	33·99	207338	27
34	793859	33·70	999158	·13	794701	33·83	205299	26
35	795881	33·54	999150	·13	796731	33·68	203269	25
36	797894	33·39	999142	·13	798752	33·52	201248	24
37	799897	33·23	999134	·13	800763	33·37	199237	23
38	801892	33·08	999126	·13	802765	33·22	197235	22
39	803876	32·93	999118	·13	804758	33·07	195242	21
40	805852	32·78	999110	·13	806742	32·92	193258	20
41	8·807819	32·63	9·999102	·13	8·808717	32·78	11·191283	19
42	809777	32·49	999094	·14	810683	32·62	189317	18
43	811726	32·34	999086	·14	812641	32·48	187359	17
44	813667	32·19	999077	·14	814589	32·33	185411	16
45	815599	32·05	999069	·14	816529	32·19	183471	15
46	817522	31·91	999061	·14	818461	32·05	181539	14
47	819436	31·77	999053	·14	820384	31·91	179616	13
48	821343	31·63	999044	·14	822298	31·77	177702	12
49	823240	31·49	999036	·14	824205	31·63	175795	11
50	825130	31·35	999027	·14	826103	31·50	173897	10
51	8·827011	31·22	9·999019	·14	8·827992	31·36	11·172008	9
52	828884	31·08	999010	·14	829874	31·23	170126	8
53	830749	30·95	999002	·14	831748	31·10	168252	7
54	832607	30·82	998993	·14	833613	30·96	166387	6
55	834456	30·69	998984	·14	835471	30·83	164529	5
56	836297	30·56	998976	·14	837321	30·70	162679	4
57	838130	30·43	998967	·15	839163	30·57	160837	3
58	839956	30·30	998958	·15	840998	30·45	159002	2
59	841774	30·17	998950	·15	842825	30·32	157175	1
60	843585	30·00	998941	·15	844644	30·19	155356	0
	Cosine	D.	Sine		Cotang.	D.	Ta g.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·843585	30·05	9·998941	·15	8·844644	30·19	11·155356	60
1	845387	29·92	998932	·15	846455	30·07	153545	59
2	847183	29·80	998923	·15	848260	29·95	151740	58
3	848971	29·67	998914	·15	850057	29·82	149943	57
4	850751	29·55	998905	·15	851846	29·70	148154	56
5	852525	29·43	998896	·15	853628	29·58	146372	55
6	854291	29·31	998887	·15	855403	29·46	144597	54
7	856049	29·19	998878	·15	857171	29·35	142829	53
8	857801	29·07	998869	·15	858932	29·23	141068	52
9	859546	28·96	998860	·15	860686	29·11	139314	51
10	861283	28·84	998851	·15	862433	29·00	137567	50
11	8·863014	28·73	9·998841	·15	8·864173	28·88	11·135827	49
12	864738	28·61	998832	·15	865906	28·77	134094	48
13	866455	28·50	998823	·16	867632	28·66	132368	47
14	868165	28·39	998813	·16	869351	28·54	130649	46
15	869868	28·28	998804	·16	871064	28·43	128936	45
16	871565	28·17	998795	·16	872770	28·32	127230	44
17	873255	28·06	998785	·16	874469	28·21	125531	43
18	874938	27·95	998776	·16	876162	28·11	123838	42
19	876615	27·86	998766	·16	877849	28·00	122151	41
20	878285	27·73	998757	·16	879529	27·89	120471	40
21	8·879949	27·63	9·998747	·16	8·881202	27·79	11·118798	39
22	881607	27·52	998738	·16	882869	27·68	117131	38
23	883258	27·42	998728	·16	884530	27·58	115470	37
24	884903	27·31	998718	·16	886185	27·47	113815	36
25	886542	27·21	998708	·16	887833	27·37	112167	35
26	888174	27·11	998699	·16	889476	27·27	110524	34
27	889801	27·00	998689	·16	891112	27·17	108888	33
28	891421	26·90	998679	·16	892742	27·07	107258	32
29	893035	26·80	998669	·17	894366	26·97	105634	31
30	894643	26·70	998659	·17	895984	26·87	104016	30
31	8·896246	26·60	9·998649	·17	8·897596	26·77	11·102404	29
32	897842	26·51	998639	·17	899203	26·67	100797	28
33	899432	26·41	998629	·17	900803	26·58	099197	27
34	901017	26·31	998619	·17	902398	26·48	097602	26
35	902596	26·22	998609	·17	903987	26·38	096013	25
36	904169	26·12	998599	·17	905570	26·29	094430	24
37	905736	26·03	998589	·17	907147	26·20	092853	23
38	907297	25·93	998578	·17	908719	26·10	091281	22
39	908853	25·84	998568	·17	910285	26·01	089715	21
40	910404	25·75	998558	·17	911846	25·92	088154	20
41	8·911949	25·66	9·998548	·17	8·913401	25·83	11·086599	19
42	913488	25·56	998537	·17	914951	25·74	085049	18
43	915022	25·47	998527	·17	916495	25·65	083505	17
44	916550	25·38	998516	·18	918034	25·56	081966	16
45	918073	25·29	998506	·18	919568	25·47	080432	15
46	919591	25·20	998495	·18	921096	25·38	078904	14
47	921103	25·12	998485	·18	922619	25·30	077381	13
48	922610	25·03	998474	·18	924136	25·21	075864	12
49	924112	24·94	998464	·18	925649	25·12	074351	11
50	925609	24·86	998453	·18	927156	25·03	072844	10
51	8·927100	24·77	9·998442	·18	8·928658	24·95	11·071342	9
52	928587	24·69	998431	·18	930155	24·86	069845	8
53	930068	24·60	998421	·18	931647	24·78	068353	7
54	931544	24·52	998410	·18	933134	24·70	066866	6
55	933015	24·43	998399	·18	934616	24·61	065384	5
56	934481	24·35	998388	·18	936093	24·53	063907	4
57	935942	24·27	998377	·18	937565	24·45	062435	3
58	937398	24·19	998366	·18	939032	24·37	060968	2
59	938850	24·11	998355	·18	940494	24·30	059506	1
60	940296	24·03	998344	·18	941952	24·21	058048	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·940296	24·03	9·998344	·19	8·941952	24·21	11·058048	60
1	941738	23·94	998333	·19	943404	24·13	056596	59
2	943174	23·87	998322	·19	944852	24·05	055148	58
3	944606	23·79	998311	·19	946295	23·97	053705	57
4	946034	23·71	998300	·19	947734	23·90	052266	56
5	947456	23·63	998289	·19	949168	23·82	050832	55
6	948874	23·55	998277	·19	950597	23·74	049403	54
7	950287	23·48	998266	·19	952021	23·66	047979	53
8	951696	23·40	998255	·19	953441	23·60	046559	52
9	953100	23·32	998243	·19	954856	23·51	045144	51
10	954499	23·25	998232	·19	956267	23·44	043733	50
11	8·955894	23·17	9·998220	·19	8·957674	23·37	11·042326	49
12	957284	23·10	998209	·19	959075	23·29	040925	48
13	958670	23·02	998197	·19	960473	23·23	039527	47
14	960052	22·95	998186	·19	961866	23·14	038134	46
15	961429	22·88	998174	·19	963255	23·07	036745	45
16	962801	22·80	998163	·19	964639	23·00	035361	44
17	964170	22·73	998151	·19	966019	22·93	033981	43
18	965534	22·66	998139	·20	967394	22·86	032606	42
19	966893	22·59	998128	·20	968766	22·79	031234	41
20	968249	22·52	998116	·20	970133	22·71	029867	40
21	8·969600	22·44	9·998104	·20	8·971496	22·65	11·028504	39
22	970947	22·38	998092	·20	972855	22·57	027145	38
23	972289	22·31	998080	·20	974209	22·51	025791	37
24	973628	22·24	998068	·20	975560	22·44	024440	36
25	974962	22·17	998056	·20	976906	22·37	023094	35
26	976293	22·10	998044	·20	978248	22·30	021752	34
27	977619	22·03	998032	·20	979586	22·23	020414	33
28	978941	21·97	998020	·20	980921	22·17	019079	32
29	980259	21·90	998008	·20	982251	22·10	017749	31
30	981573	21·83	997996	·20	983577	22·04	016423	30
31	8·982883	21·77	9·997985	·20	8·984899	21·97	11·015101	29
32	984189	21·70	997972	·20	986217	21·91	013783	28
33	985491	21·63	997959	·20	987532	21·84	012468	27
34	986789	21·57	997947	·20	988842	21·78	011158	26
35	988083	21·50	997935	·21	990149	21·71	009851	25
36	989374	21·44	997922	·21	991451	21·65	008549	24
37	990660	21·38	997910	·21	992750	21·58	007250	23
38	991943	21·31	997897	·21	994045	21·52	005955	22
39	993222	21·25	997885	·21	995337	21·46	004663	21
40	994497	21·19	997872	·21	996624	21·40	003376	20
41	8·995768	21·12	9·997860	·21	8·997908	21·34	11·002092	19
42	997036	21·06	997847	·21	999188	21·27	000812	18
43	998299	21·00	997835	·21	9·000465	21·21	10·999535	17
44	999560	20·94	997822	·21	001738	21·15	998262	16
45	9·000816	20·87	997809	·21	003007	21·09	996993	15
46	002069	20·82	997797	·21	004272	21·03	995728	14
47	003318	20·76	997784	·21	005534	20·97	994466	13
48	004563	20·70	997771	·21	006792	20·91	993208	12
49	005805	20·64	997758	·21	008047	20·85	991953	11
50	007044	20·58	997745	·21	009298	20·80	990702	10
51	9·008278	20·52	9·997732	·21	9·010546	20·74	10·989454	9
52	009510	20·46	997719	·21	011790	20·68	988210	8
53	010737	20·40	997706	·21	013031	20·62	986969	7
54	011962	20·34	997693	·22	014268	20·56	985732	6
55	013182	20·29	997680	·22	015502	20·51	984498	5
56	014400	20·23	997667	·22	016732	20·45	983268	4
57	015613	20·17	997654	·22	017959	20·40	982041	3
58	016824	20·12	997641	·22	019183	20·33	980817	2
59	018031	20·06	997628	·22	020403	20·28	979597	1
60	019235	20·00	997614	·22	021620	20·23	978380	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·019235	20·00	9·997614	·22	9·021620	20·23	10·978380	60
1	020435	19·95	997601	·22	022834	20·17	977166	59
2	021632	19·89	997588	·22	024044	20·11	975956	58
3	022825	19·84	997574	·22	025251	20·06	974749	57
4	024016	19·78	997561	·22	026455	20·00	973545	56
5	025203	19·73	997547	·22	027655	19·95	972345	55
6	026386	19·67	997534	·23	028852	19·90	971148	54
7	027567	19·62	997520	·23	030046	19·85	969954	53
8	028744	19·57	997507	·23	031237	19·79	968763	52
9	029918	19·51	997493	·23	032425	19·74	967575	51
10	031089	19·47	997480	·23	033609	19·69	966391	50
11	9·032257	19·41	9·997466	·23	9·034791	19·64	10·965209	49
12	033421	19·36	997452	·23	035969	19·58	964031	48
13	034582	19·30	997439	·23	037144	19·53	962856	47
14	035741	19·25	997425	·23	038316	19·48	961684	46
15	036896	19·20	997411	·23	039485	19·43	960515	45
16	038048	19·15	997397	·23	040651	19·38	959349	44
17	039197	19·10	997383	·23	041813	19·33	958187	43
18	040342	19·05	997369	·23	042973	19·28	957027	42
19	041485	18·99	997355	·23	044130	19·23	955870	41
20	042625	18·94	997341	·23	045284	19·18	954716	40
21	9·043762	18·89	9·997327	·24	9·046434	19·13	10·953566	39
22	044895	18·84	997313	·24	047582	19·08	952418	38
23	046026	18·79	997299	·24	048727	19·03	951273	37
24	047154	18·75	997285	·24	049869	18·98	950131	36
25	048279	18·70	997271	·24	051008	18·93	948992	35
26	049400	18·65	997257	·24	052144	18·89	947856	34
27	050519	18·60	997242	·24	053277	18·84	946723	33
28	051635	18·55	997228	·24	054407	18·79	945593	32
29	052749	18·50	997214	·24	055535	18·74	944465	31
30	053859	18·45	997199	·24	056659	18·70	943341	30
31	9·054966	18·41	9·997185	·24	9·057781	18·65	10·942219	29
32	056071	18·36	997170	·24	058900	18·60	941100	28
33	057172	18·31	997156	·24	060016	18·55	939984	27
34	058271	18·27	997141	·24	061130	18·51	938870	26
35	059367	18·22	997127	·24	062240	18·46	937760	25
36	060460	18·17	997112	·24	063348	18·42	936652	24
37	061551	18·13	997098	·24	064453	18·37	935547	23
38	062639	18·08	997083	·25	065556	18·33	934444	22
39	063724	18·04	997068	·25	066655	18·28	933345	21
40	064806	17·99	997053	·25	067752	18·24	932248	20
41	9·065885	17·94	9·997039	·25	9·068846	18·19	10·931154	19
42	066962	17·90	997024	·25	069938	18·15	930062	18
43	068036	17·86	997009	·25	071027	18·10	928973	17
44	069107	17·81	996994	·25	072113	18·06	927887	16
45	070176	17·77	996979	·25	073197	18·02	926803	15
46	071242	17·72	996964	·25	074278	17·97	925722	14
47	072306	17·68	996949	·25	075356	17·93	924644	13
48	073366	17·63	996934	·25	076432	17·89	923568	12
49	074424	17·59	996919	·25	077505	17·84	922495	11
50	075480	17·55	996904	·25	078576	17·80	921424	10
51	9·076533	17·50	9·996889	·25	9·079644	17·76	10·920356	9
52	077583	17·46	996874	·25	080710	17·72	919290	8
53	078631	17·42	996858	·25	081773	17·67	918227	7
54	079676	17·38	996843	·25	082833	17·63	917167	6
55	080719	17·33	996828	·25	083891	17·59	916109	5
56	081759	17·29	996812	·26	084947	17·55	915053	4
57	082797	17·25	996797	·26	086000	17·51	914000	3
58	083832	17·21	996782	·26	087050	17·47	912950	2
59	084864	17·17	996766	·26	088098	17·43	911902	1
60	085894	17·13	996751	·26	089144	17·38	910856	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·085894	17·13	9·996751	·26	9·089144	17·38	10·910856	60
1	086922	17·09	996735	·26	090187	17·34	909813	59
2	087947	17·04	996720	·26	091228	17·30	908772	58
3	088970	17·00	996704	·26	092266	17·27	907734	57
4	089990	16·96	996688	·26	093302	17·22	906698	56
5	091008	16·92	996673	·26	094336	17·19	905664	55
6	092024	16·88	996657	·26	095367	17·15	904633	54
7	093037	16·84	996641	·26	096395	17·11	903605	53
8	094047	16·80	996625	·26	097422	17·07	902578	52
9	095056	16·76	996610	·26	098446	17·03	901554	51
10	096062	16·73	996594	·26	099468	16·99	900532	50
11	9·097065	16·68	9·996578	·27	9·100487	16·95	10·899513	49
12	098066	16·65	996562	·27	101504	16·91	898496	48
13	099065	16·61	996546	·27	102519	16·87	897481	47
14	100062	16·57	996530	·27	103532	16·84	896468	46
15	101056	16·53	996514	·27	104542	16·80	895458	45
16	102048	16·49	996498	·27	105550	16·76	894450	44
17	103037	16·45	996482	·27	106556	16·72	893444	43
18	104025	16·41	996465	·27	107559	16·69	892441	42
19	105010	16·38	996449	·27	108560	16·65	891440	41
20	105992	16·34	996433	·27	109559	16·61	890441	40
21	9·106973	16·30	9·996417	·27	9·110556	16·58	10·889444	39
22	107951	16·27	996400	·27	111551	16·54	888449	38
23	108927	16·23	996384	·27	112543	16·50	887457	37
24	109901	16·19	996368	·27	113533	16·46	886467	36
25	110873	16·16	996351	·27	114521	16·43	885479	35
26	111842	16·12	996335	·27	115507	16·39	884493	34
27	112809	16·08	996318	·27	116491	16·36	883509	33
28	113774	16·05	996302	·28	117472	16·32	882528	32
29	114737	16·01	996285	·28	118452	16·29	881548	31
30	115698	15·97	996269	·28	119429	16·25	880571	30
31	9·116656	15·94	9·996252	·28	9·120404	16·22	10·879596	29
32	117613	15·90	996235	·28	121377	16·18	878623	28
33	118567	15·87	996219	·28	122348	16·15	877652	27
34	119519	15·83	996202	·28	123317	16·11	876683	26
35	120469	15·80	996185	·28	124284	16·07	875716	25
36	121417	15·76	996168	·28	125249	16·04	874751	24
37	122362	15·73	996151	·28	126211	16·01	873789	23
38	123306	15·69	996134	·28	127172	15·97	872828	22
39	124248	15·66	996117	·28	128130	15·94	871870	21
40	125187	15·62	996100	·28	129087	15·91	870913	20
41	9·126125	15·59	9·996083	·29	9·130041	15·87	10·869959	19
42	127060	15·56	996066	·29	130994	15·84	869006	18
43	127993	15·52	996049	·29	131944	15·81	868056	17
44	128925	15·49	996032	·29	132893	15·77	867107	16
45	129854	15·45	996015	·29	133839	15·74	866161	15
46	130781	15·42	995998	·29	134784	15·71	865216	14
47	131706	15·39	995980	·29	135726	15·67	864274	13
48	132630	15·35	995963	·29	136667	15·64	863333	12
49	133551	15·32	995946	·29	137605	15·61	862395	11
50	134470	15·29	995928	·29	138542	15·58	861458	10
51	9·135387	15·25	9·995911	·29	9·139476	15·55	10·860524	9
52	136303	15·22	995894	·29	140409	15·51	859591	8
53	137216	15·19	995876	·29	141340	15·48	858660	7
54	138128	15·16	995859	·29	142269	15·45	857731	6
55	139037	15·12	995841	·29	143196	15·42	856804	5
56	139944	15·09	995823	·29	144121	15·39	855879	4
57	140850	15·06	995806	·29	145044	15·35	854956	3
58	141754	15·03	995788	·29	145966	15·32	854034	2
59	142655	15·00	995771	·29	146885	15·29	853115	1
60	143555	14·96	995753	·29	147803	15·26	852197	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·143555	14·96	9·995753	·30	9·147803	15·26	10·852197	60
1	144453	14·93	995735	·30	148718	15·23	851282	59
2	145349	14·90	995717	·30	149632	15·20	850368	58
3	146243	14·87	995699	·30	150544	15·17	849456	57
4	147136	14·84	995681	·30	151454	15·14	848546	56
5	148026	14·81	995664	·30	152363	15·11	847637	55
6	148915	14·78	995646	·30	153269	15·08	846731	54
7	149802	14·75	995628	·30	154174	15·05	845826	53
8	150686	14·72	995610	·30	155077	15·02	844923	52
9	151569	14·69	995591	·30	155978	14·99	844022	51
10	152451	14·66	995573	·30	156877	14·96	843123	50
11	9·153330	14·63	9·995555	·30	9·157775	14·93	10·842225	49
12	154208	14·60	995537	·30	158671	14·90	841329	48
13	155083	14·57	995519	·30	159565	14·87	840435	47
14	155957	14·54	995501	·31	160457	14·84	839543	46
15	156830	14·51	995482	·31	161347	14·81	838653	45
16	157700	14·48	995464	·31	162236	14·79	837764	44
17	158569	14·45	995446	·31	163123	14·76	836877	43
18	159435	14·42	995427	·31	164008	14·73	835992	42
19	160301	14·39	995409	·31	164892	14·70	835108	41
20	161164	14·36	995390	·31	165774	14·67	834226	40
21	9·162025	14·33	9·995372	·31	9·166654	14·64	10·833346	39
22	162885	14·30	995353	·31	167532	14·61	832468	38
23	163743	14·27	995334	·31	168409	14·58	831591	37
24	164600	14·24	995316	·31	169284	14·55	830716	36
25	165454	14·22	995297	·31	170157	14·53	829843	35
26	166307	14·19	995278	·31	171029	14·50	828971	34
27	167159	14·16	995260	·31	171899	14·47	828101	33
28	168008	14·13	995241	·32	172767	14·44	827233	32
29	168856	14·10	995222	·32	173634	14·42	826366	31
30	169702	14·07	995203	·32	174499	14·39	825501	30
31	9·170547	14·05	9·995184	·32	9·175362	14·36	10·824638	29
32	171389	14·02	995165	·32	176224	14·33	823776	28
33	172230	13·99	995146	·32	177084	14·31	822916	27
34	173070	13·96	995127	·32	177942	14·28	822058	26
35	173908	13·94	995108	·32	178799	14·25	821201	25
36	174744	13·91	995089	·32	179655	14·23	820345	24
37	175578	13·88	995070	·32	180508	14·20	819492	23
38	176411	13·86	995051	·32	181360	14·17	818640	22
39	177242	13·83	995032	·32	182211	14·15	817789	21
40	178072	13·80	995013	·32	183059	14·12	816941	20
41	9·178900	13·77	9·994993	·32	9·183907	14·09	10·816093	19
42	179726	13·74	994974	·32	184752	14·07	815248	18
43	180551	13·72	994955	·32	185597	14·04	814403	17
44	181374	13·69	994935	·32	186439	14·02	813561	16
45	182196	13·66	994916	·33	187280	13·99	812720	15
46	183016	13·64	994896	·33	188120	13·96	811880	14
47	183834	13·61	994877	·33	188958	13·93	811042	13
48	184651	13·59	994857	·33	189794	13·91	810206	12
49	185466	13·56	994838	·33	190629	13·89	809371	11
50	186280	13·53	994818	·33	191462	13·86	808538	10
51	9·187092	13·51	9·994798	·33	9·192294	13·84	10·807706	9
52	187903	13·48	994779	·33	193124	13·81	806876	8
53	188712	13·46	994759	·33	193953	13·79	806047	7
54	189519	13·43	994739	·33	194780	13·76	805220	6
55	190325	13·41	994719	·33	195606	13·74	804394	5
56	191130	13·38	994700	·33	196430	13·71	803570	4
57	191933	13·36	994680	·33	197253	13·69	802747	3
58	192734	13·33	994660	·33	198074	13·66	801926	2
59	193534	13·30	994640	·33	198894	13·64	801106	1
60	194332	13·28	994620	·33	199713	13·61	800287	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

(81 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·194332	13·28	9·994620	·33	9·199713	13·61	10·800287	60
1	195129	13·26	994600	·33	200529	13·59	799471	59
2	195925	13·23	994580	·33	201345	13·56	798655	58
3	196719	13·21	994560	·34	202159	13·54	797841	57
4	197511	13·18	994540	·34	202971	13·52	797029	56
5	198302	13·16	994519	·34	203782	13·49	796218	55
6	199091	13·13	994499	·34	204592	13·47	795408	54
7	199879	13·11	994479	·34	205400	13·45	794600	53
8	200666	13·08	994459	·34	206207	13·42	793793	52
9	201451	13·06	994438	·34	207013	13·40	792987	51
10	202234	13·04	994418	·34	207817	13·38	792183	50
11	9·203017	13·01	9·994397	·34	9·208619	13·35	10·791381	49
12	203797	12·99	994377	·34	209420	13·33	790580	48
13	204577	12·96	994357	·34	210220	13·31	789780	47
14	205354	12·94	994336	·34	211018	13·28	788982	46
15	206131	12·92	994316	·34	211815	13·26	788185	45
16	206906	12·89	994295	·34	212611	13·24	787389	44
17	207679	12·87	994274	·35	213405	13·21	786595	43
18	208452	12·85	994254	·35	214198	13·19	785802	42
19	209222	12·82	994233	·35	214989	13·17	785011	41
20	209992	12·80	994212	·35	215780	13·15	784220	40
21	9·210760	12·78	9·994191	·35	9·216568	13·12	10·783432	39
22	211526	12·75	994171	·35	217356	13·10	782644	38
23	212291	12·73	994150	·35	218142	13·08	781858	37
24	213055	12·71	994129	·35	218926	13·05	781074	36
25	213818	12·68	994108	·35	219710	13·03	780290	35
26	214579	12·66	994087	·35	220492	13·01	779508	34
27	215338	12·64	994066	·35	221272	12·99	778728	33
28	216097	12·61	994045	·35	222052	12·97	777948	32
29	216854	12·59	994024	·35	222830	12·94	777170	31
30	217609	12·57	994003	·35	223606	12·92	776394	30
31	9·218363	12·55	9·993981	·35	9·224382	12·90	10·775618	29
32	219116	12·53	993960	·35	225156	12·88	774844	28
33	219868	12·50	993939	·35	225929	12·86	774071	27
34	220618	12·48	993918	·35	226700	12·84	773300	26
35	221367	12·46	993896	·36	227471	12·81	772529	25
36	222115	12·44	993875	·36	228239	12·79	771761	24
37	222861	12·42	993854	·36	229007	12·77	770993	23
38	223606	12·39	993832	·36	229773	12·75	770227	22
39	224349	12·37	993811	·36	230539	12·73	769461	21
40	225092	12·35	993789	·36	231302	12·71	768698	20
41	9·225833	12·33	9·993768	·36	9·232065	12·69	10·767935	19
42	226573	12·31	993746	·36	232826	12·67	767174	18
43	227311	12·28	993725	·36	233586	12·65	766414	17
44	228048	12·26	993703	·36	234345	12·62	765655	16
45	228784	12·24	993681	·36	235103	12·60	764897	15
46	229518	12·22	993660	·36	235859	12·58	764141	14
47	230252	12·20	993638	·36	236614	12·56	763386	13
48	230984	12·18	993616	·36	237368	12·54	762632	12
49	231714	12·16	993594	·37	238120	12·52	761880	11
50	232444	12·14	993572	·37	238872	12·50	761128	10
51	9·233172	12·12	9·993550	·37	9·239622	12·48	10·760378	9
52	233899	12·09	993528	·37	240371	12·46	759629	8
53	234625	12·07	993506	·37	241118	12·44	758882	7
54	235349	12·05	993484	·37	241865	12·42	758135	6
55	236073	12·03	993462	·37	242610	12·40	757390	5
56	236795	12·01	993440	·37	243354	12·38	756646	4
57	237515	11·99	993418	·37	244097	12·36	755903	3
58	238235	11·97	993396	·37	244839	12·34	755161	2
59	238953	11·95	993374	·37	245579	12·32	754421	1
60	239670	11·93	993351	·37	246319	12·30	753681	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·239670	11·93	9·993351	·37	9·246319	12·30	10·753681	60
1	240386	11·91	993329	·37	247057	12·28	752943	59
2	241101	11·89	993307	·37	247794	12·26	752206	58
3	241814	11·87	993285	·37	248530	12·24	751470	57
4	242526	11·85	993262	·37	249264	12·22	750736	56
5	243237	11·83	993240	·37	249998	12·20	750002	55
6	243947	11·81	993217	·38	250730	12·18	749270	54
7	244656	11·79	993195	·38	251461	12·17	748539	53
8	245363	11·77	993172	·38	252191	12·15	747809	52
9	246069	11·75	993149	·38	252920	12·13	747080	51
10	246775	11·73	993127	·38	253648	12·11	746352	50
11	9·247478	11·71	9·993104	·38	9·254374	12·09	10·745626	49
12	248181	11·69	993081	·38	255100	12·07	744900	48
13	248883	11·67	993059	·38	255824	12·05	744176	47
14	249583	11·65	993036	·38	256547	12·03	743453	46
15	250282	11·63	993013	·38	257269	12·01	742731	45
16	250980	11·61	992990	·38	257990	12·00	742010	44
17	251677	11·59	992967	·38	258710	11·98	741290	43
18	252373	11·58	992944	·38	259429	11·96	740571	42
19	253067	11·56	992921	·38	260146	11·94	739854	41
20	253761	11·54	992898	·38	260863	11·92	739137	40
21	9·254453	11·52	9·992875	·38	9·261578	11·90	10·738422	39
22	255144	11·50	992852	·38	262292	11·89	737708	38
23	255834	11·48	992829	·39	263005	11·87	736995	37
24	256523	11·46	992806	·39	263717	11·85	736283	36
25	257211	11·44	992783	·39	264428	11·83	735572	35
26	257898	11·42	992759	·39	265138	11·81	734862	34
27	258583	11·41	992736	·39	265847	11·79	734153	33
28	259268	11·39	992713	·39	266555	11·78	733445	32
29	259951	11·37	992690	·39	267261	11·76	732739	31
30	260633	11·35	992666	·39	267967	11·74	732033	30
31	9·261314	11·33	9·992643	·39	9·268671	11·72	10·731329	29
32	261994	11·31	992619	·39	269375	11·70	730625	28
33	262673	11·30	992596	·39	270077	11·69	729923	27
34	263351	11·28	992572	·39	270779	11·67	729221	26
35	264027	11·26	992549	·39	271479	11·65	728521	25
36	264703	11·24	992525	·39	272178	11·64	727822	24
37	265377	11·22	992501	·39	272876	11·62	727124	23
38	266051	11·20	992478	·40	273573	11·60	726427	22
39	266723	11·19	992454	·40	274269	11·58	725731	21
40	267395	11·17	992430	·40	274964	11·57	725036	20
41	9·268065	11·15	9·992406	·40	9·275658	11·55	10·724342	19
42	268734	11·13	992382	·40	276351	11·53	723649	18
43	269402	11·11	992359	·40	277043	11·51	722957	17
44	270069	11·10	992335	·40	277734	11·50	722266	16
45	270735	11·08	992311	·40	278424	11·48	721576	15
46	271400	11·06	992287	·40	279113	11·47	720887	14
47	272064	11·05	992263	·40	279801	11·45	720199	13
48	272726	11·03	992239	·40	280488	11·43	719512	12
49	273388	11·01	992214	·40	281174	11·41	718826	11
50	274049	10·99	992190	·40	281858	11·40	718142	10
51	9·274708	10·98	9·992166	·40	9·282542	11·38	10·717458	9
52	275367	10·96	992142	·40	283225	11·36	716775	8
53	276024	10·94	992117	·41	283907	11·35	716093	7
54	276681	10·92	992093	·41	284588	11·33	715412	6
55	277337	10·91	992069	·41	285268	11·31	714732	5
56	277991	10·89	992044	·41	285947	11·30	714053	4
57	278644	10·87	992020	·41	286624	11·28	713376	3
58	279297	10·86	991996	·41	287301	11·26	712699	2
59	279948	10·84	991971	·41	287977	11·25	712023	1
60	280599	10 82	991947	·41	288652	11·23	711348	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(79 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·280599	10·82	9·991947	·41	9·288652	11·23	10·711348	60
1	281248	10·81	991922	·41	289326	11·22	710674	59
2	281897	10·79	991897	·41	289999	11·20	710001	58
3	282544	10·77	991873	·41	290671	11·18	709329	57
4	283190	10·76	991848	·41	291342	11·17	708658	56
5	283836	10·74	991823	·41	292013	11·15	707987	55
6	284480	10·72	991799	·41	292682	11·14	707318	54
7	285124	10·71	991774	·42	293350	11·12	706650	53
8	285766	10·69	991749	·42	294017	11·11	705983	52
9	286408	10·67	991724	·42	294684	11·09	705316	51
10	287048	10·66	991699	·42	295349	11·07	704651	50
11	9·287687	10·64	9·991674	·42	9·296013	11·06	10·703987	49
12	288326	10·63	991649	·42	296677	11·04	703323	48
13	288964	10·61	991624	·42	297339	11·03	702661	47
14	289600	10·59	991599	·42	298001	11·01	701999	46
15	290236	10·58	991574	·42	298662	11·00	701338	45
16	290870	10·56	991549	·42	299322	10·98	700678	44
17	291504	10·54	991524	·42	299980	10·96	700020	43
18	292137	10·53	991498	·42	300638	10·95	699362	42
19	292768	10·51	991473	·42	301295	10·93	698705	41
20	293399	10·50	991448	·42	301951	10·92	698049	40
21	9·294029	10·48	9·991422	·42	9·302607	10·90	10·697393	39
22	294658	10·46	991397	·42	303261	10·89	696739	38
23	295286	10·45	991372	·43	303914	10·87	696086	37
24	295913	10·43	991346	·43	304567	10·86	695433	36
25	296539	10·42	991321	·43	305218	10·84	694782	35
26	297164	10·40	991295	·43	305869	10·83	694131	34
27	297788	10·39	991270	·43	306519	10·81	693481	33
28	298412	10·37	991244	·43	307168	10·80	692832	32
29	299034	10·36	991218	·43	307815	10·78	692185	31
30	299655	10·34	991193	·43	308463	10·77	691537	30
31	9·300276	10·32	9·991167	·43	9·309109	10·75	10·690891	29
32	300895	10·31	991141	·43	309754	10·74	690246	28
33	301514	10·29	991115	·43	310398	10·73	689602	27
34	302132	10·28	991090	·43	311042	10·71	688958	26
35	302748	10·26	991064	·43	311685	10·70	688315	25
36	303364	10·25	991038	·43	312327	10·68	687673	24
37	303979	10·23	991012	·43	312967	10·67	687033	23
38	304593	10·22	990986	·43	313608	10·65	686392	22
39	305207	10·20	990960	·43	314247	10·64	685753	21
40	305819	10·19	990934	·44	314885	10·62	685115	20
41	9·306430	10·17	9·990908	·44	9·315523	10·61	10·684477	19
42	307041	10·16	990882	·44	316159	10·60	683841	18
43	307650	10·14	990855	·44	316795	10·58	683205	17
44	308259	10·13	990829	·44	317430	10·57	682570	16
45	308867	10·11	990803	·44	318064	10·55	681936	15
46	309474	10·10	990777	·44	318697	10·54	681303	14
47	310080	10·08	990750	·44	319329	10·53	680671	13
48	310685	10·07	990724	·44	319961	10·51	680039	12
49	311289	10·05	990697	·44	320592	10·50	679408	11
50	311893	10·04	990671	·44	321222	10·48	678778	10
51	9·312495	10·03	9·990644	·44	9·321851	10·47	10·678149	9
52	313097	10·01	990618	·44	322479	10·45	677521	8
53	313698	10·00	990591	·44	323106	10·44	676894	7
54	314297	9·98	990565	·44	323733	10·43	676267	6
55	314897	9·97	990538	·44	324358	10·41	675642	5
56	315495	9·96	990511	·45	324983	10·40	675017	4
57	316092	9·94	990485	·45	325607	10·39	674393	3
58	316689	9·93	990458	·45	326231	10·37	673769	2
59	317284	9·91	990431	·45	326853	10·36	673147	1
60	317879	9·90	990404	·45	327475	10·35	672525	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

(78 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·317879	9·90	9·990404	·45	9·327474	10·35	10·672526	60
1	318473	9·88	990378	·45	328095	10·33	671905	59
2	319066	9·87	990351	·45	328715	10·32	671285	58
3	319658	9·86	990324	·45	329334	10·30	670666	57
4	320249	9·84	990297	·45	329953	10·29	670047	56
5	320840	9·83	990270	·45	330570	10·28	669430	55
6	321430	9·82	990243	·45	331187	10·26	668813	54
7	322019	9·80	990215	·45	331803	10·25	668197	53
8	322607	9·79	990188	·45	332418	10·24	667582	52
9	323194	9·77	990161	·45	333033	10·23	666967	51
10	323780	9·76	990134	·45	333646	10·21	666354	50
11	9·324366	9·75	9·990107	·46	9·334259	10·20	10·665741	49
12	324950	9·73	990079	·46	334871	10·19	665129	48
13	325534	9·72	990052	·46	335482	10·17	664518	47
14	326117	9·70	990025	·46	336093	10·16	663907	46
15	326700	9·69	989997	·46	336702	10·15	663298	45
16	327281	9·68	989970	·46	337311	10·13	662689	44
17	327862	9·66	989942	·46	337919	10·12	662081	43
18	328442	9·65	989915	·46	338527	10·11	661473	42
19	329021	9·64	989887	·46	339133	10·10	660867	41
20	329599	9·62	989860	·46	339739	10·08	660261	40
21	9·330176	9·61	9·989832	·46	9·340344	10·07	10·659656	39
22	330753	9·60	989804	·46	340948	10·06	659052	38
23	331329	9·58	989777	·46	341552	10·04	658448	37
24	331903	9·57	989749	·47	342155	10·03	657845	36
25	332478	9·56	989721	·47	342757	10·02	657243	35
26	333051	9·54	989693	·47	343358	10·00	656642	34
27	333624	9·53	989665	·47	343958	9·99	656042	33
28	334195	9·52	989637	·47	344558	9·98	655442	32
29	334766	9·50	989609	·47	345157	9·97	654843	31
30	335337	9·49	989582	·47	345755	9·96	654245	30
31	9·335906	9·48	9·989553	·47	9·346353	9·94	10·653647	29
32	336475	9·46	989525	·47	346949	9·93	653051	28
33	337043	9·45	989497	·47	347545	9·92	652455	27
34	337610	9·44	989469	·47	348141	9·91	651859	26
35	338176	9·43	989441	·47	348735	9·90	651265	25
36	338742	9·41	989413	·47	349329	9·88	650671	24
37	339306	9·40	989384	·47	349922	9·87	650078	23
38	339871	9·39	989356	·47	350514	9·86	649486	22
39	340434	9·37	989328	·47	351106	9·85	648894	21
40	340996	9·36	989300	·47	351697	9·83	648303	20
41	9·341558	9·35	9·989271	·47	9·352287	9·82	10·647713	19
42	342119	9·34	989243	·47	352876	9·81	647124	18
43	342679	9·32	989214	·47	353465	9·80	646535	17
44	343239	9·31	989186	·47	354053	9·79	645947	16
45	343797	9·30	989157	·47	354640	9·77	645360	15
46	344355	9·29	989128	·48	355227	9·76	644773	14
47	344912	9·27	989100	·48	355813	9·75	644187	13
48	345469	9·26	989071	·48	356398	9·74	643602	12
49	346024	9·25	989042	·48	356982	9·73	643018	11
50	346579	9·24	989014	·48	357566	9·71	642434	10
51	9·347134	9·22	9·988985	·48	9·358149	9·70	10·641851	9
52	347687	9·21	988956	·48	358731	9·69	641269	8
53	348240	9·20	988927	·48	359313	9·68	640687	7
54	348792	9·19	988898	·48	359893	9·67	640107	6
55	349343	9·17	988869	·48	360474	9·66	639526	5
56	349893	9·16	988840	·48	361053	9·65	638947	4
57	350443	9·15	988811	·49	361632	9·63	638368	3
58	350992	9·14	988782	·49	362210	9·62	637790	2
59	351540	9·13	988753	·49	362787	9·61	637213	1
60	352088	9·11	988724	·49	363364	9·60	636636	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·352088	9·11	9·988724	·49	9·363364	9·60	10·636636	60
1	352635	9·10	988695	·49	363940	9·59	636060	59
2	353181	9·09	988666	·49	364515	9·58	635485	58
3	353726	9·08	988636	·49	365090	9·57	634910	57
4	354271	9·07	988607	·49	365664	9·55	634336	56
5	354815	9·05	988578	·49	366237	9·54	633763	55
6	355358	9·04	988548	·49	366810	9·53	633190	54
7	355901	9·03	988519	·49	367382	9·52	632618	53
8	356443	9·02	988489	·49	367953	9·51	632047	52
9	356984	9·01	988460	·49	368524	9·50	631476	51
10	357524	8·99	988430	·49	369094	9·49	630906	50
11	9·358064	8·98	9·988401	·49	9·369663	9·48	10·630337	49
12	358603	8·97	988371	·49	370232	9·46	629768	48
13	359141	8·96	988342	·49	370799	9·45	629201	47
14	359678	8·95	988312	·50	371367	9·44	628633	46
15	360215	8·93	988282	·50	371933	9·43	628067	45
16	360752	8·92	988252	·50	372499	9·42	627501	44
17	361287	8·91	988223	·50	373064	9·41	626936	43
18	361822	8·90	988193	·50	373629	9·40	626371	42
19	362356	8·89	988163	·50	374193	9·39	625807	41
20	362889	8·88	988133	·50	374756	9·38	625244	40
21	9·363422	8·87	9·988103	·50	9·375319	9·37	10·624681	39
22	363954	8·85	988073	·50	375881	9·35	624119	38
23	364485	8·84	988043	·50	376442	9·34	623558	37
24	365016	8·83	988013	·50	377003	9·33	622997	36
25	365546	8·82	987983	·50	377563	9·32	622437	35
26	366075	8·81	987953	·50	378122	9·31	621878	34
27	366604	8·80	987922	·50	378681	9·30	621319	33
28	367131	8·79	987892	·50	379239	9·29	620761	32
29	367659	8·77	987862	·50	379797	9·28	620203	31
30	368185	8·76	987832	·51	380354	9·27	619646	30
31	9·368711	8·75	9·987801	·51	9·380910	9·26	10·619090	29
32	369236	8·74	987771	·51	381466	9·25	618534	28
33	369761	8·73	987740	·51	382020	9·24	617980	27
34	370285	8·72	987710	·51	382575	9·23	617425	26
35	370808	8·71	987679	·51	383129	9·22	616871	25
36	371330	8·70	987649	·51	383682	9·21	616318	24
37	371852	8·69	987618	·51	384234	9·20	615766	23
38	372373	8·67	987588	·51	384786	9·19	615214	22
39	372894	8·66	987557	·51	385337	9·18	614663	21
40	373414	8·65	987526	·51	385888	9·17	614112	20
41	9·373933	8·64	9·987496	·51	9·386438	9·15	10·613562	19
42	374452	8·63	987465	·51	386987	9·14	613013	18
43	374970	8·62	987434	·51	387536	9·13	612464	17
44	375487	8·61	987403	·52	388084	9·12	611916	16
45	376003	8·60	987372	·52	388631	9·11	611369	15
46	376519	8·59	987341	·52	389178	9·10	610822	14
47	377035	8·58	987310	·52	389724	9·09	610276	13
48	377549	8·57	987279	·52	390270	9·08	609730	12
49	378063	8·56	987248	·52	390815	9·07	609185	11
50	378577	8·54	987217	·52	391360	9·06	608640	10
51	9·379089	8·53	9·987186	·52	9·391903	9·05	10·608097	9
52	379601	8·52	987155	·52	392447	9·04	607553	8
53	380113	8·51	987124	·52	392989	9·03	607011	7
54	380624	8·50	987092	·52	393531	9·02	606469	6
55	381134	8·49	987061	·52	394073	9·01	605927	5
56	381643	8·48	987030	·52	394614	9·00	605386	4
57	382152	8·47	986998	·52	395154	8·99	604846	3
58	382661	8·46	986967	·52	395694	8·98	604306	2
59	383168	8·45	986936	·52	396233	8·97	603767	1
60	383675	8·44	986904	·52	396771	8·96	603229	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(76 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·383675	8·44	9·986904	·52	9·396771	8·96	10·603229	60
1	384182	8·43	986873	·53	397309	8·96	602691	59
2	384687	8·42	986841	·53	397846	8·95	602154	58
3	385192	8·41	986809	·53	398383	8·94	601617	57
4	385697	8·40	986778	·53	398919	8·93	601081	56
5	386201	8·39	986746	·53	399455	8·92	600545	55
6	386704	8·38	986714	·53	399990	8·91	600010	54
7	387207	8·37	986683	·53	400524	8·90	599476	53
8	387709	8·36	986651	·53	401058	8·89	598942	52
9	388210	8·35	986619	·53	401591	8·88	598409	51
10	388711	8·34	986587	·53	402124	8·87	597876	50
11	9·389211	8·33	9·986555	·53	9·402656	8·86	10·597344	49
12	389711	8·32	986523	·53	403187	8·85	596813	48
13	390210	8·31	986491	·53	403718	8·84	596282	47
14	390708	8·30	986459	·53	404249	8·83	595751	46
15	391206	8·28	986427	·53	404778	8·82	595222	45
16	391703	8·27	986395	·53	405308	8·81	594692	44
17	392199	8·26	986363	·54	405836	8·80	594164	43
18	392695	8·25	986331	·54	406364	8·79	593636	42
19	393191	8·24	986299	·54	406892	8·78	593108	41
20	393685	8·23	986266	·54	407419	8·77	592581	40
21	9·394179	8·22	9·986234	·54	9·407945	8·76	10·592055	39
22	394673	8·21	986202	·54	408471	8·75	591529	38
23	395166	8·20	986169	·54	408997	8·74	591003	37
24	395658	8·19	986137	·54	409521	8·74	590479	36
25	396150	8·18	986104	·54	410045	8·73	589955	35
26	396641	8·17	986072	·54	410569	8·72	589431	34
27	397132	8·17	986039	·54	411092	8·71	588908	33
28	397621	8·16	986007	·54	411615	8·70	588385	32
29	398111	8·15	985974	·54	412137	8·69	587863	31
30	398600	8·14	985942	·54	412658	8·68	587342	30
31	9·399088	8·13	9·985909	·55	9·413179	8·67	10·586821	29
32	399575	8·12	985876	·55	413699	8·66	586301	28
33	400062	8·11	985843	·55	414219	8·65	585781	27
34	400549	8·10	985811	·55	414738	8·64	585262	26
35	401035	8·09	985778	·55	415257	8·64	584743	25
36	401520	8·08	985745	·55	415775	8·63	584225	24
37	402005	8·07	985712	·55	416293	8·62	583707	23
38	402489	8·06	985679	·55	416810	8·61	583190	22
39	402972	8·05	985646	·55	417326	8·60	582674	21
40	403455	8·04	985613	·55	417842	8·59	582158	20
41	9·403938	8·03	9·985580	·55	9·418358	8·58	10·581642	19
42	404420	8·02	985547	·55	418873	8·57	581127	18
43	404901	8·01	985514	·55	419387	8·56	580613	17
44	405382	8·00	985480	·55	419901	8·55	580099	16
45	405862	7·99	985447	·55	420415	8·55	579585	15
46	406341	7·98	985414	·56	420927	8·54	579073	14
47	406820	7·97	985380	·56	421440	8·53	578560	13
48	407299	7·96	985347	·56	421952	8·52	578048	12
49	407777	7·95	985314	·56	422463	8·51	577537	11
50	408254	7·94	985280	·56	422974	8·50	577026	10
51	9·408731	7·94	9·985247	·56	9·423484	8·49	10·576516	9
52	409207	7·93	985213	·56	423993	8·48	576007	8
53	409682	7·92	985180	·56	424503	8·48	575497	7
54	410157	7·91	985146	·56	425011	8·47	574989	6
55	410632	7·90	985113	·56	425519	8·46	574481	5
56	411106	7·89	985079	·56	426027	8·45	573973	4
57	411579	7·88	985045	·56	426534	8·44	573466	3
58	412052	7·87	985011	·56	427041	8·43	572959	2
59	412524	7·86	984978	·56	427547	8·43	572453	1
60	412996	7·85	984944	·56	428052	8·42	571948	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·412996	7·85	9·984944	·57	9·428052	8·42	10·571948	60
1	413467	7·84	984910	·57	428557	8·41	571443	59
2	413938	7·83	984876	·57	429062	8·40	570938	58
3	414408	7·83	984842	·57	429566	8·39	570434	57
4	414878	7·82	984808	·57	430070	8·38	569930	56
5	415347	7·81	984774	·57	430573	8·38	569427	55
6	415815	7·80	984740	·57	431075	8·37	568925	54
7	416283	7·79	984706	·57	431577	8·36	568423	53
8	416751	7·78	984672	·57	432079	8·35	567921	52
9	417217	7·77	984637	·57	432580	8·34	567420	51
10	417684	7·76	984603	·57	433080	8·33	566920	50
11	9·418150	7·75	9·984569	·57	9·433580	8·32	10·566420	49
12	418615	7·74	984535	·57	434080	8·32	565920	48
13	419079	7·73	984500	·57	434579	8·31	565421	47
14	419544	7·73	984466	·57	435078	8·30	564922	46
15	420007	7·72	984432	·58	435576	8·29	564424	45
16	420470	7·71	984397	·58	436073	8·28	563927	44
17	420933	7·70	984363	·58	436570	8·28	563430	43
18	421395	7·69	984328	·58	437067	8·27	562933	42
19	421857	7·68	984294	·58	437563	8·26	562437	41
20	422318	7·67	984259	·58	438059	8·25	561941	40
21	9·422778	7·67	9·984224	·58	9·438554	8·24	10·561446	39
22	423238	7·66	984190	·58	439048	8·23	560952	38
23	423697	7·65	984155	·58	439543	8·23	560457	37
24	424156	7·64	984120	·58	440036	8·22	559964	36
25	424615	7·63	984085	·58	440529	8·21	559471	35
26	425073	7·62	984050	·58	441022	8·20	558978	34
27	425530	7·61	984015	·58	441514	8·19	558486	33
28	425987	7·60	983981	·58	442006	8·19	557994	32
29	426443	7·60	983946	·58	442497	8·18	557503	31
30	426899	7·59	983911	·58	442988	8·17	557012	30
31	9·427354	7·58	9·983875	·58	9·443479	8·16	10·556521	29
32	427809	7·57	983840	·59	443968	8·16	556032	28
33	428263	7·56	983805	·59	444458	8·15	555542	27
34	428717	7·55	983770	·59	444947	8·14	555053	26
35	429170	7·54	983735	·59	445435	8·13	554565	25
36	429623	7·53	983700	·59	445923	8·12	554077	24
37	430075	7·52	983664	·59	446411	8·12	553589	23
38	430527	7·52	983629	·59	446898	8·11	553102	22
39	430978	7·51	983594	·59	447384	8·10	552616	21
40	431429	7·50	983558	·59	447870	8·09	552130	20
41	9·431879	7·49	9·983523	·59	9·448356	8·09	10·551644	19
42	432329	7·49	983487	·59	448841	8·08	551159	18
43	432778	7·48	983452	·59	449326	8·07	550674	17
44	433226	7·47	983416	·59	449810	8·06	550190	16
45	433675	7·46	983381	·59	450294	8·06	549706	15
46	434122	7·45	983345	·59	450777	8·05	549223	14
47	434569	7·44	983309	·59	451260	8·04	548740	13
48	435016	7·44	983273	·60	451743	8·03	548257	12
49	435462	7·43	983238	·60	452225	8·02	547775	11
50	435908	7·42	983202	·60	452706	8·02	547294	10
51	9·436353	7·41	9·983166	·60	9·453187	8·01	10·546813	9
52	436798	7·40	983130	·60	453668	8·00	546332	8
53	437242	7·40	983094	·60	454148	7·99	545852	7
54	437686	7·39	983058	·60	454628	7·99	545372	6
55	438129	7·38	983022	·60	455107	7·98	544893	5
56	438572	7·37	982986	·60	455586	7·97	544414	4
57	439014	7·36	982950	·60	456064	7·96	543936	3
58	439456	7·36	982914	·60	456542	7·96	543458	2
59	439897	7·35	982878	·60	457019	7·95	542981	1
60	440338	7·34	982842	·60	457496	7·94	542504	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·440338	7·34	9·982842	·60	9·457496	7·94	10·542504	60
1	440778	7·33	982805	·60	457973	7·93	542027	59
2	441218	7·32	982769	·61	458449	7·93	541551	58
3	441658	7·31	982733	·61	458925	7·92	541075	57
4	492096	7·31	982696	·61	459400	7.91	540600	56
5	442535	7·30	982660	·61	459875	7·90	540125	55
6	442973	7·29	982624	·61	460349	7·90	539651	54
7	443410	7·28	982587	·61	460823	7·89	539177	53
8	443847	7·27	982551	·61	461297	7·88	538703	52
9	444284	7·27	982514	·61	461770	7·88	538230	51
10	444720	7·26	982477	·61	462242	7·87	537758	50
11	9·445155	7·25	9·982441	·61	9·462714	7·86	10·537286	49
12	445590	7·24	982404	·61	463186	7·85	536814	48
13	446025	7·23	982367	·61	463658	7·85	536342	47
14	446459	7·23	982331	·61	464129	7·84	535871	46
15	446893	7·22	982294	·61	464599	7·83	535401	45
16	447326	7·21	982257	·61	465069	7·83	534931	44
17	447759	7·20	982220	.62	465539	7·82	534461	43
18	448191	7·20	982183	.62	466008	7·81	533992	42
19	448623	7·19	982146	.62	466476	7·80	533524	41
20	449054	7·18	982109	.62	466945	7·80	533055	40
21	9·449485	7·17	9·982072	·62	9·467413	7·79	10·532587	39
22	449915	7·16	982035	·62	467880	7·78	532120	38
23	450345	7·16	981998	·62	468347	7·78	531653	37
24	450775	7·15	981961	·62	468814	7·77	531186	36
25	451204	7·14	981924	·62	469280	7·76	530720	35
26	451632	7·13	981886	·62	469746	7·75	530254	34
27	452060	7·13	981849	·62	470211	7·75	529789	33
28	452488	7·12	981812	·62	470676	7·74	529324	32
29	452915	7·11	981774	·62	471141	7·73	528859	31
30	453342	7·10	981737	·62	471605	7·73	528395	30
31	9·453768	7·10	9·981699	·63	9·472068	7·72	10·527932	29
32	454194	7·09	981662	·63	472532	7·71	527468	28
33	454619	7·08	981625	·63	472995	7·71	527005	27
34	455044	7.07	981587	·63	473457	7·70	526543	26
35	455469	7.07	981549	·63	473919	7·69	526081	25
36	455893	7.06	981512	·63	474381	7·69	525619	24
37	456316	7.05	981474	·63	474842	7·68	525158	23
38	456739	7·04	981436	·63	475303	7·67	524697	22
39	457162	7·04	981399	·63	475763	7·67	524237	21
40	457584	7·03	981361	·63	476223	7·66	523777	20
41	9·458006	7·02	9·981323	·63	9·476683	7·65	10·523317	19
42	458427	7·01	981285	·63	477142	7·65	522858	18
43	458848	7·01	981247	·63	477601	7·64	522399	17
44	459268	7·00	981209	·63	478059	7·63	521941	16
45	459688	6·99	981171	·63	478517	7·63	521483	15
46	460108	6·98	981133	·64	478975	7·62	521025	14
47	460527	6·98	981095	·64	479432	7·61	520568	13
48	460946	6·97	981057	·64	479889	7·61	520111	12
49	461364	6·96	981019	·64	480345	7·60	519655	11
50	461782	6·95	980981	·64	480801	7·59	519199	10
51	9·462199	6·95	9·980942	·64	9·481257	7·59	10·518743	9
52	462616	6·94	980904	·64	481712	7·58	518288	8
53	463032	6·93	980866	·64	482167	7·57	517833	7
54	463448	6·93	980827	·64	482621	7·57	517379	6
55	463864	6·92	980789	·64	483075	7·56	516925	5
56	464279	6·91	980750	·64	483529	7·55	516471	4
57	464694	6·90	980712	·64	483982	7·55	516018	3
58	465108	6·90	980673	·64	484435	7·54	515565	2
59	465522	6·89	980635	·64	484887	7·53	515113	1
60	465935	6·88	980596	·64	485339	7·53	514661	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·465935	6·88	9·980596	·64	9·485339	7·53	10·514661	60
1	466348	6·88	980558	·64	485791	7·52	514209	59
2	466761	6·87	980519	·65	486242	7·51	513758	58
3	467173	6·86	980480	·65	486693	7·51	513307	57
4	467585	6·85	980442	·65	487143	7·50	512857	56
5	467996	6·85	980403	·65	487593	7·49	512407	55
6	468407	6·84	980364	·65	488043	7·49	511957	54
7	468817	6·83	980325	·65	488492	7·48	511508	53
8	469227	6·83	980286	·65	488941	7·47	511059	52
9	469637	6·82	980247	·65	489390	7·47	510610	51
10	470046	6·81	980208	·65	489838	7·46	510162	50
11	9·470455	6·80	9·980169	·65	9·490286	7·46	10·509714	49
12	470863	6·80	980130	·65	490733	7·45	509267	48
13	471271	6·79	980091	·65	491180	7·44	508820	47
14	471679	6·78	980052	·65	491627	7·44	508373	46
15	472086	6·78	980012	·65	492073	7·43	507927	45
16	472492	6·77	979973	·65	492519	7·43	507481	44
17	472898	6·76	979934	·66	492965	7·42	507035	43
18	473304	6·76	979895	·66	493410	7·41	506590	42
19	473710	6·75	979855	·66	493854	7·40	506146	41
20	474115	6·74	979816	·66	494299	7·40	505701	40
21	9·474519	6·74	9·979776	·66	9·494743	7·40	10·505257	39
22	474923	6·73	979737	·66	495186	7·39	504814	38
23	475327	6·72	979697	·66	495630	7·38	504370	37
24	475730	6·72	979658	·66	496073	7·37	503927	36
25	476133	6·71	979618	·66	496515	7·37	503485	35
26	476536	6·70	979579	·66	496957	7·36	503043	34
27	476938	6·69	979539	·66	497399	7·36	502601	33
28	477340	6·69	979499	·66	497841	7·35	502159	32
29	477741	6·68	979459	·66	498282	7·34	501718	31
30	478142	6·67	979420	·66	498722	7·34	501278	30
31	9·478542	6·67	9·979380	·66	9·499163	7·33	10·500837	29
32	478942	6·66	979340	·66	499603	7·33	500397	28
33	479342	6·65	979300	·67	500042	7·32	499958	27
34	479741	6·65	979260	·67	500481	7·31	499519	26
35	480140	6·64	979220	·67	500920	7·31	499080	25
36	480539	6·63	979180	·67	501359	7·30	498641	24
37	480937	6·63	979140	·67	501797	7·30	498203	23
38	481334	6·62	979100	·67	502235	7·29	497765	22
39	481731	6·61	979059	·67	502672	7·28	497328	21
40	482128	6·61	979019	·67	503109	7·28	496891	20
41	9·482525	6·60	9·978979	·67	9·503546	7·27	10·496454	19
42	482921	6·59	978939	·67	503982	7·27	496018	18
43	483316	6·59	978898	·67	504418	7·26	495582	17
44	483712	6·58	978858	·67	504854	7·25	495146	16
45	484107	6·57	978817	·67	505289	7·25	494711	15
46	484501	6·57	978777	·67	505724	7·24	494276	14
47	484895	6·56	978736	·67	506159	7·24	493841	13
48	485289	6·55	978696	·68	506593	7·23	493407	12
49	485682	6·55	978655	·68	507027	7·22	492973	11
50	486075	6·54	978615	·68	507460	7·22	492540	10
51	9·486467	6·53	9·978574	·68	9·507893	7·21	10·492107	9
52	486860	6·53	978533	·68	508326	7·21	491674	8
53	487251	6·52	978493	·68	508759	7·20	491241	7
54	487643	6·51	978452	·68	509191	7·19	490809	6
55	488034	6·51	978411	·68	509622	7·19	490378	5
56	488424	6·50	978370	·68	510054	7·18	489946	4
57	488814	6·50	978329	·68	510485	7·18	489515	3
58	489204	6·49	978288	·68	510916	7·17	489084	2
59	489593	6·48	978247	·68	511346	7·16	488654	1
60	489982	6·48	978206	·68	511776	7·16	488224	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·489982	6·48	9·978206	·68	9·511776	7·16	10·488224	60
1	490371	6·48	978165	·68	512206	7·16	487794	59
2	490759	6·47	978124	·68	512635	7·15	487365	58
3	491147	6·46	978083	·69	513064	7·14	486936	57
4	491535	6·46	978042	·69	513493	7·14	486507	56
5	491922	6·45	978001	·69	513921	7·13	486079	55
6	492308	6·44	977959	·69	514349	7·13	485651	54
7	492695	6·44	977918	·69	514777	7·12	485223	53
8	493081	6·43	977877	·69	515204	7·12	484796	52
9	493466	6·42	977835	·69	515631	7·11	484369	51
10	493851	6·42	977794	·69	516057	7·10	483943	50
11	9·494236	6·41	9·977752	·69	9·516484	7·10	10·483516	49
12	494621	6·41	977711	·69	516910	7·09	483090	48
13	495005	6·40	977669	·69	517335	7·09	482665	47
14	495388	6·39	977628	·69	517761	7·08	482239	46
15	495772	6·39	977586	·69	518185	7·08	481815	45
16	496154	6·38	977544	·70	518610	7·07	481390	44
17	496537	6·37	977503	·70	519034	7·06	480966	43
18	496919	6·37	977461	·70	519458	7·06	480542	42
19	497301	6·36	977419	·70	519882	7·05	480118	41
20	497682	6·36	977377	·70	520305	7·05	479695	40
21	9·498064	6·35	9·977335	·70	9·520728	7·04	10·479272	39
22	498444	6·34	977293	·70	521151	7·03	478849	38
23	498825	6·34	977251	·70	521573	7·03	478427	37
24	499204	6·33	977209	·70	521995	7·03	478005	36
25	499584	6·32	977167	·70	522417	7·02	477583	35
26	499963	6·32	977125	·70	522838	7·02	477162	34
27	500342	6·31	977083	·70	523259	7·01	476741	33
28	500721	6·31	977041	·70	523680	7·01	476320	32
29	501099	6·30	976999	·70	524100	7·00	475900	31
30	501476	6·29	976957	·70	524520	6·99	475480	30
31	9·501854	6·29	9·976914	·70	9·524939	6·99	10·475061	29
32	502231	6·28	976872	·71	525359	6·98	474641	28
33	502607	6·28	976830	·71	525778	6·98	474222	27
34	502984	6·27	976787	·71	526197	6·97	473803	26
35	503360	6·26	976745	·71	526615	6·97	473385	25
36	503735	6·26	976702	·71	527033	6·96	472967	24
37	504110	6·25	976660	·71	527451	6·96	472549	23
38	504485	6·25	976617	·71	527868	6·95	472132	22
39	504860	6·24	976574	·71	528285	6·95	471715	21
40	505234	6·23	976532	·71	528702	6·94	471298	20
41	9·505608	6·23	9·976489	·71	9·529119	6·93	10·470881	19
42	505981	6·22	976446	·71	529535	6·93	470465	18
43	506354	6·22	976404	·71	529950	6·93	470050	17
44	506727	6·21	976361	·71	530366	6·92	469634	16
45	507099	6·20	976318	·71	530781	6·91	469219	15
46	507471	6·20	976275	·71	531196	6·91	468804	14
47	507843	6·19	976232	·72	531611	6·90	468389	13
48	508214	6·19	976189	·72	532025	6·90	467975	12
49	508585	6·18	976146	·72	532439	6·89	467561	11
50	508956	6·18	976103	·72	532853	6·89	467147	10
51	9·509326	6·17	9·976060	·72	9·533266	6·88	10·466734	9
52	509696	6·16	976017	·72	533679	6·88	466321	8
53	510065	6·16	975974	·72	534092	6·87	465908	7
54	510434	6·15	975930	·72	534504	6·87	465496	6
55	510803	6·15	975887	·72	534916	6·86	465084	5
56	511172	6·14	975844	·72	535328	6·86	464672	4
57	511540	6·13	975800	·72	535739	6·85	464261	3
58	511907	6·13	975757	·72	536150	6·85	463850	2
59	512275	6·12	975714	·72	536561	6·84	463439	1
60	512642	6·12	975670	·72	536972	6·84	463028	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·512642	6·12	9·975670	·73	9·536972	6·84	10·463028	60
1	513009	6·11	975627	·73	537382	6·83	462618	59
2	513375	6·11	975583	·73	537792	6·83	462208	58
3	513741	6·10	975539	·73	538202	6·82	461798	57
4	514107	6·09	975496	·73	538611	6·82	461389	56
5	514472	6·09	975452	·73	539020	6·81	460980	55
6	514837	6·08	975408	·73	539429	6·81	460571	54
7	515202	6·08	975365	·73	539837	6·80	460163	53
8	515566	6·07	975321	·73	540245	6·80	459755	52
9	515930	6·07	975277	·73	540653	6·79	459347	51
10	516294	6·06	975233	·73	541061	6·79	458939	50
11	9·516657	6·05	9·975189	·73	9·541468	6·78	10·458532	49
12	517020	6·05	975145	·73	541875	6·78	458125	48
13	517382	6·04	975101	·73	542281	6·77	457719	47
14	517745	6·04	975057	·73	542688	6·77	457312	46
15	518107	6·03	975013	·73	543094	6·76	456906	45
16	518468	6·03	974969	·74	543499	6·76	456501	44
17	518829	6·02	974925	·74	543905	6·75	456095	43
18	519190	6·01	974880	·74	544310	6·75	455690	42
19	519551	6·01	974836	·74	544715	6·74	455285	41
20	519911	6·00	974792	·74	545119	6·74	454881	40
21	9·520271	6·00	9·974748	·74	9·545524	6·73	10·454476	39
22	520631	5·99	974703	·74	545928	6·73	454072	38
23	520990	5·99	974659	·74	546331	6·72	453669	37
24	521349	5·98	974614	·74	546735	6·72	453265	36
25	521707	5·98	974570	·74	547138	6·71	452862	35
26	522066	5·97	974525	·74	547540	6·71	452460	34
27	522424	5·96	974481	·74	547943	6·70	452057	33
28	522781	5·96	974436	·74	548345	6·70	451655	32
29	523138	5·95	974391	·74	548747	6·69	451253	31
30	523495	5·95	974347	·75	549149	6·69	450851	30
31	9·523852	5·94	9·974302	·75	9·549550	6·68	10·450450	29
32	524208	5·94	974257	·75	549951	6·68	450049	28
33	524564	5·93	974212	·75	550352	6·67	449648	27
34	524920	5·93	974167	·75	550752	6·67	449248	26
35	525275	5·92	974122	·75	551152	6·66	448848	25
36	525630	5·91	974077	·75	551552	6·66	448448	24
37	525984	5·91	974032	·75	551952	6·65	448048	23
38	526339	5·90	973987	·75	552351	6·65	447649	22
39	526693	5·90	973942	·75	552750	6·65	447250	21
40	527046	5·89	973897	·75	553149	6·64	446851	20
41	9·527400	5·89	9·973852	·75	9·553548	6·64	10·446452	19
42	527753	5·88	973807	·75	553946	6·63	446054	18
43	528105	5·88	973761	·75	554344	6·63	445656	17
44	528458	5·87	973716	·76	554741	6·62	445259	16
45	528810	5·87	973671	·76	555139	6·62	444861	15
46	529161	5·86	972625	·76	555536	6·61	444464	14
47	529513	5·86	973580	·76	555933	6·61	444067	13
48	529864	5·85	973535	·76	556329	6·60	443671	12
49	530215	5·85	973489	·76	556725	6·60	443275	11
50	530565	5·84	973444	·76	557121	6·59	442879	10
51	9·530915	5·84	9·973398	·76	9·557517	6·59	10·442483	9
52	531265	5·83	973352	·76	557913	6·59	442087	8
53	531614	5·82	973307	·76	558308	6·58	441692	7
54	531963	5·82	973261	·76	558702	6·58	441298	6
55	532312	5·81	973215	·76	559097	6·57	440903	5
56	532661	5·81	973169	·76	559491	6·57	440509	4
57	533009	5·80	973124	·76	559885	6·56	440115	3
58	533357	5·80	973078	·76	560279	6·56	439721	2
59	533704	5·79	973032	·77	560673	6·55	439327	1
60	534052	5·78	972986	·77	561066	6·55	438934	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·534052	5·78	9·972986	·77	9·561066	6·55	10·438934	60
1	534399	5·77	972940	·77	561459	6·54	438541	59
2	534745	5·77	972894	·77	561851	6·54	438149	58
3	535092	5·77	972848	·77	562244	6·53	437756	57
4	535438	5·76	972802	·77	562636	6·53	437364	56
5	535783	5·76	972755	·77	563028	6·53	436972	55
6	536129	5·75	972709	·77	563419	6·52	436581	54
7	536474	5·74	972663	·77	563811	6·52	436189	53
8	536818	5·74	972617	·77	564202	6·51	435798	52
9	537163	5·73	972570	·77	564592	6·51	435408	51
10	537507	5·73	972524	·77	564983	6·50	435017	50
11	9·537851	5·72	9·972478	·77	9·565373	6·50	10·434627	49
12	538194	5·72	972431	·78	565763	6·49	434237	48
13	538538	5·71	972385	·78	566153	6·49	433847	47
14	538880	5·71	972338	·78	566542	6·49	433458	46
15	539223	5·70	972291	·78	566932	6·48	433068	45
16	539565	5·70	972245	·78	567320	6·48	432680	44
17	539907	5·69	972198	·78	567709	6·47	432291	43
18	540249	5·69	972151	·78	568098	6·47	431902	42
19	540590	5·68	972105	·78	568486	6·46	431514	41
20	540931	5·68	972058	·78	568873	6·46	431127	40
21	9·541272	5·67	9·972011	·78	9·569261	6·45	10·430739	39
22	541613	5·67	971964	·78	569648	6·45	430352	38
23	541953	5·66	971917	·78	570035	6·45	429965	37
24	542293	5·66	971870	·78	570422	6·44	429578	36
25	542632	5·65	971823	·78	570809	6·44	429191	35
26	542971	5·65	971776	·78	571195	6·43	428805	34
27	543310	5·64	971729	·79	571581	6·43	428419	33
28	543649	5·64	971682	·79	571967	6·42	428033	32
29	543987	5·63	971635	·79	572352	6·42	427648	31
30	544325	5·63	971588	·79	572738	6·42	427262	30
31	9·544663	5·62	9·971540	·79	9·573123	6·41	10·426877	29
32	545000	5·62	971493	·79	573507	6·41	426493	28
33	545338	5·61	971446	·79	573892	6·40	426108	27
34	545674	5·61	971398	·79	574276	6·40	425724	26
35	546011	5·60	971351	·79	574660	6·39	425340	25
36	546347	5·60	971303	·79	575044	6·39	424956	24
37	546683	5·59	971256	·79	575427	6·39	424573	23
38	547019	5·59	971208	·79	575810	6·38	424190	22
39	547354	5·58	971161	·79	576193	6·38	423807	21
40	547689	5·58	971113	·79	576576	6·37	423424	20
41	9·548024	5·57	9·971066	·80	9·576958	6·37	10·423041	19
42	548359	5·57	971018	·80	577341	6·36	422659	18
43	548693	5·56	970970	·80	577723	6·36	422277	17
44	549027	5·56	970922	·80	578104	6·36	421896	16
45	549360	5·55	970874	·80	578486	6·35	421514	15
46	549693	5·55	970827	·80	578867	6·35	421133	14
47	550026	5·54	970779	·80	579248	6·34	420752	13
48	550359	5·54	970731	·80	579629	6·34	420371	12
49	550692	5·53	970683	·80	580009	6·34	419991	11
50	551024	5·53	970635	·80	580389	6·33	419611	10
51	9·551356	5·52	9·970586	·80	9·580769	6·33	10·419231	9
52	551687	5·52	970538	·80	581149	6·32	418851	8
53	552018	5·52	970490	·80	581528	6·32	418472	7
54	552349	5·51	970442	·80	581907	6·32	418093	6
55	552680	5·51	970394	·80	582286	6·31	417714	5
56	553010	5·50	970345	·81	582665	6·31	417335	4
57	553341	5·50	970297	·81	583043	6·30	416957	3
58	553670	5·49	970249	·81	583422	6·30	416578	2
59	554000	5·49	970200	·81	583800	6·29	416200	1
60	554329	5·48	970152	·81	584177	6·29	415823	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(69 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·554329	5·48	9·970152	·81	9·584177	6·29	10·415823	60
1	554658	5·48	970103	·81	584555	6·29	415445	59
2	554987	5·47	970055	·81	584932	6·28	415068	58
3	555315	5·47	970006	·81	585309	6·28	414691	57
4	555643	5·46	969957	·81	585686	6·27	414314	56
5	555971	5·46	969909	·81	586062	6·27	413938	55
6	556299	5·45	969860	·81	586439	6·27	413561	54
7	556626	5·45	969811	·81	586815	6·26	413185	53
8	556953	5·44	969762	·81	587190	6·26	412810	52
9	557280	5·44	969714	·81	587566	6·25	412434	51
10	557606	5·43	969665	·81	587941	6·25	412059	50
11	9·557932	5·43	9·969616	·82	9·588316	6·25	10·411684	49
12	558258	5·43	969567	·82	588691	6·24	411309	48
13	558583	5·42	969518	·82	589066	6·24	410934	47
14	558909	5·42	969469	·82	589440	6·23	410560	46
15	559234	5·41	969420	·82	589814	6·23	410186	45
16	559558	5·41	969370	·82	590188	6·23	409812	44
17	559883	5·40	969321	·82	590562	6·22	409438	43
18	560207	5·40	969272	·82	590935	6·22	409065	42
19	560531	5·39	969223	·82	591308	6·22	408692	41
20	560855	5·39	969173	·82	591681	6·21	408319	40
21	9·561178	5·38	9·969124	·82	9·592054	6·21	10·407946	39
22	561501	5·38	969075	·82	592426	6·20	407574	38
23	561824	5·37	969025	·82	592798	6·20	407202	37
24	562146	5·37	968976	·82	593170	6·19	406829	36
25	562468	5·36	968926	·83	593542	6·19	406458	35
26	562790	5·36	968877	·83	593914	6·18	406086	34
27	563112	5·36	968827	·83	594285	6·18	405715	33
28	563433	5·35	968777	·83	594656	6·18	405344	32
29	563755	5·35	968728	·83	595027	6·17	404973	31
30	564075	5·34	968678	·83	595398	6·17	404602	30
31	9·564396	5·34	9·968628	·83	9·595768	6·17	10·404232	29
32	564716	5·33	968578	·83	596138	6·16	403862	28
33	565036	5·33	968528	·83	596508	6·16	403492	27
34	565356	5·32	968479	·83	596878	6·16	403122	26
35	565676	5·32	968429	·83	597247	6·15	402753	25
36	565995	5·31	968379	·83	597616	6·15	402384	24
37	566314	5·31	968329	·83	597985	6·15	402015	23
38	566632	5·31	968278	·83	598354	6·14	401646	22
39	566951	5·30	968228	·84	598722	6·14	401278	21
40	567269	5·30	968178	·84	599091	6·13	400909	20
41	9·567587	5·29	9·968128	·84	9·599459	6·13	10·400541	19
42	567904	5·29	968078	·84	599827	6·13	400173	18
43	568222	5·28	968027	·84	600194	6·12	399806	17
44	568539	5·28	967977	·84	600562	6·12	399438	16
45	568856	5·28	967927	·84	600929	6·11	399071	15
46	569172	5·27	967876	·84	601296	6·11	398704	14
47	569488	5·27	967826	·84	601662	6·11	398338	13
48	569804	5·26	967775	·84	602029	6·10	397971	12
49	570120	5·26	967725	·84	602395	6·10	397605	11
50	570435	5·25	967674	·84	602761	6·10	397239	10
51	9·570751	5·25	9·967624	·84	9·603127	6·09	10·396873	9
52	571066	5·24	967573	·84	603493	6·09	396507	8
53	571380	5·24	967522	·85	603858	6·09	396142	7
54	571695	5·23	967471	·85	604223	6·08	395777	6
55	572009	5·23	967421	·85	604588	6·08	395412	5
56	572323	5·23	967370	·85	604953	6·07	395047	4
57	572636	5·22	967319	·85	605317	6·07	394683	3
58	572950	5·22	967268	·85	605682	6·07	394318	2
59	573263	5·21	967217	·85	606046	6·06	393954	1
60	573575	5·21	967166	·85	606410	6·06	393590	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(68 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·573575	5·21	9·967166	·85	9·606410	6·06	10·393590	60
1	573888	5·20	967115	·85	606773	6·06	393227	59
2	574200	5·20	967064	·85	607137	6·05	392863	58
3	574512	5·19	967013	·85	607500	6·05	392500	57
4	574824	5·19	966961	·85	607863	6·04	392137	56
5	575136	5·19	966910	·85	608225	6·04	391775	55
6	575447	5·18	966859	·85	608588	6·04	391412	54
7	575758	5·18	966808	·85	608950	6·03	391050	53
8	576069	5·17	966756	·86	609312	6·03	390688	52
9	576379	5·17	966705	·86	609674	6·03	390326	51
10	576689	5·16	966653	·86	610036	6·02	389964	50
11	9·576999	5·16	9·966602	·86	9·610397	6·02	10·389603	49
12	577309	5·16	966550	·86	610759	6·02	389241	48
13	577618	5·15	966499	·86	611120	6·01	388880	47
14	577927	5·15	966447	·86	611480	6·01	388520	46
15	578236	5·14	966395	·86	611841	6·01	388159	45
16	578545	5·14	966344	·86	612201	6·00	387799	44
17	578853	5·13	966292	·86	612561	6·00	387439	43
18	579162	5·13	966240	·86	612921	6·00	387079	42
19	579470	5·13	966188	·86	613281	5·99	386719	41
20	579777	5·12	966136	·86	613641	5·99	386359	40
21	9·580085	5·12	9·966085	·87	9·614000	5·98	10·386000	39
22	580392	5·11	966033	·87	614359	5·98	385641	38
23	580699	5·11	965981	·87	614718	5·98	385282	37
24	581005	5·11	965928	·87	615077	5·97	384923	36
25	581312	5·10	965876	·87	615435	5·97	384565	35
26	581618	5·10	965824	·87	615793	5·97	384207	34
27	581924	5·09	965772	·87	616151	5·96	383849	33
28	582229	5·09	965720	·87	616509	5·96	383491	32
29	582525	5·09	965668	·87	616867	5·96	383133	31
30	582840	5·08	965615	·87	617224	5·95	382776	30
31	9·583145	5·08	9·965563	·87	9·617582	5·95	10·382418	29
32	583449	5·07	965511	·87	617939	5·95	382061	28
33	583754	5·07	965458	·87	618295	5·94	381705	27
34	584058	5·06	965406	·87	618652	5·94	381348	26
35	584361	5·06	965353	·88	619008	5·94	380992	25
36	584665	5·06	965301	·88	619364	5·93	380636	24
37	584968	5·05	965248	·88	619721	5·93	380279	23
38	585272	5·05	965195	·88	620076	5·93	379924	22
39	585574	5·04	965143	·88	620432	5·92	379568	21
40	585877	5·04	965090	·88	620787	5·92	379213	20
41	9·586179	5·03	9·965037	·88	9·621142	5·92	10·378858	19
42	586482	5·03	964984	·88	621497	5·91	378503	18
43	586783	5·03	964931	·88	621852	5·91	378148	17
44	587085	5·02	964879	·88	622207	5·90	377793	16
45	587386	5·02	964826	·88	622561	5·90	377439	15
46	587688	5·01	964773	·88	622915	5·90	377085	14
47	587989	5·01	964719	·88	623269	5·89	376731	13
48	588289	5·01	964666	·89	623623	5·89	376377	12
49	588590	5·00	964613	·89	623976	5·89	376024	11
50	588890	5·00	964560	·89	624330	5·88	375670	10
51	9·589190	4·99	9·964507	·89	9·624683	5·88	10·375317	9
52	589489	4·99	964454	·89	625036	5·88	374964	8
53	589789	4·99	964400	·89	625388	5·87	374612	7
54	590088	4·98	964347	·89	625741	5·87	374259	6
55	590387	4·98	964294	·89	626093	5·87	373907	5
56	590686	4·97	964240	·89	626445	5·86	373555	4
57	590984	4·97	964187	·89	626797	5·86	373203	3
58	591282	4·97	964133	·89	627149	5·86	372851	2
59	591580	4·96	964080	·89	627501	5·85	372499	1
60	591878	4·96	964026	·89	627852	5·85	372148	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·591878	4·96	9·964026	·89	9·627852	5·85	10·372148	60
1	592176	4·95	963972	·89	628203	5·85	371797	59
2	592473	4·95	963919	·89	628554	5·85	371446	58
3	592770	4·95	963865	·90	628905	5·84	371095	57
4	593067	4·94	963811	·90	629255	5·84	370745	56
5	593363	4·94	963757	·90	629606	5·83	370394	55
6	593659	4·93	963704	·90	629956	5·83	370044	54
7	593955	4·93	963650	·90	630306	5·83	369694	53
8	594251	4·93	963596	·90	630656	5·83	369344	52
9	594547	4·92	963542	·90	631005	5·82	368995	51
10	594842	4·92	963488	·90	631355	5·82	368645	50
11	9·595137	4·91	9·963434	·90	9·631704	5·82	10·368296	49
12	595432	4·91	963379	·90	632053	5·81	367947	48
13	595727	4·91	963325	·90	632401	5·81	367599	47
14	596021	4·90	963271	·90	632750	5·81	367250	46
15	596315	4·90	963217	·90	633098	5·80	366902	45
16	596609	4·89	963163	·90	633447	5·80	366553	44
17	596903	4·89	963108	·91	633795	5·80	366205	43
18	597196	4·89	963054	·91	634143	5·79	365857	42
19	597490	4·88	962999	·91	634490	5·79	365510	41
20	597783	4·88	962945	·91	634838	5·79	365162	40
21	9·598075	4·87	9·962890	·91	9·635185	5·78	10·364815	39
22	598368	4·87	962836	·91	635532	5·78	364468	38
23	598660	4·87	962781	·91	635879	5·78	364121	37
24	598952	4·86	962727	·91	636226	5·77	363774	36
25	599244	4·86	962672	·91	636572	5·77	363428	35
26	599536	4·85	962617	·91	636919	5·77	363081	34
27	599827	4·85	962562	·91	637265	5·77	362735	33
28	600118	4·85	962508	·91	637611	5·76	362389	32
29	600409	4·84	962453	·91	637956	5·76	362044	31
30	600700	4·84	962398	·92	638302	5·76	361698	30
31	9·600990	4·84	9·962343	·92	9·638647	5·75	10·361353	29
32	601280	4·83	962288	·92	638992	5·75	361008	28
33	601570	4·83	962233	·92	639337	5·75	360663	27
34	601860	4·82	962178	·92	639682	5·74	360318	26
35	602150	4·82	962123	·92	640027	5·74	359973	25
36	602439	4·82	962067	·92	640371	5·74	359629	24
37	602728	4·81	962012	·92	640716	5·73	359284	23
38	603017	4·81	961957	·92	641060	5·73	358940	22
39	603305	4·81	961902	·92	641404	5·73	358596	21
40	603594	4·80	961846	·92	641747	5·72	358253	20
41	9·603882	4·80	9·961791	·92	9·642091	5·72	10·357909	19
42	604170	4·79	961735	·92	642434	5·72	357566	18
43	604457	4·79	961680	·92	642777	5·72	357223	17
44	604745	4·79	961624	·93	643120	5·71	356880	16
45	605032	4·78	961569	·93	643463	5·71	356537	15
46	605319	4·78	961513	·93	643806	5·71	356194	14
47	605606	4·78	961458	·93	644148	5·70	355852	13
48	605892	4·77	961402	·93	644490	5·70	355510	12
49	606179	4·77	961346	·93	644832	5·70	355168	11
50	606465	4·76	961290	·93	645174	5·69	354826	10
51	9·606751	4·76	9·961235	·93	9·645516	5·69	10·354484	9
52	607036	4·76	961179	·93	645857	5·69	354143	8
53	607322	4·75	961123	·93	646199	5·69	353801	7
54	607607	4·75	961067	·93	646540	5·68	353460	6
55	607892	4·74	961011	·93	646881	5·68	353119	5
56	608177	4·74	960955	·93	647222	5·68	352778	4
57	608461	4·74	960899	·93	647562	5·67	352438	3
58	608745	4·73	960843	·94	647903	5·67	352097	2
59	609029	4·73	960786	·94	648243	5·67	351757	1
60	609313	4·73	960730	·94	648583	5·66	351417	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(66 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·609313	4·73	9·960730	·94	9·648583	5·66	10·351417	60
1	609597	4·72	960674	·94	648923	5·66	351077	59
2	609880	4·72	960618	·94	649263	5·66	350737	58
3	610164	4·72	960561	·94	649602	5·66	350398	57
4	610447	4·71	960505	·94	649942	5·65	350058	56
5	610729	4·71	960448	·94	650281	5·65	349719	55
6	611012	4·70	960392	·94	650620	5·65	349380	54
7	611294	4·70	960335	·94	650959	5·64	349041	53
8	611576	4·70	960279	·94	651297	5·64	348703	52
9	611858	4·69	960222	·94	651636	5·64	348364	51
10	612140	4·69	960165	·94	651974	5·63	348026	50
11	9·612421	4·69	9·960109	·95	9·652312	5·63	10·347688	49
12	612702	4·68	960052	·95	652650	5·63	347350	48
13	612983	4·68	959995	·95	652988	5·63	347012	47
14	613264	4·67	959938	·95	653326	5·62	346674	46
15	613545	4·67	959882	·95	653663	5·62	346337	45
16	613825	4·67	959825	·95	654000	5·62	346000	44
17	614105	4·66	959768	·95	654337	5·61	345663	43
18	614385	4·66	959711	·95	654674	5·61	345326	42
19	614665	4·66	959654	·95	655011	5·61	344989	41
20	614944	4·65	959596	·95	655348	5·61	344652	40
21	9·615223	4·65	9·959539	·95	9·655684	5·60	10·344316	39
22	615502	4·65	959482	·95	656020	5·60	343980	38
23	615781	4·64	959425	·95	656356	5·60	343644	37
24	616060	4·64	959368	·95	656692	5·59	343308	36
25	616338	4·64	959310	·96	657028	5·59	342972	35
26	616616	4·63	959253	·96	657364	5·59	342636	34
27	616894	4·63	959195	·96	657699	5·59	342301	33
28	617172	4·62	959138	·96	658034	5·58	341966	32
29	617450	4·62	959081	·96	658369	5·58	341631	31
30	617727	4·62	959023	·96	658704	5·58	341296	30
31	9·618004	4·61	9·958965	·96	9·659039	5·58	10·340961	29
32	618281	4·61	958908	·96	659373	5·57	340627	28
33	618558	4·61	958850	·96	659708	5·57	340292	27
34	618834	4·60	958792	·96	660042	5·57	339958	26
35	619110	4·60	958734	·96	660376	5·57	339624	25
36	619386	4·60	958677	·96	660710	5·56	339290	24
37	619662	4·59	958619	·96	661043	5·56	338957	23
38	619938	4·59	958561	·96	661377	5·56	338623	22
39	620213	4·59	958503	·97	661710	5·55	338290	21
40	620488	4·58	958445	·97	662043	5·55	337957	20
41	9·620763	4·58	9·958387	·97	9·662376	5·55	10·337624	19
42	621038	4·57	958329	·97	662709	5·54	337291	18
43	621313	4·57	958271	·97	663042	5·54	336958	17
44	621587	4·57	958213	·97	663375	5·54	336625	16
45	621861	4·56	958154	·97	663707	5·54	336293	15
46	622135	4·56	958096	·97	664039	5·53	335961	14
47	622409	4·56	958038	·97	664371	5·53	335629	13
48	622682	4·55	957979	·97	664703	5·53	335297	12
49	622956	4·55	957921	·97	665035	5·53	334965	11
50	623229	4·55	957863	·97	665366	5·52	334634	10
51	9·623502	4·54	9·957804	·97	9·665697	5·52	10·334303	9
52	623774	4·54	957746	·98	666029	5·52	333971	8
53	624047	4·54	957687	·98	666360	5·51	333640	7
54	624319	4·53	957628	·98	666691	5·51	333309	6
55	624591	4·53	957570	·98	667021	5·51	332979	5
56	624863	4·53	957511	·98	667352	5·51	332648	4
57	625135	4·52	957452	·98	667682	5·50	332318	3
58	625406	4·52	957393	·98	668013	5·50	331987	2
59	625677	4·52	957335	·98	668343	5·50	331657	1
60	625948	4·51	957276	·98	668672	5·50	331328	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·625948	4·51	9·957276	·98	9·668673	5·50	10·331327	60
1	626219	4·51	957217	·98	669002	5·49	330998	59
2	626490	4·51	957158	·98	669332	5·49	330668	58
3	626760	4·50	957099	·98	669661	5·49	330339	57
4	627030	4·50	957040	·98	669991	5·48	330009	56
5	627300	4·50	956981	·98	670320	5·48	329680	55
6	627570	4·49	956921	·99	670649	5·48	329351	54
7	627840	4·49	956862	·99	670977	5·48	329023	53
8	628109	4·49	956803	·99	671306	5·47	328694	52
9	628378	4·48	956744	·99	671634	5·47	328366	51
10	628647	4·48	956684	·99	671963	5·47	328037	50
11	9·628916	4·47	9·956625	·99	9·672291	5·47	10·327709	49
12	629185	4·47	956566	·99	672619	5·46	327381	48
13	629453	4·47	956506	·99	672947	5·46	327053	47
14	629721	4·46	956447	·99	673274	5·46	326726	46
15	629989	4·46	956387	·99	673602	5·46	326398	45
16	630257	4·46	956327	·99	673929	5·45	326071	44
17	630524	4·46	956268	·99	674257	5·45	325743	43
18	630792	4·45	956208	1·00	674584	5·45	325416	42
19	631059	4·45	956148	1·00	674910	5·44	325090	41
20	631326	4·45	956089	1·00	675237	5·44	324763	40
21	9·631593	4·44	9·956029	1·00	9·675564	5·44	10·324436	39
22	631859	4·44	955969	1·00	675890	5·44	324110	38
23	632125	4·44	955909	1·00	676216	5·43	323784	37
24	632392	4·43	955849	1·00	676543	5·43	323457	36
25	632658	4·43	955789	1·00	676869	5·43	323131	35
26	632923	4·43	955729	1·00	677194	5·43	322806	34
27	633189	4·42	955669	1·00	677520	5·42	322480	33
28	633454	4·42	955609	1·00	677846	5·42	322154	32
29	633719	4·42	955548	1·00	678171	5·42	321829	31
30	633984	4·41	955488	1·00	678496	5·42	321504	30
31	9·634249	4·41	9·955428	1·01	9·678821	5·41	10·321179	29
32	634514	4·40	955368	1·01	679146	5·41	320854	28
33	634778	4·40	955307	1·01	679471	5·41	320529	27
34	635042	4·40	955247	1·01	679795	5·41	320205	26
35	635306	4·39	955186	1·01	680120	5·40	319880	25
36	635570	4·39	955126	1·01	680444	5·40	319556	24
37	635834	4·39	955065	1·01	680768	5·40	319232	23
38	636097	4·38	955005	1·01	681092	5·40	318908	22
39	636360	4·38	954944	1·01	681416	5·39	318584	21
40	636623	4·38	954883	1·01	681740	5·39	318260	20
41	9·636886	4·37	9·954823	1·01	9·682063	5·39	10·317937	19
42	637148	4·37	954762	1·01	682387	5·39	317613	18
43	637411	4·37	954701	1·01	682710	5·38	317290	17
44	637673	4·37	954640	1·01	683033	5·38	316967	16
45	637935	4·36	954579	1·01	683356	5·38	316644	15
46	638197	4·36	954518	1·02	683679	5·38	316321	14
47	638458	4·36	954457	1·02	684001	5·37	315999	13
48	638720	4·35	954396	1·02	684324	5·37	315676	12
49	638981	4·35	954335	1·02	684646	5·37	315354	11
50	639242	4·35	954274	1·02	684968	5·37	315032	10
51	9·639503	4·34	9·954213	1·02	9·685290	5·36	10·314710	9
52	639764	4·34	954152	1·02	685612	5·36	314388	8
53	640024	4·34	954090	1·02	685934	5·36	314066	7
54	640284	4·33	954029	1·02	686255	5·36	313745	6
55	640544	4·33	953968	1·02	686577	5·35	313423	5
56	640804	4·33	953906	1·02	686898	5·35	313102	4
57	641064	4·32	953845	1·02	687219	5·35	312781	3
58	641324	4·32	953783	1·02	687540	5·35	312460	2
59	641584	4·32	953722	1·03	687861	5·34	312139	1
60	641842	4·31	953660	1·03	688182	5·34	311818	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(64 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·641842	4·31	9·953660	1·03	9·688182	5·34	10·311818	60
1	642101	4·31	953599	1·03	688502	5·34	311498	59
2	642360	4·31	953537	1·03	688823	5·34	311177	58
3	642618	4·30	953475	1·03	689143	5·33	310857	57
4	642877	4·30	953413	1·03	689463	5·33	310537	56
5	643135	4·30	953352	1·03	689783	5·33	310217	55
6	643393	4·30	953290	1·03	690103	5·33	309897	54
7	643650	4·29	953228	1·03	690423	5·33	309577	53
8	643908	4·29	953166	1·03	690742	5·32	309258	52
9	644165	4·29	953104	1·03	691062	5·32	308938	51
10	644423	4·28	953042	1·03	691381	5·32	308619	50
11	9·644680	4·28	9·952980	1·04	9·691700	5·31	10·308300	49
12	644936	4·28	952918	1·04	692019	5·31	307981	48
13	645193	4·27	952855	1·04	692338	5·31	307662	47
14	645450	4·27	952793	1·04	692656	5·31	307344	46
15	645706	4·27	952731	1·04	692975	5·31	307025	45
16	645962	4·26	952669	1·04	693293	5·30	306707	44
17	646218	4·26	952606	1·04	693612	5·30	306388	43
18	646474	4·26	952544	1·04	693930	5·30	306070	42
19	646729	4·25	952481	1·04	694248	5·30	305752	41
20	646984	4·25	952419	1·04	694566	5·29	305434	40
21	9·647240	4·25	9·952356	1·04	9·694883	5·29	10·305117	39
22	647494	4·24	952294	1·04	695201	5·29	304799	38
23	647749	4·24	952231	1·04	695518	5·29	304482	37
24	648004	4·24	952168	1·05	695836	5·29	304164	36
25	648258	4·24	952106	1·05	696153	5·28	303847	35
26	648512	4·23	952043	1·05	696470	5·28	303530	34
27	648766	4·23	951980	1·05	696787	5·28	303213	33
28	649020	4·23	951917	1·05	697103	5·28	302897	32
29	649274	4·22	951854	1·05	697420	5·27	302580	31
30	649527	4·22	951791	1·05	697736	5·27	302264	30
31	9·649781	4·22	9·951728	1·05	9·698053	5·27	10·301947	29
32	650034	4·22	951665	1·05	698369	5·27	301631	28
33	650287	4·21	951602	1·05	698685	5·26	301315	27
34	650539	4·21	951539	1·05	699001	5·26	300999	26
35	650792	4·21	951476	1·05	699316	5·26	300684	25
36	·651044	4·20	951412	1·05	699632	5·26	300368	24
37	651297	4·20	951349	1·06	699947	5·26	300053	23
38	651549	4·20	951286	1·06	700263	5·25	299737	22
39	651800	4·19	951222	1·06	700578	5·25	299422	21
40	652052	4·19	951159	1·06	700893	5·25	299107	20
41	9·652304	4·19	9·951096	1·06	9·701208	5·24	10·298792	19
42	652555	4·18	951032	1·06	701523	5·24	298477	18
43	652806	4·18	950968	1·06	701837	5·24	298163	17
44	653057	4·18	950905	1·06	702152	5·24	297848	16
45	653308	4·18	950841	1·06	702466	5·24	297534	15
46	653558	4·17	950778	1·06	702780	5·23	297220	14
47	653808	4·17	950714	1·06	703095	5·23	296905	13
48	654059	4·17	950650	1·06	703409	5·23	296591	12
49	654309	4·16	950586	1·06	703723	5·23	296277	11
50	654558	4·16	950522	1·07	704036	5·22	295964	10
51	9·654808	4·16	9·950458	1·07	9·704350	5·22	10·295650	9
52	655058	4·16	950394	1·07	704663	5·22	295337	8
53	655307	4·15	950330	1·07	704977	5·22	295023	7
54	655556	4·15	950266	1·07	705290	5·22	294710	6
55	655805	4·15	950202	1·07	705603	5·21	294397	5
56	656054	4·14	950138	1·07	705916	5·21	294084	4
57	656302	4·14	950074	1·07	706228	5·21	293772	3
58	656551	4·14	950010	1·07	706541	5·21	293459	2
59	656799	4·13	949945	1·07	706854	5·21	293146	1
60	657047	4·13	949881	1·07	707166	5·20	292834	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·657047	4·13	9·949881	1·07	9·707166	5·20	10·292834	60
1	657295	4·13	949816	1·07	707478	5·20	292522	59
2	657542	4·12	949752	1·07	707790	5·20	292210	58
3	657790	4·12	949688	1·08	708102	5·20	291898	57
4	658037	4·12	949623	1·08	708414	5·19	291586	56
5	658284	4·12	949558	1·08	708726	5·19	291274	55
6	658531	4·11	949494	1·08	709037	5·19	290963	54
7	658778	4·11	949429	1·08	709349	5·19	290651	53
8	659025	4·11	949364	1·08	709660	5·19	290340	52
9	659271	4·10	949300	1·08	709971	5·18	290029	51
10	659517	4·10	949235	1·08	710282	5·18	289718	50
11	9·659763	4·10	9·949170	1·08	9·710593	5·18	10·289407	49
12	660009	4·09	949105	1·08	710904	5·18	289096	48
13	660255	4·09	949040	1·08	711215	5·18	288785	47
14	660501	4·09	948975	1·08	711525	5·17	288475	46
15	660746	4·09	948910	1·08	711836	5·17	288164	45
16	660991	4·08	948845	1·08	712146	5·17	287854	44
17	661236	4·08	948780	1·09	712456	5·17	287544	43
18	661481	4·08	948715	1·09	712766	5·16	287234	42
19	661726	4·07	948650	1·09	713076	5·16	286924	41
20	661970	4·07	948584	1·09	713386	5·16	286614	40
21	9·662214	4·07	9·948519	1·09	9·713696	5·16	10·286304	39
22	662459	4·07	948454	1·09	714005	5·16	285995	38
23	662703	4·06	948388	1·09	714314	5·15	285686	37
24	662946	4·06	948323	1·09	714624	5·15	285376	36
25	663190	4·06	948257	1·09	714933	5·15	285067	35
26	663433	4·05	948192	1·09	715242	5·15	284758	34
27	663677	4·05	948126	1·09	715551	5·14	284449	33
28	663920	4·05	948060	1·09	715860	5·14	284140	32
29	664163	4·05	947995	1·10	716168	5·14	283832	31
30	664406	4·04	947929	1·10	716477	5·14	283523	30
31	9·664648	4·04	9·947863	1·10	9·716785	5·14	10·283215	29
32	664891	4·04	947797	1·10	717093	5·13	282907	28
33	665133	4·03	947731	1·10	717401	5·13	282599	27
34	665375	4·03	947665	1·10	717709	5·13	282291	26
35	665617	4·03	947600	1·10	718017	5·13	281983	25
36	665859	4·02	947533	1·10	718325	5·13	281670	24
37	666100	4·02	947467	1·10	718633	5·12	281367	23
38	666342	4·02	947401	1·10	718940	5·12	281060	22
39	666583	4·02	947335	1·10	719248	5·12	280752	21
40	666824	4·01	947269	1·10	719555	5·12	280445	20
41	9·667065	4·01	9·947203	1·10	9·719862	5·12	10·280138	19
42	667305	4·01	947136	1·11	720169	5·11	279831	18
43	667546	4·01	947070	1·11	720476	5·11	279524	17
44	667786	4·00	947004	1·11	720783	5·11	279217	16
45	668027	4·00	946937	1·11	721089	5·11	278911	15
46	668267	4·00	946871	1·11	721396	5·11	278604	14
47	668506	3·99	946804	1·11	721702	5·10	278298	13
48	668746	3·99	946738	1·11	722009	5·10	277991	12
49	668986	3·99	946671	1·11	722315	5·10	277685	11
50	669225	3·99	946604	1·11	722621	5·10	277379	10
51	9·669464	3·98	9·946538	1·11	9·722927	5·10	10·277073	9
52	669703	3·98	946471	1·11	723232	5·09	276768	8
53	669942	3·98	946404	1·11	723538	5·09	276462	7
54	670181	3·97	946337	1·11	723844	5·09	276156	6
55	670419	3·97	946270	1·12	724149	5·09	275851	5
56	670658	3·97	946203	1·12	724454	5·09	275546	4
57	670896	3·97	946136	1·12	724759	5·08	275241	3
58	671134	3·96	946069	1·12	725065	5·08	274935	2
59	671372	3·96	946002	1·12	725369	5·08	274631	1
60	671609	3·96	945935	1·12	725674	5·08	274326	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·671609	3·96	9·945935	1·12	9·725674	5·08	10·274326	60
1	671847	3·95	945868	1·12	725979	5·08	274021	59
2	672084	3·95	945800	1·12	726284	5·07	273716	58
3	672321	3·95	945733	1·12	726588	5·07	273412	57
4	672558	3·95	945666	1·12	726892	5·07	273108	56
5	672795	3·94	945598	1·12	727197	5·07	272803	55
6	673032	3·94	945531	1·12	727501	5·07	272499	54
7	673268	3·94	945464	1·13	727805	5·06	272195	53
8	673505	3·94	945396	1·13	728109	5·06	271891	52
9	673741	3·93	945328	1·13	728412	5·06	271588	51
10	673977	3·93	945261	1·13	728716	5·06	271284	50
11	9·674213	3·93	9·945193	1·13	9·729020	5·06	10·270980	49
12	674448	3·92	945125	1·13	729323	5·05	270677	48
13	674684	3·92	945058	1·13	729626	5·05	270374	47
14	674919	3·92	944990	1·13	729929	5·05	270071	46
15	675155	3·92	944922	1·13	730233	5·05	269767	45
16	675390	3·91	944854	1·13	730535	5·05	269465	44
17	675624	3·91	944786	1·13	730838	5·04	269162	43
18	675859	3·91	944718	1·13	731141	5·04	268859	42
19	676094	3·91	944650	1·13	731444	5·04	268556	41
20	676328	3·90	944582	1·14	731746	5·04	268254	40
21	9·676562	3·90	9·944514	1·14	9·732048	5·04	10·267952	39
22	676796	3·90	944446	1·14	732351	5·03	267649	38
23	677030	3·90	944377	1·14	732653	5·03	267347	37
24	677264	3·89	944309	1·14	732955	5·03	267045	36
25	677498	3·89	944241	1·14	733257	5·03	266743	35
26	677731	3·89	944172	1·14	733558	5·03	266442	34
27	677964	3·88	944104	1·14	733860	5·02	266140	33
28	678197	3·88	944036	1·14	734162	5·02	265838	32
29	678430	3·88	943967	1·14	734463	5·02	265537	31
30	678663	3·88	943899	1·14	734764	5·02	265236	30
31	9·678895	3·87	9·943830	1·14	9·735066	5·02	10·264934	29
32	679128	3·87	943761	1·14	735367	5·02	264633	28
33	679360	3·87	943693	1·15	735668	5·01	264332	27
34	679592	3·87	943624	1·15	735969	5·01	264031	26
35	679824	3·86	943555	1·15	736269	5·01	263731	25
36	680056	3·86	943486	1·15	736570	5·01	263430	24
37	680288	3·86	943417	1·15	736871	5·01	263129	23
38	680519	3·85	943348	1·15	737171	5·00	262829	22
39	680750	3·85	943279	1·15	737471	5·00	262529	21
40	680982	3·85	943210	1·15	737771	5·00	262229	20
41	9·681213	3·85	9·943141	1·15	9·738071	5·00	10·261929	19
42	681443	3·84	943072	1·15	738371	5·00	261629	18
43	681674	3·84	943003	1·15	738671	4·99	261329	17
44	681905	3·84	942934	1·15	738971	4·99	261029	16
45	682135	3·84	942864	1·15	739271	4·99	260729	15
46	682365	3·83	942795	1·16	739570	4·99	260430	14
47	682595	3·83	942726	1·16	739870	4·99	260130	13
48	682825	3·83	942656	1·16	740169	4·99	259831	12
49	683055	3·83	942587	1·16	740468	4·98	259532	11
50	683284	3·82	942517	1·16	740767	4·98	259233	10
51	9·683514	3·82	9·942448	1·16	9·741066	4·98	10·258934	9
52	683743	3·82	942378	1·16	741365	4·98	258635	8
53	683972	3·82	942308	1·16	741664	4·98	258336	7
54	684201	3·81	942239	1·16	741962	4·97	258038	6
55	684430	3·81	942169	1·16	742261	4·97	257739	5
56	684658	3·81	942099	1·16	742559	4·97	257441	4
57	684887	3·80	942029	1·16	742858	4·97	257142	3
58	685115	3·80	941959	1·16	743156	4·97	256844	2
59	685343	3·80	941889	1·17	743454	4·97	256546	1
60	685571	3·80	941819	1·17	743752	4·96	256248	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·685571	3·80	9·941819	1·17	9·743752	4·96	10·256248	60
1	685799	3·79	941749	1·17	744050	4·96	255950	59
2	686027	3·79	941679	1·17	744348	4·96	255652	58
3	686254	3·79	941609	1·17	744645	4·96	255355	57
4	686482	3·79	941539	1·17	744943	4·96	255057	56
5	686709	3·78	941469	1·17	745240	4·96	254760	55
6	686936	3·78	941398	1·17	745538	4·95	254462	54
7	687163	3·78	941328	1·17	745835	4·95	254165	53
8	687389	3·78	941258	1·17	746132	4·95	253868	52
9	687616	3·77	941187	1·17	746429	4·95	253571	51
10	687843	3·77	941117	1·17	746726	4·95	253274	50
11	9·688069	3·77	9·941046	1·18	9·747023	4·94	10·252977	49
12	688295	3·77	940975	1·18	747319	4·94	252681	48
13	688521	3·76	940905	1·18	747616	4·94	252384	47
14	688747	3·76	940834	1·18	747913	4·94	252087	46
15	688972	3·76	940763	1·18	748209	4·94	251791	45
16	689198	3·76	940693	1·18	748505	4·93	251495	44
17	689423	3·75	940622	1·18	748801	4·93	251199	43
18	689648	3·75	940551	1·18	749097	4·93	250903	42
19	689873	3·75	940480	1·18	749393	4·93	250607	41
20	690098	3·75	940409	1·18	749689	4·93	250311	40
21	9·690323	3·74	9·940338	1·18	9·749985	4·93	10·250015	39
22	690548	3·74	940267	1·18	750281	4·92	249719	38
23	690772	3·74	940196	1·18	750576	4·92	249424	37
24	690996	3·74	940125	1·19	750872	4·92	249128	36
25	691220	3·73	940054	1·19	751167	4·92	248833	35
26	691444	3·73	939982	1·19	751462	4·92	248538	34
27	691668	3·73	939911	1·19	751757	4·92	248243	33
28	691892	3·73	939840	1·19	752052	4·91	247948	32
29	692115	3·72	939768	1·19	752347	4·91	247653	31
30	692339	3·72	939697	1·19	752642	4·91	247358	30
31	9·692562	3·72	9·939625	1·19	9·752937	4·91	10·247063	29
32	692785	3·71	939554	1·19	753231	4·91	246769	28
33	693008	3·71	939482	1·19	753526	4·91	246474	27
34	693231	3·71	939410	1·19	753820	4·90	246180	26
35	693453	3·71	939339	1·19	754115	4·90	245885	25
36	693676	3·70	939267	1·20	754409	4·90	245591	24
37	693898	3·70	939195	1·20	754703	4·90	245297	23
38	694120	3·70	939123	1·20	754997	4·90	245003	22
39	694342	3·70	939052	1·20	755291	4·90	244709	21
40	694564	3·69	938980	1·20	755585	4·89	244415	20
41	9·694786	3·69	9·938908	1·20	9·755878	4·89	10·244122	19
42	695007	3·69	938836	1·20	756172	4·89	243828	18
43	695229	3·69	938763	1·20	756465	4·89	243535	17
44	695450	3·68	938691	1·20	756759	4·89	243241	16
45	695671	3·68	938619	1·20	757052	4·89	242948	15
46	695892	3·68	938547	1·20	757345	4·88	242655	14
47	696113	3·68	938475	1·20	757638	4·88	242362	13
48	696334	3·67	938402	1·21	757931	4·88	242069	12
49	696554	3·67	938330	1·21	758224	4·88	241776	11
50	696775	3·67	938258	1·21	758517	4·88	241483	10
51	9·696995	3·67	9·938185	1·21	9·758810	4·88	10·241190	9
52	697215	3·66	938113	1·21	759102	4·87	240898	8
53	697435	3·66	938040	1·21	759395	4·87	240605	7
54	697654	3·66	937967	1·21	759687	4·87	240313	6
55	697874	3·66	937895	1·21	759979	4·87	240021	5
56	698094	3·65	937822	1·21	760272	4·87	239728	4
57	698313	3·65	937749	1·21	760564	4·87	239436	3
58	698532	3·65	937676	1·21	760856	4·86	239144	2
59	698751	3·65	937604	1·21	761148	4·86	238852	1
60	698970	3·64	937531	1·21	761439	4·86	238561	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·698970	3·64	9·937531	1·21	9·761439	4·86	10·238561	60
1	699189	3·64	937458	1·22	761731	4·86	238269	59
2	699407	3·64	937385	1·22	762023	4·86	237977	58
3	699626	3·64	937312	1·22	762314	4·86	237686	57
4	699844	3·63	937238	1·22	762606	4·85	237394	56
5	700062	3·63	937165	1·22	762897	4·85	237103	55
6	700280	3·63	937092	1·22	763188	4·85	236812	54
7	700498	3·63	937019	1·22	763479	4·85	236521	53
8	700716	3·63	936946	1·22	763770	4·85	236230	52
9	700933	3·62	936872	1·22	764061	4·85	235939	51
10	701151	3·62	936799	1·22	764352	4·84	235648	50
11	9·701368	3·62	9·936725	1·22	9·764643	4·84	10·235357	49
12	701585	3·62	936652	1·23	764933	4·84	235067	48
13	701802	3·61	936578	1·23	765224	4·84	234776	47
14	702019	3·61	936505	1·23	765514	4·84	234486	46
15	702236	3·61	936431	1·23	765805	4·84	234195	45
16	702452	3·61	936357	1·23	766095	4·84	233905	44
17	702669	3·60	936284	1·23	766385	4·83	233615	43
18	702885	3·60	936210	1·23	766675	4·83	233325	42
19	703101	3·60	936136	1·23	766965	4·83	233035	41
20	703317	3·60	936062	1·23	767255	4·83	232745	40
21	9·703533	3·59	9·935988	1·23	9·767545	4·83	10·232455	39
22	703749	3·59	935914	1·23	767834	4·83	232166	38
23	703964	3·59	935840	1·23	768124	4·82	231876	37
24	704179	3·59	935766	1·24	768413	4·82	231587	36
25	704395	3·59	935692	1·24	768703	4·82	231297	35
26	704610	3·58	935618	1·24	768992	4·82	231008	34
27	704825	3·58	935543	1·24	769281	4·82	230719	33
28	705040	3·58	935469	1·24	769570	4·82	230430	32
29	705254	3·58	935395	1·24	769860	4·81	230140	31
30	705469	3·57	935320	1·24	770148	4·81	229852	30
31	9·705683	3·57	9·935246	1·24	9·770437	4·81	10·229563	29
32	705898	3·57	935171	1·24	770726	4·81	229274	28
33	706112	3·57	935097	1·24	771015	4·81	228985	27
34	706326	3·56	935022	1·24	771303	4·81	228697	26
35	706539	3·56	934948	1·24	771592	4·81	228408	25
36	706753	3·56	934873	1·24	771880	4·80	228120	24
37	706967	3·56	934798	1·25	772168	4·80	227832	23
38	707180	3·55	934723	1·25	772457	4·80	227543	22
39	707393	3·55	934649	1·25	772745	4·80	227255	21
40	707606	3·55	934574	1·25	773033	4·80	226967	20
41	9·707819	3·55	9·934499	1·25	9·773321	4·80	10·226679	19
42	708032	3·54	934424	1·25	773608	4·79	226392	18
43	708245	3·54	934349	1·25	773896	4·79	226104	17
44	708458	3·54	934274	1·25	774184	4·79	225816	16
45	708670	3·54	934199	1·25	774471	4·79	225529	15
46	708882	3·53	934123	1·25	774759	4·79	225241	14
47	709094	3·53	934048	1·25	775046	4·79	224954	13
48	709306	3·53	933973	1·25	775333	4·79	224667	12
49	709518	3·53	933898	1·26	775621	4·78	224379	11
50	709730	3·53	933822	1·26	775908	4·78	224092	10
51	9·709941	3·52	9·933747	1·26	9·776195	4·78	10·223805	9
52	710153	3·52	933671	1·26	776482	4·78	223518	8
53	710364	3·52	933596	1·26	776769	4·78	223231	7
54	710575	3·52	933520	1·26	777055	4·78	222945	6
55	710786	3·51	933445	1·26	777342	4·78	222658	5
56	710997	3·51	933369	1·26	777628	4·77	222372	4
57	711208	6·51	933293	1·26	777915	4·77	222085	3
58	711419	3·51	933217	1·26	778201	4·77	221799	2
59	711629	3·50	933141	1·26	778487	4·77	221512	1
60	711839	3·50	933066	1·26	778774	4·77	221226	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·711839	3·50	9·933066	1·26	9·778774	4·77	10·221226	60
1	712050	3·50	932990	1·27	779060	4·77	220940	59
2	712260	3·50	932914	1·27	779346	4·76	220654	58
3	712469	3·49	932838	1·27	779632	4·76	220368	57
4	712679	3·49	932762	1·27	779918	4·76	220082	56
5	712889	3·49	932685	1·27	780203	4·76	219797	55
6	713098	3·49	932609	1·27	780489	4·76	219511	54
7	713308	3·49	932533	1·27	780775	4·76	219225	53
8	713517	3·48	932457	1·27	781060	4·76	218940	52
9	713726	3·48	932380	1·27	781346	4·75	218654	51
10	713935	3·48	932304	1·27	781631	4·75	218369	50
11	9·714144	3·48	9·932228	1·27	9·781916	4·75	10·218084	49
12	714352	3·47	932151	1·27	782201	4·75	217799	48
13	714561	3·47	932075	1·28	782486	4·75	217514	47
14	714769	3·47	931998	1·28	782771	4·75	217229	46
15	714978	3·47	931921	1·28	783056	4·75	216944	45
16	715186	3·47	931845	1·28	783341	4·75	216659	44
17	715394	3·46	931768	1·28	783626	4·74	216374	43
18	715602	3·46	931691	1·28	783910	4·74	216090	42
19	715809	3·46	931614	1·28	784195	4·74	215805	41
20	716017	3·46	931537	1·28	784479	4·74	215521	40
21	9·716224	3·45	9·931460	1·28	9·784764	4·74	10·215236	39
22	716432	3·45	931383	1·28	785048	4·74	214952	38
23	716639	3·45	931306	1·28	785332	4·73	214668	37
24	716846	3·45	931229	1·29	785616	4·73	214384	36
25	717053	3·45	931152	1·29	785900	4·73	214100	35
26	717259	3·44	931075	1·29	786184	4·73	213816	34
27	717466	3·44	930998	1·29	786468	4·73	213532	33
28	717673	3·44	930921	1·29	786752	4·73	213248	32
29	717879	3·44	930843	1·29	787036	4·73	212964	31
30	718085	3·43	930766	1·29	787319	4·72	212681	30
31	9·718291	3·43	9·930688	1·29	9·787603	4·72	10·212397	29
32	718497	3·43	930611	1·29	787886	4·72	212114	28
33	718703	3·43	930533	1·29	788170	4·72	211830	27
34	718909	3·43	930456	1·29	788453	4·72	211547	26
35	719114	3·42	930378	1·29	788736	4·72	211264	25
36	719320	3·42	930300	1·30	789019	4·72	210981	24
37	719525	3·42	930223	1·30	789302	4·71	210698	23
38	719730	3·42	930145	1·30	789585	4·71	210415	22
39	719935	3·41	930067	1·30	789868	4·71	210132	21
40	720140	3·41	929989	1·30	790151	4·71	209849	20
41	9·720345	3·41	9·929911	1·30	9·790433	4·71	10·209567	19
42	720549	3·41	929833	1·30	790716	4·71	209284	18
43	720754	3·40	929755	1·30	790999	4·71	209001	17
44	720958	3·40	929677	1·30	791281	4·71	208719	16
45	721162	3·40	929599	1·30	791563	4·70	208437	15
46	721366	3·40	929521	1·30	791846	4·70	208154	14
47	721570	3·40	929442	1·30	792128	4·70	207872	13
48	721774	3·39	929364	1·31	792410	4·70	207590	12
49	721978	3·39	929286	1·31	792692	4·70	207308	11
50	722181	3·39	929207	1·31	792974	4·70	207026	10
51	9·722385	3·39	9·929129	1·31	9·793256	4·70	10·206744	9
52	722588	3·39	929050	1·31	793538	4·69	206462	8
53	722791	3·38	928972	1·31	793819	4·69	206181	7
54	722994	3·38	928893	1·31	794101	4·69	205899	6
55	723197	3·38	928815	1·31	794383	4·69	205617	5
56	723400	3·38	928736	1·31	794664	4·69	205336	4
57	723603	3·37	928657	1·31	794945	4·69	205055	3
58	723805	3·37	928578	1·31	795227	4·69	204773	2
59	724007	3·37	928499	1·31	795508	4·68	204492	1
60	724210	3·37	928420	1·31	795789	4·68	204211	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·724210	3·37	9·928420	1·32	9·795789	4·68	10·204211	60
1	724412	3·37	928342	1·32	796070	4·68	203930	59
2	724614	3·36	928263	1·32	796351	4·68	203649	58
3	724816	3·36	928183	1·32	796632	4·68	203368	57
4	725017	3·36	928104	1·32	796913	4·68	203087	56
5	725219	3·36	928025	1·32	797194	4·68	202806	55
6	725420	3·35	927946	1·32	797475	4·68	202525	54
7	725622	3·35	927867	1·32	797755	4·68	202245	53
8	725823	3·35	927787	1·32	798036	4·67	201964	52
9	726024	3·35	927708	1·32	798316	4·67	201684	51
10	726225	3·35	927629	1·32	798596	4·67	201404	50
11	9·726426	3·34	9·927549	1·32	9·798877	4·67	10·201123	49
12	726626	3·34	927470	1·33	799157	4·67	200843	48
13	726827	3·34	927390	1·33	799437	4·67	200563	47
14	727027	3·34	927310	1·33	799717	4·67	200283	46
15	727228	3·34	927231	1·33	799997	4·66	200003	45
16	727428	3·33	927151	1·33	800277	4·66	199723	44
17	727628	3·33	927071	1·33	800557	4·66	199443	43
18	727828	3·33	926991	1·33	800836	4·66	199164	42
19	728027	3·33	926911	1·33	801116	4·66	198884	41
20	728227	3·33	926831	1·33	801396	4·66	198604	40
21	9·728427	3·32	9·926751	1·33	9·801675	4·66	10·198325	39
22	728626	3·32	926671	1·33	801955	4·66	198045	38
23	728825	3·32	926591	1·33	802234	4·65	197766	37
24	729024	3·32	926511	1·34	802513	4·65	197487	36
25	729223	3·31	926431	1·34	802792	4·65	197208	35
26	729422	3·31	926351	1·34	803072	4·65	196928	34
27	729621	3·31	926270	1·34	803351	4·65	196649	33
28	729820	3·31	926190	1·34	803630	4·65	196370	32
29	730018	3·30	926110	1·34	803908	4·65	196092	31
30	730216	3·30	926029	1·34	804187	4·65	195813	30
31	9·730415	3·30	9·925949	1·34	9·804466	4·64	10·195534	29
32	730613	3·30	925868	1·34	804745	4·64	195255	28
33	730811	3·30	925788	1·34	805023	4·64	194977	27
34	731009	3·29	925707	1·34	805302	4·64	194698	26
35	731206	3·29	925626	1·34	805580	4·64	194420	25
36	731404	3·29	925545	1·35	805859	4·64	194141	24
37	731602	3·29	925465	1·35	806137	4·64	193863	23
38	731799	3·29	925384	1·35	806415	4·63	193585	22
39	731996	3·28	925303	1·35	806693	4·63	193307	21
40	732193	3·28	925222	1·35	806971	4·63	193029	20
41	9·732390	3·28	9·925141	1·35	9·807249	4·63	10·192751	19
42	732587	3·28	925060	1·35	807527	4·63	192473	18
43	732784	3·28	924979	1·35	807805	4·63	192195	17
44	732980	3·27	924897	1·35	808083	4·63	191917	16
45	733177	3·27	924816	1·35	808361	4·63	191639	15
46	733373	3·27	924735	1·36	808638	4·62	191362	14
47	733569	3·27	924654	1·36	808916	4·62	191084	13
48	733765	3·27	924572	1·36	809193	4·62	190807	12
49	733961	3·26	924491	1·36	809471	4·62	190529	11
50	734157	3·26	924409	1·36	809748	4·62	190252	10
51	9·734353	3·26	9·924328	1·36	9·810025	4·62	10·189975	9
52	734549	3·26	924246	1·36	810302	4·62	189698	8
53	734744	3·25	924164	1·36	810580	4·62	189420	7
54	734939	3·25	924083	1·36	810857	4·62	189143	6
55	735135	3·25	924001	1·36	811134	4·61	188866	5
56	735330	3·25	923919	1·36	811410	4·61	188590	4
57	735525	3·25	923837	1·36	811687	4·61	188313	3
58	735719	3·24	923755	1·37	811964	4·61	188036	2
59	735914	3·24	923673	1·37	812241	4·61	187759	1
60	736109	3·24	923591	1·37	812517	4·61	187483	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(57 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·736109	3·24	9·923591	1·37	9·812517	4·61	10·187482	60
1	736303	3·24	923509	1·37	812794	4·61	187206	59
2	736498	3·24	923427	1·37	813070	4·61	186930	58
3	736692	3·23	923345	1·37	813347	4·60	186653	57
4	736886	3·23	923263	1·37	813623	4·60	186377	56
5	737080	3·23	923181	1·37	813899	4·60	186101	55
6	737274	3·23	923098	1·37	814175	4·60	185825	54
7	737467	3·23	923016	1·37	814452	4·60	185548	53
8	737661	3·22	922933	1·37	814728	4·60	185272	52
9	737855	3·22	922851	1·37	815004	4·60	184996	51
10	738048	3·22	922768	1·38	815279	4·60	184721	50
11	9·738241	3·22	9·922686	1·38	9·815555	4·59	10·184445	49
12	738434	3·22	922603	1·38	815831	4·59	184169	48
13	738627	3·21	922520	1·38	816107	4·59	183893	47
14	738820	3·21	922438	1·38	816382	4·59	183618	46
15	739013	3·21	922355	1·38	816658	4·59	183342	45
16	739206	3·21	922272	1·38	816933	4·59	183067	44
17	739398	3·21	922189	1·38	817209	4·59	182791	43
18	739590	3·20	922106	1·38	817484	4·59	182516	42
19	739783	3·20	922023	1·38	817759	4·59	182241	41
20	739975	3·20	921940	1·38	818035	4·58	181965	40
21	9·740167	3·20	9·921857	1·39	9·818310	4·58	10·181690	39
22	740359	3·20	921774	1·39	818585	4·58	181415	38
23	740550	3·19	921691	1·39	818860	4·58	181140	37
24	740742	3·19	921607	1·39	819135	4·58	180865	36
25	740934	3·19	921524	1·39	819410	4·58	180590	35
26	741125	3·19	921441	1·39	819684	4·58	180316	34
27	741316	3·19	921357	1·39	819959	4·58	180041	33
28	741508	3·18	921274	1·39	820234	4·58	179766	32
29	741699	3·18	921190	1·39	820508	4·57	179492	31
30	741889	3·18	921107	1·39	820783	4·57	179217	30
31	9·742080	3·18	9·921023	1·39	9·821057	4·57	10·178943	29
32	742271	3·18	920939	1·40	821332	4·57	178668	28
33	742462	3·17	920856	1·40	821606	4·57	178394	27
34	742652	3·17	920772	1·40	821880	4·57	178120	26
35	742842	3·17	920688	1·40	822154	4·57	177846	25
36	743033	3·17	920604	1·40	822429	4·57	177571	24
37	743223	3·17	920520	1·40	822703	4·57	177297	23
38	743413	3·16	920436	1·40	822977	4·56	177023	22
39	743602	3·16	920352	1·40	823250	4·56	176750	21
40	743792	3·16	920268	1·40	823524	4·56	176476	20
41	9·743982	3·16	9·920184	1·40	9·823798	4·56	10·176202	19
42	744171	3·16	920099	1·40	824072	4·56	175928	18
43	744361	3·15	920015	1·40	824345	4·56	175655	17
44	744550	3·15	919931	1·41	824619	4·56	175381	16
45	744739	3·15	919846	1·41	824893	4·56	175107	15
46	744928	3·15	919762	1·41	825166	4·56	174834	14
47	745117	3·15	919677	1·41	825439	4·55	174561	13
48	745306	3·14	919593	1·41	825713	4·55	174287	12
49	745494	3·14	919508	1·41	825986	4·55	174014	11
50	745683	3·14	919424	1·41	826259	4·55	173741	10
51	9·745871	3·14	9·919339	1·41	9·826532	4·55	10·173468	9
52	746059	3·14	919254	1·41	826805	4·55	173195	8
53	746248	3·13	919169	1·41	827078	4·55	172922	7
54	746436	3·13	919085	1·41	827351	4·55	172649	6
55	746624	3·13	919000	1·41	827624	4·55	172376	5
56	746812	3·13	918915	1·42	827897	4·54	172103	4
57	746999	3·13	918830	1·42	828170	4·54	171830	3
58	747187	3·12	918745	1·42	828442	4·54	171558	2
59	747374	3·12	918659	1·42	828715	4·54	171285	1
60	747562	3·12	918574	1·42	828987	4·54	171013	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

(56 DEGREES.)

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·747562	3·12	9·918574	1·42	9·828987	4·54	10·171013	60
1	747749	3·12	918489	1·42	829260	4·54	170740	59
2	747936	3·12	918404	1·42	829532	4·54	170468	58
3	748123	3·11	918318	1·42	829805	4·54	170195	57
4	748310	3·11	918233	1·42	830077	4·54	169923	56
5	748497	3·11	918147	1·42	830349	4·53	169651	55
6	748683	3·11	918062	1·42	830621	4·53	169379	54
7	748870	3·11	917976	1·43	830893	4·53	169107	53
8	749056	3·10	917891	1·43	831165	4·53	168835	52
9	749243	3·10	917805	1·43	831437	4·53	168563	51
10	749429	3·10	917719	1·43	831709	4·53	168291	50
11	9·749615	3·10	9·917634	1·43	9·831981	4·53	10·168019	49
12	749801	3·10	917548	1·43	832253	4·53	167747	48
13	749987	3·09	917462	1·43	832525	4·53	167475	47
14	750172	3·09	917376	1·43	832796	4·53	167204	46
15	750358	3·09	917290	1·43	833068	4·52	166932	45
16	750543	3·09	917204	1·43	833339	4·52	166661	44
17	750729	3·09	917118	1·44	833611	4·52	166389	43
18	750914	3·08	917032	1·44	833882	4·52	166118	42
19	751099	3·08	916946	1·44	834154	4·52	165846	41
20	751284	3·08	916859	1·44	834425	4·52	165575	40
21	9·751469	3·08	9·916773	1·44	9·834696	4·52	10·165304	39
22	751654	3·08	916687	1·44	834967	4·52	165033	38
23	751839	3·08	916600	1·44	835238	4·52	164762	37
24	752023	3·07	916514	1·44	835509	4·52	164491	36
25	752208	3·07	916427	1·44	835780	4·51	164220	35
26	752392	3·07	916341	1·44	836051	4·51	163949	34
27	752576	3·07	916254	1·44	836322	4·51	163678	33
28	752760	3·07	916167	1·45	836593	4·51	163407	32
29	752944	3·06	916081	1·45	836864	4·51	163136	31
30	753128	3·06	915994	1·45	837134	4·51	162866	30
31	9·753312	3·06	9·915907	1·45	9·837405	4·51	10·162595	29
32	753495	3·06	915820	1·45	837675	4·51	162325	28
33	753679	3·06	915733	1·45	837946	4·51	162054	27
34	753862	3·05	915646	1·45	838216	4·51	161784	26
35	754046	3·05	915559	1·45	838487	4·50	161513	25
36	754229	3·05	915472	1·45	838757	4·50	161243	24
37	754412	3·05	915385	1·45	839027	4·50	160973	23
38	754595	3·05	915297	1·45	839297	4·50	160703	22
39	754778	3·04	915210	1·45	839568	4·50	160432	21
40	754960	3·04	915123	1·46	839838	4·50	160162	20
41	9·755143	3·04	9·915035	1·46	9·840108	4·50	10·159892	19
42	755326	3·04	914948	1·46	840378	4·50	159622	18
43	755508	3·04	914860	1·46	840647	4·50	159353	17
44	755690	3·04	914773	1·46	840917	4·49	159083	16
45	755872	3·03	914685	1·46	841187	4·49	158813	15
46	756054	3·03	914598	1·46	841457	4·49	158543	14
47	756236	3·03	914510	1·46	841726	4·49	158274	13
48	756418	3·03	914422	1·46	841996	4·49	158004	12
49	756600	3·03	914334	1·46	842266	4·49	157734	11
50	756782	3·02	914246	1·47	842535	4·49	157465	10
51	9·756963	3·02	9·914158	1·47	9·842805	4·49	10·157195	9
52	757144	3·02	914070	1·47	843074	4·49	156926	8
53	757326	3·02	913982	1·47	843343	4·49	156657	7
54	757507	3·02	913894	1·47	843612	4·49	156388	6
55	757688	3·01	913806	1·47	843882	4·48	156118	5
56	757869	3·01	913718	1·47	844151	4·48	155849	4
57	758050	3·01	913630	1·47	844420	4·48	155580	3
58	758230	3·01	913541	1·47	844689	4·48	155311	2
59	758411	3·01	913453	1·47	844958	4·48	155042	1
60	758591	3·01	913365	1·47	845227	4·48	154773	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·758591	3·01	9·913365	1·47	9·845227	4·48	10·154773	60
1	758772	3·00	913276	1·47	845496	4·48	154504	59
2	758952	3·00	913187	1·48	845764	4·48	154236	58
3	759132	3·00	913099	1·48	846033	4·48	153967	57
4	759312	3·00	913010	1·48	846302	4·48	153698	56
5	759492	3·00	912922	1·48	846570	4·47	153430	55
6	759672	2·99	912833	1·48	846839	4·47	153161	54
7	759852	2·99	912744	1·48	847107	4·47	152893	53
8	760031	2·99	912655	1·48	847376	4·47	152624	52
9	760211	2·99	912566	1·48	847644	4·47	152356	51
10	760390	2·99	912477	1·48	847913	4·47	152087	50
11	9·760569	2·98	9·912388	1·48	9·848181	4·47	10·151819	49
12	760748	2·98	912299	1·49	848449	4·47	151551	48
13	760927	2·98	912210	1·49	848717	4·47	151283	47
14	761106	2·98	912121	1·49	848986	4·47	151014	46
15	761285	2·98	912031	1·49	849254	4·47	150746	45
16	761464	2·98	911942	1·49	849522	4·47	150478	44
17	761642	2·97	911853	1·49	849790	4·46	150210	43
18	761821	2·97	911763	1·49	850058	4·46	149942	42
19	761999	2·97	911674	1·49	850325	4·46	149675	41
20	762177	2·97	911584	1·49	850593	4·46	149407	40
21	9·762356	2·97	9·911495	1·49	9·850861	4·46	10·149139	39
22	762534	2·96	911405	1·49	851129	4·46	148871	38
23	762712	2·96	911315	1·50	851396	4·46	148604	37
24	762889	2·96	911226	1·50	851664	4·46	148336	36
25	763067	2·96	911136	1·50	851931	4·46	148069	35
26	763245	2·96	911046	1·50	852199	4·46	147801	34
27	763422	2·96	910956	1·50	852466	4·46	147534	33
28	763600	2·95	910866	1·50	852733	4·45	147267	32
29	763777	2·95	910776	1·50	853001	4·45	146999	31
30	763954	2·95	910686	1·50	853268	4·45	146732	30
31	9·764131	2·95	9·910596	1·50	9·853535	4·45	10·146465	29
32	764308	2·95	910506	1·50	853802	4·45	146198	28
33	764485	2·94	910415	1·50	854069	4·45	145931	27
34	764662	2·94	910325	1·51	854336	4·45	145664	26
35	764838	2·94	910235	1·51	854603	4·45	145397	25
36	765015	2·94	910144	1·51	854870	4·45	145130	24
37	765191	2·94	910054	1·51	855137	4·45	144863	23
38	765367	2·94	909963	1·51	855404	4·45	144596	22
39	765544	2·93	909873	1·51	855671	4·44	144329	21
40	765720	2·93	909782	1·51	855938	4·44	144062	20
41	9·765896	2·93	9·909691	1·51	9·856204	4·44	10·143796	19
42	766072	2·93	909601	1·51	856471	4·44	143529	18
43	766247	2·93	909510	1·51	856737	4·44	143263	17
44	766423	2·93	909419	1·51	857004	4·44	142996	16
45	766598	2·92	909328	1·52	857270	4·44	142730	15
46	766774	2·92	909237	1·52	857537	4·44	142463	14
47	766949	2·92	909146	1·52	857803	4·44	142197	13
48	767124	2·92	909055	1·52	858069	4·44	141931	12
49	767300	2·92	908964	1·52	858336	4·44	141664	11
50	767475	2·91	908873	1·52	858602	4·43	141398	10
51	9·767649	2·91	9·908781	1·52	9·858868	4·43	10·141132	9
52	767824	2·91	908690	1·52	859134	4·43	140866	8
53	767999	2·91	908599	1·52	859400	4·43	140600	7
54	768173	2·91	908507	1·52	859666	4·43	140334	6
55	768348	2·90	908416	1·53	859932	4·43	140068	5
56	768522	2·90	908324	1·53	860198	4·43	139802	4
57	768697	2·90	908233	1·53	860464	4·43	139536	3
58	768871	2·90	908141	1·53	860730	4·43	139270	2
59	769045	2·90	908049	1·53	860995	4·43	139005	1
60	769219	2·90	907958	1·53	861261	4·43	138739	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·769219	2·90	9·907958	1·53	9·861261	4·43	10·138739	60
1	769393	2·89	907866	1·53	861527	4·43	138473	59
2	769566	2·89	907774	1·53	861792	4·42	138208	58
3	769740	2·89	907682	1·53	862058	4·42	137942	57
4	769913	2·89	907590	1·53	862323	4·42	137677	56
5	770087	2·89	907498	1·53	862589	4·42	137411	55
6	770260	2·88	907406	1·53	862854	4·42	137146	54
7	770433	2·88	907314	1·54	863119	4·42	136881	53
8	770606	2·88	907222	1·54	863385	4·42	136615	52
9	770779	2·88	907129	1·54	863650	4·42	136350	51
10	770952	2·88	907037	1·54	863915	4·42	136085	50
11	9·771125	2·88	9·906945	1·54	9·864180	4·42	10·135820	49
12	771298	2·87	906852	1·54	864445	4·42	135555	48
13	771470	2·87	906760	1·54	864710	4·42	135290	47
14	771643	2·87	906667	1·54	864975	4·41	135025	46
15	771815	2·87	906575	1·54	865240	4·41	134760	45
16	771987	2·87	906482	1·54	865505	4·41	134495	44
17	772159	2·87	906389	1·55	865770	4·41	134230	43
18	772331	2·86	906296	1·55	866035	4·41	133965	42
19	772503	2·86	906204	1·55	866300	4·41	133700	41
20	772675	2·86	906111	1·55	866564	4·41	133436	40
21	9·772847	2·86	9·906018	1·55	9·866829	4·41	10·133171	39
22	773018	2·86	905925	1·55	867094	4·41	132906	38
23	773190	2·86	905832	1·55	867358	4·41	132642	37
24	773361	2·85	905739	1·55	867623	4·41	132377	36
25	773533	2·85	905645	1·55	867887	4·41	132113	35
26	773704	2·85	905552	1·55	868152	4·40	131848	34
27	773875	2·85	905459	1·55	868416	4·40	131584	33
28	774046	2·85	905366	1·56	868680	4·40	131320	32
29	774217	2·85	905272	1·56	868945	4·40	131055	31
30	774388	2·84	905179	1·56	869209	4·40	130794	30
31	9·774558	2·84	9·905085	1·56	9·869473	4·40	10·130527	29
32	774729	2·84	904992	1·56	869737	4·40	130263	28
33	774899	2·84	904898	1·56	870001	4·40	129999	27
34	775070	2·84	904804	1·56	870265	4·40	129735	26
35	775240	2·84	904711	1·56	870529	4·40	129471	25
36	775410	2·83	904617	1·56	870793	4·40	129207	24
37	775580	2·83	904523	1·56	871057	4·40	128943	23
38	775750	2·83	904429	1·57	871321	4·40	128679	22
39	775920	2·83	904335	1·57	871585	4·40	128415	21
40	776090	2·83	904241	1·57	871849	4·39	128151	20
41	9·776259	2·83	9·904147	1·57	9·872112	4·39	10·127888	19
42	776429	2·82	904053	1·57	872376	4·39	127624	18
43	776598	2·82	903959	1·57	872640	4·39	127360	17
44	776768	2·82	903864	1·57	872903	4·39	127097	16
45	776937	2·82	903770	1·57	873167	4·39	126833	15
46	777106	2·82	903676	1·57	873430	4·39	126570	14
47	777275	2·81	903581	1·57	873694	4·39	126306	13
48	777444	2·81	903487	1·57	873957	4·39	126043	12
49	777613	2·81	903392	1·58	874220	4·39	125780	11
50	777781	2·81	903298	1·58	874484	4·39	125516	10
51	9·777950	2·81	9·903203	1·58	9·874747	4·39	10·125253	9
52	778119	2·81	903108	1·58	875010	4·39	124990	8
53	778287	2·80	903014	1·58	875273	4·38	124727	7
54	778455	2·80	902919	1·58	875536	4·38	124464	6
55	778624	2·80	902824	1·58	875800	4·38	124200	5
56	778792	2·80	902729	1·58	876063	4·38	123937	4
57	778960	2·80	902634	1·58	876326	4·38	123674	3
58	779128	2·80	902539	1·59	876589	4·38	123411	2
59	779295	2·79	902444	1·59	876851	4·38	123149	1
60	779463	2·79	902349	1·59	877114	4·38	122886	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·779463	2·79	9·902349	1·59	9·877114	4·38	10·122886	60
1	779631	2·79	902253	1·59	877377	4·38	122623	59
2	779798	2·79	902158	1·59	877640	4·38	122360	58
3	779966	2·79	902063	1·59	877903	4·38	122097	57
4	780133	2·79	901967	1·59	878165	4·38	121835	56
5	780300	2·78	901872	1·59	878428	4·38	121572	55
6	780467	2·78	901776	1·59	878691	4·38	121309	54
7	780634	2·78	901681	1·59	878953	4·37	121047	53
8	780801	2·78	901585	1·59	879216	4·37	120784	52
9	780968	2·78	901490	1·59	879478	4·37	120522	51
10	781134	2·78	901394	1·60	879741	4·37	120259	50
11	9·781301	2·77	9·901298	1·60	9·880003	4·37	10·119997	49
12	781468	2·77	901202	1·60	880265	4·37	119735	48
13	781634	2·77	901106	1·60	880528	4·37	119472	47
14	781800	2·77	901010	1·60	880790	4·37	119210	46
15	781966	2·77	900914	1·60	881052	4·37	118948	45
16	782132	2·77	900818	1·60	881314	4·37	118686	44
17	782298	2·76	900722	1·60	881576	4·37	118424	43
18	782464	2·76	900626	1·60	881839	4·37	118161	42
19	782630	2·76	900529	1·60	882101	4·37	117899	41
20	782796	2·76	900433	1·61	882363	4·36	117637	40
21	9·782961	2·76	9·900337	1·61	9·882625	4·36	10·117375	39
22	783127	2·76	900240	1·61	882887	4·36	117113	38
23	783292	2·75	900144	1·61	883148	4·36	116852	37
24	783458	2·75	900047	1·61	883410	4·36	116590	36
25	783623	2·75	899951	1·61	883672	4·36	116328	35
26	783788	2·75	899854	1·61	883934	4·36	116066	34
27	783953	2·75	899757	1·61	884196	4·36	115804	33
28	784118	2·75	899660	1·61	884457	4·36	115543	32
29	784282	2·74	899564	1·61	884719	4·36	115281	31
30	784447	2·74	899467	1·62	884980	4·36	115020	30
31	9·784612	2·74	9·899370	1·62	9·885242	4·36	10·114758	29
32	784776	2·74	899273	1·62	885503	4·36	114497	28
33	784941	2·74	899176	1·62	885765	4·36	114235	27
34	785105	2·74	899078	1·62	886026	4·36	113974	26
35	785269	2·73	898981	1·62	886288	4·36	113712	25
36	785433	2·73	898884	1·62	886549	4·35	113451	24
37	785597	2·73	898787	1·62	886810	4·35	113190	23
38	785761	2·73	898689	1·62	887072	4·35	112928	22
39	785925	2·73	898592	1·62	887333	4·35	112667	21
40	786089	2·73	898494	1·63	887594	4·35	112406	20
41	9·786252	2·72	9·898397	1·63	9·887855	4·35	10·112145	19
42	786416	2·72	898299	1·63	888116	4·35	111884	18
43	786579	2·72	898202	1·63	888377	4·35	111623	17
44	786742	2·72	898104	1·63	888639	4·35	111361	16
45	786906	2·72	898006	1·63	888900	4·35	111100	15
46	787069	2·72	897908	1·63	889160	4·35	110840	14
47	787232	2·71	897810	1·63	889421	4·35	110579	13
48	787395	2·71	897712	1·63	889682	4·35	110318	12
49	787557	2·71	897614	1·63	889943	4·35	110057	11
50	787720	2·71	897516	1·63	890204	4·34	109796	10
51	9·787883	2·71	9·897418	1·64	9·890465	4·34	10·109535	9
52	788045	2·71	897320	1·64	890725	4·34	109275	8
53	788208	2·71	897222	1·64	890986	4·34	109014	7
54	788370	2·70	897123	1·64	891247	4·34	108753	6
55	788532	2·70	897025	1·64	891507	4·34	108493	5
56	788694	2·70	896926	1·64	891768	4·34	108232	4
57	788856	2·70	896828	1·64	892028	4·34	107972	3
58	789018	2·70	896729	1·64	892289	4·34	107711	2
59	789180	2·70	896631	1·64	892549	4·34	107451	1
60	789342	2·69	896532	1·64	892810	4·34	107190	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·789342	2·69	9·896532	1·64	9·892810	4·34	10·107190	60
1	789504	2·69	896433	1·65	893070	4·34	106930	59
2	789665	2·69	896335	1·65	893331	4·34	106669	58
3	789827	2·69	896236	1·65	893591	4·34	106409	57
4	789988	2·69	896137	1·65	893851	4·34	106149	56
5	790149	2·69	896038	1·65	894111	4·34	105889	55
6	790310	2·68	895939	1·65	894371	4·34	105629	54
7	790471	2·68	895840	1·65	894632	4·33	105368	53
8	790632	2·68	895741	1·65	894892	4·33	105108	52
9	790793	2·68	895641	1·65	895152	4·33	104848	51
10	790954	2·68	895542	1·65	895412	4·33	104588	50
11	9·791115	2·68	9·895443	1·66	9·895672	4·33	10·104328	49
12	791275	2·67	895343	1·66	895932	4·33	104068	48
13	791436	2·67	895244	1·66	896192	4·33	103808	47
14	791596	2·67	895145	1·66	896452	4·33	103548	46
15	791757	2·67	895045	1·66	896712	4·33	103288	45
16	791917	2·67	894945	1·66	896971	4·33	103029	44
17	792077	2·67	894846	1·66	897231	4·33	102769	43
18	792237	2·66	894746	1·66	897491	4·33	102509	42
19	792397	2·66	894646	1·66	897751	4·33	102249	41
20	792557	2·66	894546	1·66	898010	4·33	101990	40
21	9·792716	2·66	9·894446	1·67	9·898270	4·33	10·101730	39
22	792876	2·66	894346	1·67	898530	4·33	101470	38
23	793035	2·66	894246	1·67	898789	4·33	101211	37
24	793195	2·65	894146	1·67	899049	4·32	100951	36
25	793354	2·65	894046	1·67	899308	4·32	100692	35
26	793514	2·65	893946	1·67	899568	4·32	100432	34
27	793673	2·65	893846	1·67	899827	4·32	100173	33
28	793832	2·65	893745	1·67	900086	4·32	099914	32
29	793991	2·65	893645	1·67	900346	4·32	099654	31
30	794150	2·64	893544	1·67	900605	4·32	099395	30
31	9·794308	2·64	9·893444	1·68	9·900864	4·32	10·099136	29
32	794467	2·64	893343	1·68	901124	4·32	098876	28
33	794626	2·64	893243	1·68	901383	4·32	098617	27
34	794784	2·64	893142	1·68	901642	4·32	098358	26
35	794942	2·64	893041	1·68	901901	4·32	098099	25
36	795101	2·64	892940	1·68	902160	4·32	097840	24
37	795259	2·63	892839	1·68	902419	4·32	097581	23
38	795417	2·63	892739	1·68	902679	4·32	097321	22
39	795575	2·63	892638	1·68	902938	4·32	097062	21
40	795733	2·63	892536	1·68	903197	4·31	096803	20
41	9·795891	2·63	9·892435	1·69	9·903455	4·31	10·096545	19
42	796049	2·63	892334	1·69	903714	4·31	096286	18
43	796206	2·63	892233	1·69	903973	4·31	096027	17
44	796364	2·62	892132	1·69	904232	4·31	095768	16
45	796521	2·62	892030	1·69	904491	4·31	095509	15
46	796679	2·62	891929	1·69	904750	4·31	095250	14
47	796836	2·62	891827	1·69	905008	4·31	094992	13
48	796993	2·62	891726	1·69	905267	4·31	094733	12
49	797150	2·61	891624	1·69	905526	4·31	094474	11
50	797307	2·61	891523	1·70	905784	4·31	094216	10
51	9·797464	2·61	9·891421	1·70	9·906043	4·31	10·093957	9
52	797621	2·61	891319	1·70	906302	4·31	093698	8
53	797777	2·61	891217	1·70	906560	4·31	093440	7
54	797934	2·61	891115	1·70	906819	4·31	093181	6
55	798091	2·61	891013	1·70	907077	4·31	092923	5
56	798247	2·61	890911	1·70	907336	4·31	092664	4
57	798403	2·60	890809	1·70	907594	4·31	092406	3
58	798560	2·60	890707	1·70	907852	4·31	092148	2
59	798716	2·60	890605	1·70	908111	4·30	091889	1
60	798872	2·60	890503	1·70	908369	4·30	091631	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·798872	2·60	9·890503	1·70	9·908369	4·30	10·091631	60
1	799028	2·60	890400	1·71	908628	4·30	091372	59
2	799184	2·60	890298	1·71	908886	4·30	091114	58
3	799339	2·59	890195	1·71	909144	4·30	090856	57
4	799495	2·59	890093	1·71	909402	4·30	090598	56
5	799651	2·59	889990	1·71	909660	4·30	090340	55
6	799806	2·59	889888	1·71	909918	4·30	090082	54
7	799962	2·59	889785	1·71	910177	4·30	089823	53
8	u00117	2·59	889682	1·71	910435	4·30	089565	52
9	800272	2·58	889579	1·71	910693	4·39	089307	51
10	800427	2·58	889477	1·71	910951	4·30	089049	50
11	9·800582	2·58	9·889374	1·72	9·911209	4·30	10·088791	49
12	800737	2·58	889271	1·72	911467	4·30	088533	48
13	800892	2·58	889168	1·72	911724	4·30	088276	47
14	801047	2·58	889064	1·72	911982	4·30	088018	46
15	801201	2·58	888961	1·72	912240	4·30	087760	45
16	801356	2·57	888858	1·72	912498	4·30	087502	44
17	801511	2·57	888755	1·72	912756	4·30	087244	43
18	801665	2·57	888651	1·72	913014	4·29	086986	42
19	801819	2·57	888548	1·72	913271	4·29	086729	41
20	801973	2·57	888444	1·73	913529	4·29	086471	40
21	9·802128	2·57	9·888341	1·73	9·913787	4·29	10·086213	39
22	802282	2·56	888237	1·73	914044	4·29	085956	38
23	802436	2·56	888134	1·73	914302	4·29	085698	37
24	802589	2·56	888030	1·73	914560	4·29	085440	36
25	802743	2·56	887926	1·73	914817	4·29	085183	35
26	802897	2·56	887822	1·73	915075	4·29	084925	34
27	803050	2·56	887718	1·73	915332	4·29	084668	33
28	803204	2·56	887614	1·73	915590	4·29	084410	32
29	803357	2·55	887510	1·73	915847	4·29	084153	31
30	803511	2·55	887406	1·74	916104	4·29	083896	30
31	9·803664	2·55	9·887302	1·74	9·916362	4·29	10·083638	29
32	803817	2·55	887198	1·74	916619	4·29	083381	28
33	803970	2·55	887093	1·74	916877	4·29	083123	27
34	804123	2·55	886989	1·74	917134	4·29	082866	26
35	804276	2·54	886885	1·74	917391	4·29	082609	25
36	804428	2·54	886780	1·74	917648	4·29	082352	24
37	804581	2·54	886676	1·74	917905	4·29	082095	23
38	804734	2·54	886571	1·74	918163	4·28	081837	22
39	804886	2·54	886466	1·74	918420	4·28	081580	21
40	805039	2·54	886362	1·75	918677	4·28	081323	20
41	9·805191	2·54	9·886257	1·75	9·918934	4·28	10·081066	19
42	805343	2·53	886152	1·75	919191	4·28	080809	18
43	805495	2·53	886047	1·75	919448	4·28	080552	17
44	805647	2·53	885942	1·75	919705	4·28	080295	16
45	805799	2·53	885837	1·75	919962	4·28	080038	15
46	805951	2·53	885732	1·75	920219	4·28	079781	14
47	806103	2·53	885627	1·75	920476	4·28	079524	13
48	806254	2·53	885522	1·75	920733	4·28	079267	12
49	806406	2·52	885416	1·75	920990	4·28	079010	11
50	806557	2·52	885311	1·76	921247	4·28	078753	10
51	9·806709	2·52	9·885205	1·76	9·921503	4·28	10·078497	9
52	806860	2·52	885100	1·76	921760	4·28	078240	8
53	807011	2·52	884994	1·76	922017	4·28	077983	7
54	807163	2·52	884889	1·76	922274	4·28	077726	6
55	807314	2·52	884783	1·76	922530	4·28	077470	5
56	807465	2·51	884677	1·76	922787	4·28	077213	4
57	807615	2·51	884572	1·76	923044	4·28	076956	3
58	807766	2·51	884466	1·76	923300	4·28	076700	2
59	807917	2·51	884360	1·76	923557	4·27	076443	1
60	808067	2·51	884254	1·77	923813	4·27	076187	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·808067	2·51	9·884254	1·77	9·923813	4·27	10·076187	60
1	808218	2·51	884148	1·77	924070	4·27	075930	59
2	808368	2·51	884042	1·77	924327	4·27	075673	58
3	808519	2·50	883936	1·77	924583	4·27	075417	57
4	808669	2·50	883829	1·77	924840	4·27	075160	56
5	808819	2·50	883723	1·77	925096	4·27	074904	55
6	808969	2·50	883617	1·77	925352	4·27	074648	54
7	809119	2·50	883510	1·77	925609	4·27	074391	53
8	809269	2·50	883404	1·77	925865	4·27	074135	52
9	809419	2·49	883297	1·78	926122	4·27	073878	51
10	809569	2·49	883191	1·78	926378	4·27	073622	50
11	9·809718	2·49	9·883084	1·78	9·926634	4·27	10·073366	49
12	809868	2·49	882977	1·78	926890	4·27	073110	48
13	810017	2·49	882871	1·78	927147	4·27	072853	47
14	810167	2·49	882764	1·78	927403	4·27	072597	46
15	810316	2·48	882657	1·78	927659	4·27	072341	45
16	810465	2·48	882550	1·78	927915	4·27	072085	44
17	810614	2·48	882443	1·78	928171	4·27	071829	43
18	810763	2·48	882336	1·79	928427	4·27	071573	42
19	810912	2·48	882229	1·79	928683	4·27	071317	41
20	811061	2·48	882121	1·79	928940	4·27	071060	40
21	9·811210	2·48	9·882014	1·79	9·929196	4·27	10·070804	39
22	811358	2·47	881907	1·79	929452	4·27	070548	38
23	811507	2·47	881799	1·79	929708	4·27	070292	37
24	811655	2·47	881692	1·79	929964	4·26	070036	36
25	811804	2·47	881584	1·79	930220	4·26	069780	35
26	811952	2·47	881477	1·79	930475	4·26	069525	34
27	812100	2·47	881369	1·79	930731	4·26	069269	33
28	812248	2·47	881261	1·80	930987	4·26	069013	32
29	812396	2·46	881153	1·80	931243	4·26	068757	31
30	812544	2·46	881046	1·80	931499	4·26	068501	30
31	9·812692	2·46	9·880938	1·80	9·931755	4·26	10·068245	29
32	812840	2·46	880830	1·80	932010	4·26	067990	28
33	812988	2·46	880722	1·80	932266	4·26	067734	27
34	813135	2·46	880613	1·80	932522	4·26	067478	26
35	813283	2·46	880505	1·80	932778	4·26	067222	25
36	813430	2·45	880397	1·80	933033	4·26	066967	24
37	813578	2·45	880289	1·81	933289	4·26	066711	23
38	813725	2·45	880180	1·81	933545	4·26	066455	22
39	813872	2·45	880072	1·81	933800	4·26	066200	21
40	814019	2·45	879963	1·81	934056	4·26	065944	20
41	9·814166	2·45	9·879855	1·81	9·934311	4·26	10·065689	19
42	814313	2·45	879746	1·81	934567	4·26	065433	18
43	814460	2·44	879637	1·81	934823	4·26	065177	17
44	814607	2·44	879529	1·81	935078	4·26	064922	16
45	814753	2·44	879420	1·81	935333	4·26	064667	15
46	814900	2·44	879311	1·81	935589	4·26	064411	14
47	815046	2·44	879202	1·82	935844	4·26	064156	13
48	815193	2·44	879093	1·82	936100	4·26	063900	12
49	815339	2·44	878984	1·82	936355	4·26	063645	11
50	815485	2·43	878875	1·82	936610	4·26	063390	10
51	9·815631	2·43	9·878766	1·82	9·936866	4·25	10·063134	9
52	815778	2·43	878656	1·82	937121	4·25	062879	8
53	815924	2·43	878547	1·82	937376	4·25	062624	7
54	816069	2·43	878438	1·82	937632	4·25	062368	6
55	816215	2·43	878328	1·82	937887	4·25	062113	5
56	816361	2·43	878219	1·83	938142	4·25	061858	4
57	816507	2·42	878109	1·83	938398	4·25	061602	3
58	816652	2·42	877999	1·83	938653	4·25	061347	2
59	816798	2·42	877890	1·83	938908	4·25	061092	1
60	816943	2·42	877780	1·83	939163	4·25	060837	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·816943	2·42	9·877780	1·83	9·939163	4·25	10·060837	60
1	817088	2·42	877670	1·83	939418	4·25	060582	59
2	817233	2·42	877560	1·83	939673	4·25	060327	58
3	817379	2·42	877450	1·83	939928	4·25	060072	57
4	817524	2·41	877340	1·83	940183	4·25	059817	56
5	817668	2·41	877230	1·84	940438	4·25	059562	55
6	817813	2·41	877120	1·84	940694	4·25	059306	54
7	817958	2·41	877010	1·84	940949	4·25	059051	53
8	818103	2·41	876899	1·84	941204	4·25	058796	52
9	818247	2·41	876789	1·84	941458	4·25	058542	51
10	818392	2·41	876678	1·84	941714	4·25	058286	50
11	9·818536	2·40	9·876568	1·84	9·941968	4·25	10·058032	49
12	818681	2·40	876457	1·84	942223	4·25	057777	48
13	818825	2·40	876347	1·84	942478	4·25	057522	47
14	818969	2·40	876236	1·85	942733	4·25	057267	46
15	819113	2·40	876125	1·85	942988	4·25	057012	45
16	819257	2·40	876014	1·85	943243	4·25	056757	44
17	819401	2·40	875904	1·85	943498	4·25	056502	43
18	819545	2·39	875793	1·85	943752	4·25	056248	42
19	819689	2·39	875682	1·85	944007	4·25	055993	41
20	819832	2·39	875571	1·85	944262	4·25	055738	40
21	9·819976	2·39	9·875459	1·85	9·944517	4·25	10·055483	39
22	820120	2·39	875348	1·85	944771	4·24	055229	38
23	820263	2·39	875237	1·85	945026	4·24	054974	37
24	820406	2·39	875126	1·86	945281	4·24	054719	36
25	820550	2·38	875014	1·86	945535	4·24	054465	35
26	820693	2·38	874903	1·86	945790	4·24	054210	34
27	820836	2·38	874791	1·86	946045	4·24	053955	33
28	820979	2·38	874680	1·86	946299	4·24	053701	32
29	821122	2·38	874568	1·86	946554	4·24	053446	31
30	821265	2·38	874456	1·86	946808	4·24	053192	30
31	9·821407	2·38	9·874344	1·86	9·947063	4·24	10·052937	29
32	821550	2·38	874232	1·87	947318	4·24	052682	28
33	821693	2·37	874121	1·87	947572	4·24	052428	27
34	821835	2·37	874009	1·87	947826	4·24	052174	26
35	821977	2·37	873896	1·87	948081	4·24	051919	25
36	822120	2·37	873784	1·87	948336	4·24	051664	24
37	822262	2·37	873672	1·87	948590	4·24	051410	23
38	822404	2·37	873560	1·87	948844	4·24	051156	22
39	822546	2·37	873448	1·87	949099	4·24	050901	21
40	822688	2·36	873335	1·87	949353	4·24	050647	20
41	9·822830	2·36	9·873223	1·87	9·949607	4·24	10·050393	19
42	822972	2·36	873110	1·88	949862	4·24	050138	18
43	823114	2·36	872998	1·88	950116	4·24	049884	17
44	823255	2·36	872885	1·88	950370	4·24	049630	16
45	823397	2·36	872772	1·88	950625	4·24	049375	15
46	823539	2·36	872659	1·88	950879	4·24	049121	14
47	823680	2·35	872547	1·88	951133	4·24	048867	13
48	823821	2·35	872434	1·88	951388	4·24	048612	12
49	823963	2·35	872321	1·88	951642	4·24	048358	11
50	824104	2·35	872208	1·88	951896	4·24	048104	10
51	9·824245	2·35	9·872095	1·89	9·952150	4·24	10·047850	9
52	824386	2·35	871981	1·89	952405	4·24	047595	8
53	824527	2·35	871868	1·89	952659	4·24	047341	7
54	824668	2·34	871755	1·89	952913	4·24	047087	6
55	824808	2·34	871641	1·89	953167	4·23	046833	5
56	824949	2·34	871528	1·89	953421	4·23	046579	4
57	825090	2·34	871414	1·89	953675	4·23	046325	3
58	825230	2·34	871301	1·89	953929	4·23	046071	2
59	825371	2·34	871187	1·89	954183	4·23	045817	1
60	825511	2·34	871073	1·90	954437	4·23	045563	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·825511	2·34	9·871073	1·90	9·954437	4·23	10·045563	60
1	825651	2·33	870960	1·90	954691	4·23	045309	59
2	825791	2·33	870846	1·90	954945	4·23	045055	58
3	825931	2·33	870732	1·90	955200	4·23	044800	57
4	826071	2·33	870618	1·90	955454	4·23	044546	56
5	826211	2·33	870504	1·90	955707	4·23	044293	55
6	826351	2·33	870390	1·90	955961	4·23	044039	54
7	826491	2·33	870276	1·90	956215	4·23	043785	53
8	826631	2·33	870161	1·90	956469	4·23	043531	52
9	826770	2·32	870047	1·91	956723	4·23	043277	51
10	826910	2·32	869933	1·91	956977	4·23	043023	50
11	9·827049	2·32	9·869818	1·91	9·957231	4·23	10·042769	49
12	827189	2·32	869704	1·91	957485	4·23	042515	48
13	827328	2·32	869589	1·91	957739	4·23	042261	47
14	827467	2·32	869474	1·91	957993	4·23	042007	46
15	827606	2·32	869360	1·91	958246	4·23	041754	45
16	827745	2·32	869245	1·91	958500	4·23	041500	44
17	827884	2·31	869130	1·91	958754	4·23	041246	43
18	828023	2·31	869015	1·92	959008	4·23	040992	42
19	828162	2·31	868900	1·92	959262	4·23	040738	41
20	828301	2·31	868785	1·92	959516	4·23	040484	40
21	9·828439	2·31	9·868670	1·92	9·959769	4·23	10·040231	39
22	828578	2·31	868555	1·92	960023	4·23	039977	38
23	828716	2·31	868440	1·92	960277	4·23	039723	37
24	828855	2·30	868324	1·92	960531	4·23	039469	36
25	828993	2·30	868209	1·92	960784	4·23	039216	35
26	829131	2·30	868093	1·92	961038	4·23	038962	34
27	829269	2·30	867978	1·93	961291	4·23	038709	33
28	829407	2·30	867862	1·93	961545	4·23	038455	32
29	829545	2·30	867747	1·93	961799	4·23	038201	31
30	829683	2·30	867631	1·93	962052	4·23	037948	30
31	9·829821	2·29	9·867515	1·93	9·962306	4·23	10·037694	29
32	829959	2·29	867399	1·93	962560	4·23	037440	28
33	830097	2·29	867283	1·93	962813	4·23	037187	27
34	830234	2·29	867167	1·93	963067	4·23	036933	26
35	830372	2·29	867051	1·93	963320	4·23	036680	25
36	830509	2·29	866935	1·94	963574	4·23	036426	24
37	830646	2·29	866819	1·94	963827	4·23	036173	23
38	830784	2·29	866703	1·94	964081	4·23	035919	22
39	830921	2·28	866586	1·94	964335	4·23	035665	21
40	831058	2·28	866470	1·94	964588	4·22	035412	20
41	9·831195	2·28	9·866353	1·94	9·964842	4·22	10·035158	19
42	831332	2·28	866237	1·94	965095	4·22	034905	18
43	831469	2·28	866120	1·94	965349	4·22	034651	17
44	831606	2·28	866004	1·95	965602	4·22	034398	16
45	831742	2·28	865887	1·95	965855	4·22	034145	15
46	831879	2·28	865770	1·95	966105	4·22	033891	14
47	832015	2·27	865653	1·95	966362	4·22	033638	13
48	832152	2·27	865536	1·95	966616	4·22	033384	12
49	832288	2·27	865419	1·95	966869	4·22	033131	11
50	832425	2·27	865302	1·95	967123	4·22	032877	10
51	9·832561	2·27	9·865185	1·95	9·967376	4·22	10·032624	9
52	832697	2·27	865068	1·95	967629	4·22	032371	8
53	832833	2·27	864950	1·95	967883	4·22	032117	7
54	832969	2·26	864833	1·96	968136	4·22	031864	6
55	833105	2·26	864716	1·96	968389	4·22	031611	5
56	833241	2·26	864598	1·96	968643	4·22	031357	4
57	833377	2·26	864481	1·96	968896	4·22	031104	3
58	833512	2·26	864363	1·96	969149	4·22	030851	2
59	833648	2·26	864245	1·96	969403	4·22	030597	1
60	833783	2·26	864127	1·96	969656	4·22	030344	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·833783	2·26	9·864127	1·96	9·969656	4·22	10·030344	60
1	833919	2·25	864010	1·96	969909	4·22	030091	59
2	834054	2·25	863892	1·97	970162	4·22	029838	58
3	834189	2·25	863774	1·97	970416	4·22	029584	57
4	834325	2·25	863656	1·97	970669	4·22	029331	56
5	834460	2·25	863538	1·97	970922	4·22	029078	55
6	834595	2·25	863419	1·97	971175	4·22	028825	54
7	834730	2·25	863301	1·97	971429	4·22	028571	53
8	834865	2·25	863183	1·97	971682	4·22	028318	52
9	834999	2·24	863064	1·97	971935	4·22	028065	51
10	835134	2·24	862946	1·98	972188	4·22	027812	50
11	9·835269	2·24	9·862827	1·98	9·972441	4·22	10·027559	49
12	835403	2·24	862709	1·98	972694	4·22	027306	48
13	835538	2·24	862590	1·98	972948	4·22	027052	47
14	835672	2·24	862471	1·98	973201	4·22	026799	46
15	835807	2·24	862353	1·98	973454	4·22	026546	45
16	835941	2·24	862234	1·98	973707	4·22	026293	44
17	836075	2·23	862115	1·98	973960	4·22	026040	43
18	836209	2·23	861996	1·98	974213	4·22	025787	42
19	836343	2·23	861877	1·98	974466	4·22	025534	41
20	836477	2·23	861758	1·99	974719	4·22	025281	40
21	9·836611	2·23	9·861638	1·99	9·974973	4·22	10·025027	39
22	836745	2·23	861519	1·99	975226	4·22	024774	38
23	836878	2·23	861400	1·99	975479	4·22	024521	37
24	837012	2·22	861280	1·99	975732	4·22	024268	36
25	837146	2·22	861161	1·99	975985	4·22	024015	35
26	837279	2·22	861041	1·99	976238	4·22	023762	34
27	837412	2·22	860922	1·99	976491	4·22	023509	33
28	837546	2·22	860802	1·99	976744	4·22	023256	32
29	837679	2·22	860682	2·00	976997	4·22	023003	31
30	837812	2·22	860562	2·00	977250	4·22	022750	30
31	9·837945	2·22	9·860442	2·00	9·977503	4·22	10·022497	29
32	838078	2·21	860322	2·00	977756	4·22	022244	28
33	838211	2·21	860202	2·00	978009	4·22	021991	27
34	838344	2·21	860082	2·00	978262	4·22	021738	26
35	838477	2·21	859962	2·00	978515	4·22	021485	25
36	838610	2·21	859842	2·00	978768	4·22	021232	24
37	838742	2·21	859721	2·01	979021	4·22	020979	23
38	838875	2·21	859601	2·01	979274	4·22	020726	22
39	839007	2·21	859480	2·01	979527	4·22	020473	21
40	839140	2·20	859360	2·01	979780	4·22	020220	20
41	9·839272	2·20	9·859239	2·01	9·980033	4·22	10·019967	19
42	839404	2·20	859119	2·01	980286	4·22	019714	18
43	839536	2·20	858998	2·01	980538	4·22	019462	17
44	839668	2·20	858877	2·01	980791	4·21	019209	16
45	839800	2·20	858756	2·02	981044	4·21	018956	15
46	839932	2·20	858635	2·02	981297	4·21	018703	14
47	840064	2·19	858514	2·02	981550	4·21	018450	13
48	840196	2·19	858393	2·02	981803	4·21	018197	12
49	840328	2·19	858272	2·02	982056	4·21	017944	11
50	840459	2·19	858151	2·02	982309	4·21	017691	10
51	9·840591	2·19	9·858029	2·02	9·982562	4·21	10·017438	9
52	840722	2·19	857908	2·02	982814	4·21	017186	8
53	840854	2·19	857786	2·02	983067	4·21	016933	7
54	840985	2·19	857665	2·03	983320	4·21	016680	6
55	841116	2·18	857543	2·03	983573	4·21	016427	5
56	841247	2·18	857422	2·03	983826	4·21	016174	4
57	841378	2·18	857300	2·03	984079	4·21	015921	3
58	841509	2·18	857178	2·03	984331	4·21	015669	2
59	841640	2·18	857056	2·03	984584	4·21	015416	1
60	841771	2·18	856934	2·03	984837	4·21	015163	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·841771	2·18	9·856934	2·03	9·984837	4·21	10·015163	60
1	841902	2·18	856812	2·03	985090	4·21	014910	59
2	842033	2·18	856690	2·04	985343	4·21	014657	58
3	842163	2·17	856568	2·04	985596	4·21	014404	57
4	842294	2·17	856446	2·04	985848	4·21	014152	56
5	842424	2·17	856323	2·04	986101	4·21	013899	55
6	842555	2·17	856201	2·04	986354	4·21	013646	54
7	842685	2·17	856078	2·04	986607	4·21	013393	53
8	842815	2·17	855956	2·04	986860	4·21	013140	52
9	842946	2·17	855833	2·04	987112	4·21	012888	51
10	843076	2·17	855711	2·05	987365	4·21	012635	50
11	9·843206	2·16	9·855588	2·05	9·987618	4·21	10·012382	49
12	843336	2·16	855465	2·05	987871	4·21	012129	48
13	843466	2·16	855342	2·05	988123	4·21	011877	47
14	843595	2·16	855219	2·05	988376	4·21	011624	46
15	843725	2·16	855096	2·05	988629	4·21	011371	45
16	843855	2·16	854973	2·05	988882	4·21	011118	44
17	843984	2·16	854850	2·05	989134	4·21	010866	43
18	844114	2·15	854727	2·06	989387	4·21	010613	42
19	844243	2·15	854603	2·06	989640	4·21	010360	41
20	844372	2·15	854480	2·06	989893	4·21	010107	40
21	9·844502	2·15	9·854356	2·06	9·990145	4·21	10·009855	39
22	844631	2·15	854233	2·06	990398	4·21	009602	38
23	844760	2·15	854109	2·06	990651	4·21	009349	37
24	844889	2·15	853986	2·06	990903	4·21	009097	36
25	845018	2·15	853862	2·06	991156	4·21	008844	35
26	845147	2·15	853738	2·06	991409	4·21	008591	34
27	845276	2·14	853614	2·07	991662	4·21	008338	33
28	845405	2·14	853490	2·07	991914	4·21	008086	32
29	845533	2·14	853366	2·07	992167	4·21	007833	31
30	845662	2·14	853242	2·07	992420	4·21	007580	30
31	9·845790	2·14	9·853118	2·07	9·992672	4·21	10·007328	29
32	845919	2·14	852994	2·07	992925	4·21	007075	28
33	846047	2·14	852869	2·07	993178	4·21	006822	27
34	846175	2·14	852745	2·07	993430	4·21	006570	26
35	846304	2·14	852620	2·07	993683	4·21	006317	25
36	846432	2·13	852496	2·08	993936	4·21	006064	24
37	846560	2·13	852371	2·08	994189	4·21	005811	23
38	846688	2·13	852247	2·08	994441	4·21	005559	22
39	846816	2·13	852122	2·08	994694	4·21	005306	21
40	846944	2·13	851997	2·08	994947	4·21	005053	20
41	9·847071	2·13	9·851872	2·08	9·995199	4·21	10·004801	19
42	847199	2·13	851747	2·08	995452	4·21	004548	18
43	847327	2·13	851622	2·08	995705	4·21	004295	17
44	847454	2·12	851497	2·09	995957	4·21	004043	16
45	847582	2·12	851372	2·09	996210	4·21	003790	15
46	847709	2·12	851246	2·09	996463	4·21	003537	14
47	847836	2·12	851121	2·09	996715	4·21	003285	13
48	847964	2·12	850996	2·09	996968	4·21	003032	12
49	848091	2·12	850870	2·09	997221	4·21	002779	11
50	848218	2·12	850745	2·09	997473	4·21	002527	10
51	9·848345	2·12	9·850619	2·09	9·997726	4·21	10·002274	9
52	848472	2·11	850493	2·10	997979	4·21	002021	8
53	848599	2·11	850368	2·10	998231	4·21	001769	7
54	848726	2·11	850242	2·10	998484	4·21	001516	6
55	848852	2·11	850116	2·10	998737	4·21	001263	5
56	848979	2·11	849990	2·10	998989	4·21	001011	4
57	849106	2·11	849864	2·10	999242	4·21	000758	3
58	849232	2·11	849738	2·10	999495	4·21	000505	2
59	849359	2·11	849611	2·10	999748	4·21	000253	1
60	849485	2·11	849485	2·10	10·000000	4·21	10·000000	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.